CHENNAI MATHEMATICAL INSTITUTE

M.Sc. / Ph.D. Programme in Computer Science

Entrance Examination, 15 May 2019

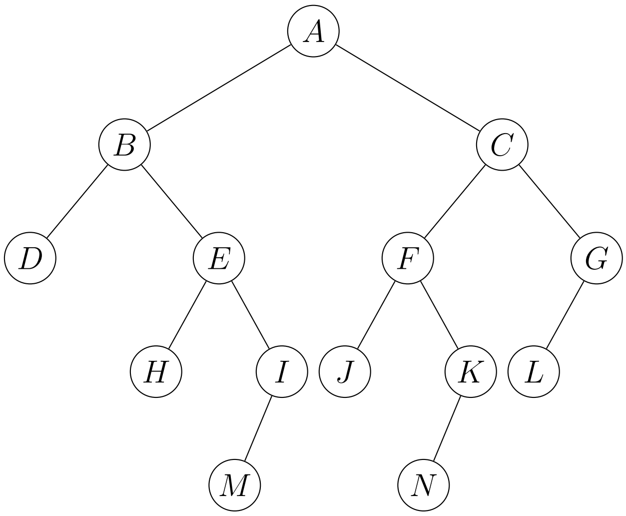
Part A has 10 questions of 3 marks each. Part B has 7 questions of 10 marks each. The total marks are 100.
Answers to Part A must be given in the special answer sheet provided for it. Answers to Part B must be written only in the designated space after the corresponding question.

Part A

1. Let $L_1 := \{\ a^n b^m \mid m, n \geq 0 \text{ and } m \geq n\ \}$ and $L_2 := \{\ a^n b^m \mid m, n \geq 0 \text{ and } m < n\ \}$. The language $L_1 \cup L_2$ is:
   (a) regular, but not context-free
   (b) context-free, but not regular
   (c) both regular and context-free
   (d) neither regular nor context-free

2. Let A be an NFA with n states.Which of the following is necessarily true?
   (a) The shortest word in $L(A)$ has length at most $n-1$.
   (b) The shortest word in $L(A)$ has length at least n.
   (c) The shortest word *not* in $L(A)$ has length at most $n-1$.
   (d) The shortest word *not* in $L(A)$ has length at least n.

3. Suppose that the figure to the right is a binary search tree. The letters indicate the names of the nodes, not the values that are stored. What is the predecesor node, in terms of value, of the root node A?



   (a) D
   (b) H
   (c) I
   (d) M

4. There are five buckets, each of which can be open or closed. An arrangement is a specification of which buckets are open and which are closed. Every person likes some of the arrangements and dislikes the rest. You host a party, and amazingly, no two people on the guest list have the same likes and dislikes. What is the maximum number of guests possible?
   (a) 5^2
   (b) $\binom{5}{2}$
   (c) 2^5
   (d) 2^{2^5}

5. Let $\pi = [x_1, x_2, \ldots, x_n]$ be a permutation of $\{1, 2, \ldots, n\}$. For $k < n$, we say that π has its first ascent at k if $x_1 > x_2 \ldots > x_k$ and $x_k < x_{k+1}$. Home many permutations have their first ascent at k?
   (a) $\binom{n}{k}$ - $\binom{n}{(k+1)}$
   (b) $\frac{n!}{k!} - \frac{n!}{(k+1)!}$
   (c) $\frac{n!}{(k+1)!} - \frac{n!}{(k+2)!}$
   (d) $\binom{n}{(k+1)}$ - $\binom{n}{(k+2)}$

6. Suppose you alternate between throwing a normal six-sided fair die and tossing a fair coin. You start by throwing the die. What is the probability that you will see a 5 on the die before you see tails on the coin?

   (a) $\frac{1}{12}$ (b) $\frac{1}{6}$ (c) $\frac{2}{9}$ (d) $\frac{2}{7}$

7. An interschool basketball tournament is being held at the Olympic sports complex. There are multiple basketball courts. Matches are scheduled in parallel, with staggered timings, to ensure that spectators always have some match or other available to watch. Each match requires a team of referees and linesmen. Two matches that overlap require disjoint teams of referees and linesmen. The tournament organizers would like to determine how many teams of referees and linesmen they need to mobilize to effectively conduct the tournament. To determine this, which graph theoretic problem do the organizers have to solve?

