CHENNAI MATHEMATICAL INSTITUTE

M.Sc. / Ph.D. Programme in Computer Science

Entrance Examination, 2023

Part A has 10 questions of 3 marks each. Each question in Part A has four choices, of which exactly one is correct. Part B has 7 questions of 10 marks each. The total marks are 100. Answers to Part A must be entered directly on the computer.

In all questions related to graphs, unless otherwise specified, we use the word "graph" to mean an undirected graph with no self-loops, and at most one edge between any pair of vertices.

Part A

1. A candy factory uses 5 fruit flavours $\{A, B, C, D, E\}$. Each candy is made using one or more of these flavours. The taste of a candy depends on which flavours are included.

   A kid may prefer some combination of flavours more than others. For example, the kid may prefer the combination $\{A, B, C\}$ over $\{B, C\}$. The *preference order* of a kid is a total *ordering* of all the combinations, the ones occurring earlier being preferred more.

   Suppose you want to throw a party and do not want more than one kid with the same preference order. What is the maximum number of kids that can attend such a party?

   (a) $(2^{(5!)}) - 1$

   (b) $((2^5) - 1)!$

   (c) $((5 * 4)/2) - 1$

   (d) $5 * 5$

2. How many elements are in the following set?

$$\{(A, B) \mid A \subseteq B \subseteq \{1, 2, 3, \ldots, n\}\}$$

   (a) 2^{n-1}

   (b) 3^n

   (c) 2^{n+1}

   (d) 2^{2n}

3. At a kindergarten, 2024 kids sit in a circle. Suddenly, each kid randomly pokes either the kid to their left or the one to their right with equal probability. What is the expected number of unpoked kids?

   (a) 1

   (b) 253

   (c) 506

   (d) 1012

4. A graph is k-regular if all the vertices have degree exactly k. What is the minimum number of vertices in a k-regular graph that has no 3-length cycles?

   (a) k
   (b) $k+1$
   (c) $2k$
   (d) $2k+1$

5. In a connected graph, any two paths of maximum length:

   (a) have at least one vertex in common, but not necessarily an edge in common
   (b) have at least one edge in common
   (c) have at least two common vertices, but not necessarily an edge in common
   (d) have at least two edges in common

6. Consider the automaton over the alphabet $\{a, b, c\}$ shown in Figure 1. The initial state is the leftmost state. States with a double circle are accepting states.

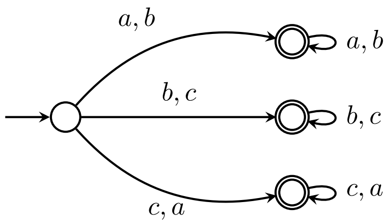

Figure 1: Automaton for Question 6

   What is the *complement* of the language accepted by this automaton?

   (a) $\{\epsilon\}$
   (b) $c^* + a^* + b^*$
   (c) $(a+b)^* + (b+c)^* + (c+a)^*$
   (d) None of the above

7. Let L be a regular language, and let $n = 10$. Which of the following statements is true?

   (a) $L \cup \{a^n b^n\}$ is regular
   (b) $L \cup \{a^n b^n\}$ is context-free, but not regular
   (c) $L \cup \{a^n b^n\}$ is regular, but not context-free
   (d) $L \cup \{a^n b^n\}$ is not context-free

8. Let $f : \mathbb{N} \to \mathbb{N}$ and $g : \mathbb{N} \to \mathbb{N}$ be functions over the set $\mathbb{N}$ of natural numbers. We will say:

   - $f(n) = O(g(n))$ if there exist natural numbers c and x_0 such that $f(n) \leq c\ g(n)$ for all $n \geq x_0$
   - $f(n) = 2^{O(g(n))}$ if there exist natural numbers c and x_0 such that $f(n) \leq 2^{(c\ g(n))}$ for all $n \geq x_0$.

   Consider the following statements

   (I) $3^n = O(2^n)$ (II) $3^n = 2^{O(n)}$

   (a) Both (I) and (II) are true
   (b) (I) is true and (II) is false
   (c) (I) is false and (II) is true
   (d) Both (I) and (II) are false

Questions 9 and 10 refer to the following two functions. We assume that all the arguments are non-negative integers. The operation `x div 2` divides `x` by 2 and returns an integer, discarding the fractional part.

```
int f(int x, int i) {
    if (i == 0) {
        if (even x)
            return 0;
        else
            return 1;
    } else {
        return f(x div 2, i-1);
    }
}
```

```
int g(int x) {
    if (x <= 2)
        return 1;
    else
        return 1 + g(x div 3);
}
```

9. For how many values of `i` will `f(1000,i)` return `1`?

   (a) 2
   (b) 5
   (c) 6
   (d) 10

10. What is the value of `g(10000)`?

   (a) 3
   (b) 9
   (c) 12
   (d) 15

Part B

1. Let $\Sigma = \{a, b\}$ be an alphabet. A palindrome is a word which reads the same when read from left-to-right, or from right-to-left. For example, the words *abba*, *aba* and *aa* are palindromes, whereas *aabb* and *ab* are not. A palindrome is said to be non-trivial if it has length at least 2.

