**CHENNAI MATHEMATICAL INSTITUTE**

**Graduate Programme in Mathematics - M.Sc./Ph.D.**

*Entrance Examination, 2012*

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## Part A

State whether True or False and give brief reasons in the sheets provided. Marks will be given only when reasons are provided. Try to answer 10 questions. Each question carries 5 marks.

1. The function $f : \mathbb{R}^n \to \mathbb{R}$, defined as $f(x_1, \cdots, x_n) = Max\{|x_i|\}, i = 1, \cdots, n$, is uniformly continuous.
2. Let $x_n$ be a sequence with the following property: Every subsequence of $x_n$ has a further subsequence which converges to $x$. Then the sequence $x_n$ converges to $x$.
3. Let $f : (0, \infty) \longrightarrow \mathbb{R}$ be a continuous function. Then $f$ maps any Cauchy sequence to a Cauchy sequence.
4. Let $\{f_n : \mathbb{R} \longrightarrow \mathbb{R}\}$ be a sequence of continuous functions. Let $x_n \longrightarrow x$ be a convergent sequence of reals. If $f_n \longrightarrow f$ uniformly then $f_n(x_n) \longrightarrow f(x)$.
5. Let $K \subset \mathbb{R}^n$ such that every real valued continuous function on $K$ is bounded. Then $K$ is compact (i.e closed and bounded).
6. If $A \subset \mathbb{R}^2$ is a countable set, then $\mathbb{R}^2 \setminus A$ is connected.
7. The set $A = \{(z, w) \in \mathbb{C}^2 \mid z^2 + w^2 = 1\}$ is bounded in $\mathbb{C}^2$.
8. Let $f, g : \mathbb{C} \longrightarrow \mathbb{C}$ be complex analytic, and let $h : [0, 1] \longrightarrow \mathbb{C}$ be a non-constant continuous map. Suppose $f(z) = g(z)$ for every $z \in \text{Im } h$, then $f = g$. (Here Im $h$ denotes the image of the function $h$.)
9. There is a field with 121 elements.
10. The matrix $\begin{pmatrix} \pi & \pi \\ 0 & \frac{22}{7} \end{pmatrix}$ is diagonalizable over $\mathbb{C}$.
11. There are no infinite group with subgroups of index 5.
12. Every finite group of odd order is isomorphic to a subgroup of $A_n$, the group of all even permutations.
13. Every group of order 6 abelian.

14. Two abelian groups of the same order are isomorphic.

15. There is a non-constant continuous function $f : \mathbb{R} \to \mathbb{R}$ whose image is contained in $\mathbb{Q}$.

## Part B

Each question carries 10 marks. Try to answer 5 questions.

1. Suppose $f : \mathbb{R} \mapsto \mathbb{R}^n$ be a differentiable mapping satisfying $\|f(t)\| = 1$ for all $t \in \mathbb{R}$. Show that $\langle f'(t), f(t) \rangle = 0$ for all $t \in \mathbb{R}$. (Here $\|.\|$ denotes standard norm or length of a vector in $\mathbb{R}^n$, and $\langle ., . \rangle$ denotes the standard inner product (or scalar product) in $\mathbb{R}^n$.)

2. Let $A, B \subset \mathbb{R}^n$ and define $A + B = \{a + b;\ a \in A,\ b \in B\}$. If $A$ and $B$ are open, is $A + B$ open? If $A$ and $B$ are closed, is $A + B$ closed? Justify your answers.

3. Let $f : X \mapsto Y$ be continuous map onto $Y$, and let $X$ be compact. Also $g : Y \mapsto Z$ is such that $g \circ f$ is continuous. Show $g$ is continuous.

4. Let $A$ be a $n \times m$ matrix with real entries, and let $B = AA^t$ and let $\alpha$ be the supremum of $x^t Bx$ where supremum is taken over all vectors $x \in \mathbb{R}^n$ with norm less than or equal to 1. Consider
$$C_k = I + \sum_{j=1}^{k} B^j.$$
Show that the sequence of matrices $C_k$ converges if and only if $\alpha < 1$.

5. Show that a power series $\sum_{n \geq 0} a_n z^n$ where $a_n \to 0$ as $n \to \infty$ cannot have a pole on the unit circle. Is the statement true with the hypothesis that $(a_n)$ is a bounded sequence?

6. Show that a biholomorphic map of the unit ball onto itself which fixes the origin is necessarily a rotation.

7. (i) Let $G = GL(2, \mathbb{F}_p)$. Prove that there is a Sylow $p-$subgroup $H$ of $G$ whose normalizer $N_G(H)$ is the group of all upper triangular matrices in $G$.

   (ii) Hence prove that the number of Sylow subgroups of $G$ is $1 + p$.

8. Calculate the minimal polynomial of $\sqrt{2}e^{\frac{2\pi i}{3}}$ over $\mathbb{Q}$.

9. Let $G$ be a group $\mathbb{F}$ a field and $n$ a positive integer. A *linear action* of $G$ on $\mathbb{F}^n$ is a map $\alpha : G \times \mathbb{F}^n \to \mathbb{F}^n$ such that $\alpha(g, v) = \rho(g)v$ for some group homomorphism $\rho : G \to \mathrm{GL}_n(\mathbb{F})$. Show that for every finite group $G$, there is an $n$ such that there is a linear action $\alpha$ of $G$ on $\mathbb{F}^n$ and such that there is a nonzero vector $v \in \mathbb{F}^n$ such that $\alpha(g, v) = v$ for all $g \in G$.

10. Let $R$ be an integral domain containing a field $F$ as a subring. Show that if $R$ is a finite-dimensional vector space over $F$, then $R$ is a field.