CHENNAI Mathematical Institute

Graduate Programme in Mathematics

Entrance Examination, 2010

Part A

State whether True or False and give brief reasons in the sheets provided (e.g., if you feel that a statement is "False" then give a counter-example). Marks will be given only when reasons are provided.

1. Suppose A is an $m \times n$ matrix, V an $m \times 1$ matrix, with both A and V having rational entries. If the equation $AX = V$ has a solution in $\mathbb{R}^n$, then the equation has a solution with rational entries. (Here and in Question 5 below of Part A, $\mathbb{R}^n$ is identified with the space of $n \times 1$ real matrices.)

2. A closed and bounded subset of a complete metric space is compact.

3. Let p be a prime number. If P is a p-Sylow subgroup of some finite group G, then for every subgroup H of G, $H \cap P$ is a p-Sylow subgroup of H.

4. There exists a real 3×3 orthogonal matrix with only non-zero entries.

5. A 5×5 real matrix has an eigenvector in $\mathbb{R}^5$.

6. A continuous function on $\mathbb{Q} \cap [0,1]$ can be extended to a continuous function on $[0,1]$.

7. Let $f\colon \mathbb{R} \to \mathbb{R}$ be a differentiable function. Then $f'(x)$ is continuous.

8. There is a continuous onto function from the unit sphere in $\mathbb{R}^3$ to the complex plane $\mathbb{C}$.

9. $f\colon \mathbb{C} \to \mathbb{C}$ is an entire function such that the function $g(z)$ given by $g(z) = f(\frac{1}{z})$ has a pole at 0. Then f is a surjective map.

10. Every finite group of order 17 is abelian.

11. Let $n \geq 2$ be an integer. Given an integer k there exists an $n \times n$ matrix A with integer entries such that $\det A = k$ and the first row of A is $(1, 2, \ldots, n)$.

12. There is a finite Galois extension of $\mathbb{R}$ whose Galois group is nonabelian.

13. There is a non-constant continuous function from the open unit disc

$$D = \{z \in \mathbb{C} \mid |z| < 1\}$$

to $\mathbb{R}$ which takes only irrational values.

14. There is a field of order 121.

Part B

Answer all questions.

1. Let α, β be two complex numbers with $\beta \neq 0$, and $f(z)$ a polynomial function on $\mathbb{C}$ such that $f(z) = \alpha$ whenever $z^5 = \beta$. What can you say about the degree of the polynomial $f(z)$?

2. Let $f, g \colon \mathbb{Z}/5\mathbb{Z} \to S_5$ be two non-trivial group homomorphisms. Show that there is a $\sigma \in S_5$ such that $f(x) = \sigma g(x) \sigma^{-1}$, for every $x \in \mathbb{Z}/5\mathbb{Z}$.

3. Suppose f is continuous on $[0, \infty)$, differentiable on $(0, \infty)$ and $f(0) \geq 0$. Suppose $f'(x) \geq f(x)$ for all $x \in (0, \infty)$. Show that $f(x) \geq 0$ for all $x \in (0, \infty)$.

4. A linear transformation $T \colon \mathbb{R}^8 \to \mathbb{R}^8$ is defined on the standard basis $e_1, \ldots, e_8$ by

$$\begin{aligned} Te_j &= e_{j+1} \qquad j = 1, \ldots, 5 \\ Te_6 &= e_7 \\ Te_7 &= e_6 \\ Te_8 &= e_2 + e_4 + e_6 + e_8. \end{aligned}$$

   What is the nullity of T?

5. If f and g are continuous functions on $[0, 1]$ satisfying $f(x) \geq g(x)$ for every $0 \leq x \leq 1$, and if $\int_0^1 f(x)dx = \int_0^1 g(x)dx$, then show that $f = g$.

6. Let $\{a_n\}$ and $\{b_n\}$ be sequences of complex numbers such that each a_n is non-zero, $\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n = 0$, and such that for every natural number k,

$$\lim_{n\to\infty} \frac{b_n}{a_n^k} = 0.$$

   Suppose f is an analytic function on a connected open subset U of $\mathbb{C}$ which contains 0 and all the a_n. Show that if $f(a_n) = b_n$ for every natural number n, then $b_n = 0$ for every natural number n.

7. Let $T \colon \mathbb{R}^3 \to \mathbb{R}^3$ be an orthogonal transformation such that $\det T = 1$ and T is not the identity linear transformation. Let $S \subset \mathbb{R}^3$ be the unit sphere, i.e.,

$$S = \{(x, y, z) \mid x^2 + y^2 + z^2 = 1\}.$$

   Show that T fixes exactly two points on S.

8. Compute

$$\int_0^\infty \frac{x^{1/3}}{1+x^2}\, dx.$$

9. Let $f(x) = x^n + a_{n-1}x^{n-1} + \cdots + a_0$ be a polynomial with integer coefficients and whose degree is at least 2. Suppose each a_i $(0 \le i \le n-1)$ is of the form

$$a_i = \pm \frac{17!}{r!(17-r)!}$$

with $1 \le r \le 16$. Show that $f(m)$ is not equal to zero for any integer m.

10. Suppose $\varphi = (\varphi_2, \ldots, \varphi_n)\colon \mathbb{R}^n \to \mathbb{R}^{n-1}$ is a C^2 function, i.e. all second order partial derivatives of the φ_i exist and are continuous. Show that the symbolic determinant

$$\begin{vmatrix} \frac{\partial}{\partial x_1} & \frac{\partial \varphi_2}{\partial x_1} & \cdots & \frac{\partial \varphi_n}{\partial x_1} \\ \vdots & \vdots & & \vdots \\ \frac{\partial}{\partial x_n} & \frac{\partial \varphi_2}{\partial x_n} & \cdots & \frac{\partial \varphi_n}{\partial x_n} \end{vmatrix}$$

vanishes indentically.