   Let L be the set of all words that contain a non-trivial palindrome as a prefix, that is,

   $$L = \{w \in (a+b)^* \mid w = uv \text{ for some non-trivial palindrome } u \text{ and some } v \in (a+b)^* \}$$

   (a) Give an example of a word in L, which is not a palindrome.

   (b) Give three examples of words not in L.

   (c) Suppose $ax \notin L$ for some $x \in \Sigma^*$. What can you say about x?

   (d) Show that L is regular.

2. A non-empty collection $\mathcal{S}$ of subsets of a set U is a *scattering* of U if it satisfies the following condition:

   $$\text{for all } A, B \in \mathcal{S},\ A \text{ is not a proper subset of } B.$$

   (a) Let $U = \{1, 2, 3, 4, 5\}$. Give an example of a scattering of U.

   (b) Is the following statement true? Justify your answer with a proof or a counterexample.

   - $\mathcal{F}$ is a scattering of U if and only if $\{U \setminus A \mid A \in \mathcal{F}\}$ is a scattering of U, where $U \setminus A$ denotes the *complement* of A with respect to U.

3. Let $a_1, \ldots, a_n$ be integers. Show that for some k, m such that $1 \leq k \leq m \leq n$, the sum $a_k + a_{k+1} + \cdots + a_m$ is divisible by n. (*Hint:* Consider the sums $a_1 + \cdots + a_i$ modulo n, for $1 \leq i \leq n$.)

4. On the island of Knights, Knaves, and Normals, Knights always tell the truth, Knaves always lie, and Normals sometimes tell the truth and lie sometimes.

   One day Professor Raymond visited this island and met two inhabitants, A and B. He already knew that one of them was a Knight and the other was a Normal, but he didn't know which was which. He asked A whether B was normal, and received a Yes-or-No answer. Professor Raymond was then able to figure out who the Knight was.

   Who was the Knight – A or B?

5. You are starting a new bus service. You are hiring drivers and conductors. A driver and a conductor can run a bus only if they can speak a common language. There are n drivers and m conductors who have applied for a job. Their CV has a list of languages they speak.

   (a) Given the CVs of the drivers and conductors, we want to calculate the maximum number of buses that can be run. Show how to use graph matching to solve this problem. (*Note:* You do not need to solve graph matching itself. Just demonstrate how to use it for this problem.)

   (b) You are in a hurry to get the bus service running. Whenever you get a new application from a driver (or a conductor), you check if you can team this new candidate with an existing conductor (or driver) who is free. If yes, you assign the pair of them to a bus. If no, then the new candidate is added to the list of free driver (or conductors).

   Provide an example scenario where the above procedure does not compute the maximum number of buses that can be run.

6. The input to the problem consists of (i) an array $A[1, 2, \ldots, n]$ of n positive integers and (ii) a positive integer T. We are given the guarantee that at least one element of the array is less than or equal to T. The task is to find the *maximum* sum of a non-empty sub-collection of the integers from A which is less than or equal to T.

   Describe an algorithm that solves this problem in $O(nT)$ time. The algorithm should take an array $A[1, 2, \ldots, n]$ and an integer T as described above, and should output a number $T' \leq T$ that is closest to T and can be realized as the sum of some sub-collection of A. It is *not* required that the algorithm find the subset of indices which forms the sum T'.

7. Consider the following code which computes a function `f`. The input to `f` is an array `A[1..m]` which represents a number `N` in ternary. For example, `A = [1, 2, 0]` represents the number 15 since $15 = 1 * 3^2 + 2 * 3^1 + 0 * 3^0$; and `A = [1,0,0,0]` represents the number 27. Function `f` uses the function `pow(x,y)` that returns x^y. You may assume that the first element in the array is not zero.

```
int f(A[1..m]) {
    x = 1;
    for j = 1 to m {
        x = pow(x, 3);
        y = pow(3, A[j]);
        x = x * y;
    }
    return x;
}
```

   (a) What are `f([1,2])`, `f([1,2,0])` and `f([1,2,2])`?

   (b) Suppose array `A` represents the number `N`. What is `f(A)` in terms of `N`?