State whether True or False and give brief reasons in the sheets provided. Marks will be given only when reasons are provided. Try to answer 10 questions. Each question carries 5 marks.

1. The function \( f : \mathbb{R}^n \to \mathbb{R} \), defined as \( f(x_1, \cdots, x_n) = \max\{|x_i|, i = 1, \cdots, n\} \), is uniformly continuous.

2. Let \( x_n \) be a sequence with the following property: Every subsequence of \( x_n \) has a further subsequence which converges to \( x \). Then the sequence \( x_n \) converges to \( x \).

3. Let \( f : (0, \infty) \to \mathbb{R} \) be a continuous function. Then \( f \) maps any Cauchy sequence to a Cauchy sequence.

4. Let \( \{f_n : \mathbb{R} \to \mathbb{R}\} \) be a sequence of continuous functions. Let \( x_n \to x \) be a convergent sequence of reals. If \( f_n \to f \) uniformly then \( f_n(x_n) \to f(x) \).

5. Let \( K \subset \mathbb{R}^n \) such that every real valued continuous function on \( K \) is bounded. Then \( K \) is compact (i.e closed and bounded).

6. If \( A \subset \mathbb{R}^2 \) is a countable set, then \( \mathbb{R}^2 \setminus A \) is connected.

7. The set \( A = \{(z, w) \in \mathbb{C}^2 \mid z^2 + w^2 = 1\} \) is bounded in \( \mathbb{C}^2 \).

8. Let \( f, g : \mathbb{C} \to \mathbb{C} \) be complex analytic, and let \( h : [0, 1] \to \mathbb{C} \) be a non-constant continuous map. Suppose \( f(z) = g(z) \) for every \( z \in \text{Im} \ h \), then \( f = g \). (Here \( \text{Im} \ h \) denotes the image of the function \( h \).)

9. There is a field with 121 elements.

10. The matrix \( \begin{pmatrix} \pi & \pi \\ 0 & \frac{22}{7} \end{pmatrix} \) is diagonalizable over \( \mathbb{C} \).

11. There are no infinite group with subgroups of index 5.

12. Every finite group of odd order is isomorphic to a subgroup of \( A_n \), the group of all even permutations.

13. Every group of order 6 abelian.
14. Two abelian groups of the same order are isomorphic.

15. There is a non-constant continuous function $f : \mathbb{R} \to \mathbb{R}$ whose image is contained in $\mathbb{Q}$. 
Part B

Each question carries 10 marks. Try to answer 5 questions.

1. Suppose $f : \mathbb{R} \mapsto \mathbb{R}^n$ be a differentiable mapping satisfying $\|f(t)\| = 1$ for all $t \in \mathbb{R}$.
   Show that $\langle f'(t), f(t) \rangle = 0$ for all $t \in \mathbb{R}$. (Here $\|\|$ denotes standard norm or length of a vector in $\mathbb{R}^n$, and $\langle.,.\rangle$ denotes the standard inner product (or scalar product) in $\mathbb{R}^n$.)

2. Let $A, B \subset \mathbb{R}^n$ and define $A + B = \{a + b; \ a \in A, \ b \in B\}$. If $A$ and $B$ are open, is $A + B$ open? If $A$ and $B$ are closed, is $A + B$ closed? Justify your answers.

3. Let $f : X \mapsto Y$ be continuous map onto $Y$, and let $X$ be compact. Also $g : Y \mapsto Z$ is such that $g \circ f$ is continuous. Show $g$ is continuous.

4. Let $A$ be a $n \times m$ matrix with real entries, and let $B = AA^t$ and let $\alpha$ be the supremum of $x^tBx$ where supremum is taken over all vectors $x \in \mathbb{R}^n$ with norm less than or equal to 1. Consider
   \[
   C_k = I + \sum_{j=1}^{k} B^j. \]
   Show that the sequence of matrices $C_k$ converges if and only if $\alpha < 1$.

5. Show that a power series $\sum_{n \geq 0} a_n z^n$ where $a_n \to 0$ as $n \to \infty$ cannot have a pole on the unit circle. Is the statement true with the hypothesis that $(a_n)$ is a bounded sequence?

6. Show that a biholomorphic map of the unit ball onto itself which fixes the origin is necessarily a rotation.

7. (i) Let $G = GL(2, \mathbb{F}_p)$. Prove that there is a Sylow $p$--subgroup $H$ of $G$ whose normalizer $N_G(H)$ is the group of all upper triangular matrices in $G$.

   (ii) Hence prove that the number of Sylow subgroups of $G$ is $1 + p$.

8. Calculate the minimal polynomial of $\sqrt{2} e^{\frac{2\pi}{3}i}$ over $\mathbb{Q}$.

9. Let $G$ be a group $\mathbb{F}$ a field and $n$ a positive integer. A linear action of $G$ on $\mathbb{F}^n$ is a map $\alpha : G \times \mathbb{F}^n \mapsto \mathbb{F}^n$ such that $\alpha(g, v) = \rho(g)v$ for some group homomorphism $\rho : G \mapsto GL_n(\mathbb{F})$. Show that for every finite group $G$, there is an $n$ such that there is a linear action $\alpha$ of $G$ on $\mathbb{F}^n$ and such that there is a nonzero vector $v \in \mathbb{F}^n$ such that $\alpha(g, v) = v$ for all $g \in G$.

10. Let $R$ be an integral domain containing a field $F$ as a subring. Show that if $R$ is a finite-dimensional vector space over $F$, then $R$ is a field.