CHENNAI MATHEMATICAL INSTITUTE

M.Sc. / Ph.D. Programme in Computer Science

Entrance Examination, 2024

Part A has 10 questions of 3 marks each. Each question in Part A has four choices, of which exactly one is correct. Part B has 7 questions of 10 marks each. The total marks are 100. Answers to Part A must be filled in the answer sheet provided.
In all questions related to graphs, unless otherwise specified, we use the word "graph" to mean an undirected graph with no self-loops, and at most one edge between any pair of vertices.

Part A

1. All inhabitants of the Old Forest are either Ents or Bents. Ents always tell the truth and Bents always lie. During a visit to the Old Forest, you encounter four inhabitants – A, B, C and D. They make the following assertions.

   A: Exactly one of us is a Bent.
   B: Exactly two of us are Bents.
   C: Exactly three of us are Bents.
   D: Exactly four of us are Bents.

   How many of them are Bents?

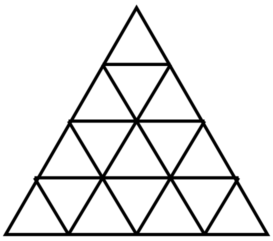
   (a) 1 (b) 2 (c) 3 (d) 4

2. Friends Akshay and Bipasha go to a fun fair, and decide to explore the stalls separately. Each of them visits a stall every 10 minutes, starting at 8am. If a stall that Akshay visits is one that he has not seen before, he texts *new* to Bipasha; otherwise he texts *old*. If a stall that Bipasha visits is new to her she texts *new* to Akshay; otherwise she texts *old*. A *round* refers to such a pair of messages, one sent by Akshay and the other sent by Bipasha. If there are 50 stalls, what is the maximum number of rounds they can go before both of them text *old* to each other in the same round?

   (a) 99 (b) 100 (c) 2500 (d) 50

3. You live in Δ-City, which is a large equilateral triangle with each side of length 2km. The corners are named A, B and C. The city is partitioned into triangular blocks with sides of 500m each, and there are roads at the boundaries of the blocks.

   You live in corner A and your office is at corner B.



   You have decided to walk from home to the office everyday, and you are willing to change your route, as long as the total distance is at most 2.5kms. How many such routes are possible from A to B?

   (a) 5 (b) 11 (c) 21 (d) 45

4. Rohit and Ben participate in a new toss system introduced by the ICC. The umpire repeatedly tosses a fair coin (which comes up heads with probability $\frac{1}{2}$ and tails with probability $\frac{1}{2}$), until one of them wins. Rohit wins if the result of two consecutive tosses is heads (i.e., the pattern HH is seen), while Ben wins if the pattern TH is seen. What are the winning probabilities for Rohit and Ben?

   (a) $\frac{1}{4}$ and $\frac{3}{4}$ (b) $\frac{1}{4}$ and $\frac{1}{4}$ (c) $\frac{1}{2}$ and $\frac{1}{2}$ (d) $\frac{1}{3}$ and $\frac{2}{3}$

5. Let $\Sigma = \{a, b, c\}$. What is the language generated by the following grammar?

$$S := \epsilon \mid aS \mid Sb \mid cS$$

   (a) $(a + b + c)^*b^*$ (b) $(a + b + c)^*c^*$
   (c) $(a + c)^*b^*$ (d) $(a + b)^*c^*$

6. Let $\Sigma = \{a_1, a_2, \ldots, a_n\}$ for some $n \geq 1$. Consider the two languages:

$$L_1 = \{w \in \Sigma^* \mid \exists a_i \in \Sigma \text{ such that } a_i \text{ occurs at least two times in } w\}$$
$$L_2 = \Sigma^* \ (a_1a_1 + a_2a_2 + \cdots a_na_n) \ \Sigma^*$$

   Which of the following statements are true?

   (I) There is an NFA with $O(n)$ states accepting L_1.

   (II) There is a DFA with $O(n)$ states accepting L_1.

   (III) There is an NFA with $O(n)$ states accepting L_2.

   (IV) There is a DFA with $O(n)$ states accepting L_2.

   (a) All of the above (b) I, III, IV
   (c) I and III (d) I, II and III

7. Consider the following two recurrence relations:

   - $T_1(n) = T_1(\frac{n}{2}) + T_1(\frac{n}{3}) + \Theta(n)$, $T_1(1) = 2$
   - $T_2(n) = T_2(\frac{2n}{3}) + T_2(\frac{n}{3}) + \Theta(n)$, $T_2(1) = 2$

   Which of the following statements is true?

   (a) $T_1(n) = \Theta(n)$ and $T_2(n) = \Theta(n)$
   (b) $T_1(n) = \Theta(n \log n)$ and $T_2(n) = \Theta(n \log n)$
   (c) $T_1(n) = \Theta(n)$ and $T_2(n) = \Theta(n \log n)$
   (d) $T_1(n) = \Theta(n \log n)$ and $T_2(n) = \Theta(n)$

8. Let G be an undirected connected graph with distinct edge weights. Let $e_{\max}$ be the edge with maximum weight and $e_{\min}$ the edge with minimum weight. What can you say about the following statements?

   (I) Every minimum spanning tree of G must contain $e_{\min}$
   (II) Every minimum spanning tree must exclude $e_{\max}$

   (a) I is True, but II is False
   (b) I is False, but II is True
   (c) Both I and II are False
   (d) Both I and II are True

9. Let G be a directed graph with distinct and nonnegative edge weights. Let s be a starting vertex and t a destination vertex. Assume that G has at least one s-t path. What can you say about the following statements?

