Part A

State whether True or False and give brief reasons in the sheets provided. Marks will be given only when reasons are provided.

1. Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function such that $f(x+1) = f(x)$, $\forall x$. Then f is uniformly continuous.
2. Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function such that $|f(x) - f(y)| \geq |x - y|$, $\forall x, y$. Then f is surjective.
3. Let $f : \mathbb{R} \to \mathbb{R}$ be a differentiable function. Then $f'(x)$ is continuous.
4. The group $(\mathbb{Q}, +)$ is a direct product of two nontrivial subgroups.
5. A group of order 10 which has a normal subgroup of order 2 is abelian.
6. Let G be a cyclic group and $a, b \in G$ be elements which are not squares. Then $a \cdot b \in G$ is a square.
7. There are infinite groups with subgroups of index 5.
8. Two abelian groups of the same order are isomorphic.
9. Let $K \subset \mathbb{R}^n$ such that every real valued continuous function on K is bounded. Then K is compact (i.e closed and bounded).
10. There exist nonempty connected subset of $X \subset \mathbb{R}$ such that X has more than two elements and has only rational numbers in it.
11. A linear transformation A on $\mathbb{R}^2$ has $\frac{2}{3}$ and $\frac{7}{4}$ as its eigen values. Then for a real vector v, one has $\|Av\| > 2\|v\|$?
12. Let $f(x)$ and $g(x)$ be two monic polynomials of the same degree such that adding 1 to the roots of $f(x)$ we get the roots of $g(x)$. Then the constant term of $g(x)$ can be found from the constant term of $f(x)$.
13. G is a non-abelian group with nontrivial centre C. The the centre of G/C trivial.
    .
14. H is a normal cyclic subgroup of a finite group G such that G/H is cyclic. Then G is cyclic? .

15. Let $f\colon \mathbb{C} \to \mathbb{C}$ be the function given by

$$f(z) = (7 - 11i)^2 z^3 + 6z^2 - (4 + i)z - \frac{8}{3}.$$

Then the complex number $\cos \pi/7 + i \sin \pi/7$ belongs to the image of f.

16. Let X, Y be two disjoint connected subsets of the plane $\mathbb{R}^2$. The $X \cup Y$ can be the whole of $\mathbb{R}^2$.

Part B

Answer all questions.

1. Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function. Show that
$$\int_0^1 f(x)x^2 dx = \frac{1}{3}f(c)$$
for some $c \in [0, 1]$.

2. Let $f : [0, 1] \to \mathbb{R}$ be continuous. Evaluate
$$\lim_{n\to\infty} \int_0^1 x^n f(x)dx$$

3. Let $a_1, a_2, \ldots a_n$ be distinct real numbers. Show that $e^{a_1 x}, \ldots, e^{a_n x}$ are linearly independent over $\mathbb{R}$.

4. Let V, W be finite dimensional vector spaces over a field k and let $Z \subset W$ be a subspace. Let $T : V \to W$ be a linear map. Prove that
$$dim(T^{-1}(Z)) \leq dim(V) - dim(W) + dim(Z)$$

5. Let $f : \mathbb{C} \to \mathbb{C}$ be an analytic function such that $h(0) = 3 + 4i$ and $|h(z)| \leq 5$ if $|z| < 1$. What is $h'(0)$?

6. Let $M(n)$ be the space of real $n \times n$–matrices. Regard this as a metric space with distance function
$$d(A, B) = \sum_{i,j=}^{n} |a_{ij} - b_{ij}|$$
where $A = (a_i j)$ and $B = (b_i j)$. Prove that the subset $N \subset M(n)$ consisting of matrices A such that $A^k = 0$ for some k is a closed subset of $M(n)$.

7. Show that there exist infinitely many quadratic extensions of $\mathbb{Q}$ which are pairwise nonisomorphic.

8. Let A, B be two linear maps $\mathbb{R}^3 \to \mathbb{R}^3$, corresponding to rotations by 30° around x-axis, and rotation by $22\frac{1}{2}^\circ$ around z-axis respectively. For the composed map $B \circ A$ what is the norm of the image of the vector $v = (-3, 5, 1)$?

9. Let A be an $n \times n$–matrix with complex entries such that $Trace(A) = 0$. Then show that A is similar to a matrix with 0's in the diagonal entries.