   (I) Every shortest s-t path (minimum weight) must include the minimum-weight edge of G.
   (II) Every shortest s-t path must exclude the maximum-weight edge of G.

   (a) I is True, but II is False
   (b) I is False, but II is True
   (c) Both I and II are False
   (d) Both I and II are True

10. In the following code, A is an array indexed from 0, and for two integers a, b the expression $a//b$ returns $\lfloor \frac{a}{b} \rfloor$, the largest integer which is not larger than a/b.

```
FOO(A, first, last)
if first ≥ last
    then return A[first]
    else
        mid ← (first + last)//2
        l ← FOO(A, first, mid)
        r ← FOO(A, mid+1, last)
        return l + r
```

   If $A = [1, 2, 3, 4, 5, 6]$, what will FOO$(A, 0, 5)$ return?

   (a) 3 (b) 7 (c) 15 (d) 21

Part B

1. A *binary search tree* is a binary tree whose proper subtrees are binary search trees, and whose root is strictly greater than all elements in the left subtree, and strictly less than all elements in the right subtree. A *preorder listing* of a tree is obtained by listing the root first, then recursively listing all elements of the left subtree in preorder, followed by all elements of the right subtree in preorder.

   Provide algorithms for the following two problems, and calculate their worst-case running times.

   (a) Input is an array of integers $A[1..n]$. Output is "Yes" if A is the preorder listing of a binary search tree, and "No" otherwise.

   (b) Input is an array of integers $A[1..n]$ which is guaranteed to be the preorder listing of a binary search tree. Output is a binary tree t such that A is the preorder listing of t.

2. Let $M_1 = (Q_1, \{q_1\}, \Delta_1, F_1)$, where $\Delta_1 \subseteq Q_1 \times (\Sigma \cup \{\epsilon\}) \times Q_1$, be a non deterministic finite automaton (NFA) accepting a language $L_1 \subseteq \{0,1\}^*$. Let ϵ denote the null string. We construct a new NFA $M_2 = (Q,\{q_2\}, \Delta_2, F_2)$, where $\Delta_2 \subseteq Q_2 \times (\Sigma \cup \{\epsilon\}) \times Q_2$, as follows.

   - $Q_2 = Q_1$.
   - $q_2 = q_1$.
   - $F_2 = F_1 \cup \{q_1\}$.
   - $(p, a, p') \in \Delta_2$ iff either $(p, a, p') \in \Delta_1$ or ($p \in F_1$ and $a = \epsilon$ and $p' = q_1$)

   Prove or disprove: The language L_2 accepted by M_2 is L_1^*.

3. You are using a fair coin to walk along the number line, starting at position 0. You toss the coin. If you get heads you move two steps to your right, to position 2. On the other hand if you get tails you move one step to your right, to position 1. You continue tossing the coin. Every time you get heads you move two steps to the right of the position you are in currently, and if you get tails you move one step to the right of the position you are in currently.

   Show that the probability of landing in position n is

   $$\frac{1}{3}[2 + (-\frac{1}{2})^n]$$

   .

4. Suppose all points on the 2D plane with integer coordinates are coloured either blue or green. Show that there is an isosceles right angled triangle all of whose vertices are the same colour.

5. Let Σ be a finite alphabet. The *reverse* of a word is defined inductively as follows: $\mathsf{rev}(\epsilon) = \epsilon$ and $\mathsf{rev}(wa) = a \cdot \mathsf{rev}(w)$ for $w \in \Sigma^*$ and $a \in \Sigma$. For example, $\mathsf{rev}(aab) = baa$.

   For a language L, we define $\mathsf{rev}(L) := \{\mathsf{rev}(w) \mid w \in L\}$.

   (a) Is $\mathsf{rev}(L_1 \cap L_2)$ equal to $\mathsf{rev}(L_1) \cap \mathsf{rev}(L_2)$?

(b) Prove or disprove the following statement: $w \in L \cap \mathsf{rev}(L)$ iff $w = \mathsf{rev}(w)$.

(c) Show that if L is regular, $\mathsf{rev}(L)$ is also regular.

6. A *tournament* is a directed graph that has exactly one directed edge between each pair of vertices. A king in a tournament is a vertex v such that every other vertex is reachable from v via a directed path of length at most 2.

   (a) Prove that in any tournament there is at least one king.

   (b) Can there be more than one king in a tournament? Justify your answer.

7. A *subsequence* of an array A is any sub-array of A, obtained by deleting zero or more elements of A *without* changing the order of the remaining elements. The input to the SUBSEQUENCE SUM problem consists of (i) an array $A[1 \ldots n]$ of n positive integers, and (ii) a target integer $T \geq 0$. The problem is to decide if there exists a subsequence B of A such that the sum of all the elements of B is exactly T. We define the sum of the *empty* subsequence (one with no elements) to be zero, and the sum of a subsequence with one element, to be that element.

   Describe an algorithm that solves this problem in $O(n^c T^d)$ time for some constants c, d. The algorithm should take an array $A[1 \ldots n]$ and an integer T as described above. It should output True if there is a subsequence B of A such that the sum of all the elements of B is exactly T, and False otherwise. It is *not* required that the algorithm find a subsequence whose sum is T.

   Clearly explain why your algorithm correctly solves the problem, and why it runs in time $O(n^c T^d)$. What are the constants c, d that you get?