   (a) Find a minimal colouring.
   (b) Find a minimal spanning tree.
   (c) Find a minimal cut.
   (d) Find a minimal vertex cover.

8. We have constructed a polynomial time reduction from problem A to problem B. Which of the following is *not* a possible scenario?

   (a) We know of polynomial time algorithms for both A and B.
   (b) We only know of exponential time algorithms for both A and B.
   (c) We only know an exponential time algorithm for A, but we have a polynomial time algorithm for B.
   (d) We only know an exponential time algorithm for B, but we have a polynomial time algorithm for A.

The next two questions refer to the following program.

In the code below `reverse(A,i,j)` takes an array `A`, indices `i` and `j` with `i` $\leq$ `j`, and reverses the segment `A[i]`, `A[i+1]`,...,`A[j]`. For instance if `A = [0,1,2,3,4,5,6,7]` then, after we apply `reverse(A,3,6)`, the contents of the array will be `A = [0,1,2,6,5,4,3,7]`.

```
function mystery (A[0..99]) {
  int i,j,m;
  for (i = 0; i < 100; i++) {
    m = i;
    for (j = i; j < 100, j++) {
      if (A[j] > A[m]) {
        m = j;
      }
    }
    reverse(A,i,m);
  }
  return;
}
```

9. When the procedure terminates, the array `A` has been:

   (a) Sorted in descending order (b) Sorted in ascending order
   (c) Reversed (d) Left unaltered

10. The number of times the test `A[j] > A[m]` is executed is:

    (a) 4950 (b) 5050 (c) 10000 (d) Depends on the contents of `A`

Part B

Answers to Part B must be written only in the designated space after the corresponding question.

1. Consider an alphabet $\Sigma = \{a, b\}$.

   Let L_1 be the language given by the regular expression $(a+b)^*bb(a+b)^*$ and let L_2 be the language baa^*b.

   Define $L := \{u \in \Sigma^* \mid \exists w \in L_2 \text{ s.t. } uw \in L_1\}$.

   (a) Give an example of a word in L.

   (b) Give an example of a word *not in* L.

   (c) Construct an NFA for L.

2. Let us assume a binary alphabet $\Sigma = \{a, b\}$. Two words $u, v \in \Sigma^*$ are said to be conjugates if there exist $w_1, w_2 \in \Sigma^*$ such that $u = w_1 w_2$ and $v = w_2 w_1$. Prove that u and v are conjugates if and only if there exists $w \in \Sigma^*$ such that $uw = wv$.

3. There is a party of n people. Each attendee has at most r friends in the party. The friend circle of a person includes the person and all her friends. You are required to pick some people for a party game, with the restriction that at most one person is picked from each friend circle. Show that you can pick $\frac{n}{r^2+1}$ people for the game.

4. Consider the assertion: *Any* connected undirected graph G with at least two vertices contains a vertex v such that deleting v from G results in a connected graph.

   Either give a proof of the assertion, or give a counterexample (thereby disproving the assertion).

5. In the land of Twitter, there are two kinds of people: knights (also called *outragers*), who always tell the truth, and knaves (also called *trolls*), who always lie. It so happened that a person with handle `@anand` tweeted something offensive. It was not known whether `@anand` was a knight or a knave. A crack team, headed by Inspector Chitra, rounded up three suspects and interrogated them.

   The first interrogation went as follows.

Chitra	:	What do you know about `@anand`?
Suspect 1	:	`@anand` once claimed that I was a knave.
Chitra	:	Are you by any chance `@anand`?
Suspect 1	:	Yes.

   The second interrogation:

Chitra	:	Have you ever claimed you were `@anand`?
Suspect 2	:	No.
Chitra	:	Did you ever claim you are *not* `@anand`?
Suspect 2	:	Yes.

   The third suspect arrived with a defense lawyer (also on Twitter):

Lawyer	:	My client is indeed a knave, but he is not `@anand`.
Suspect 3	:	My lawyer always tells the truth.

   Which of the above suspects are innocent, and which are guilty? Explain your reasoning.

6. Let A be an $n \times n$ matrix of integers such that each row *and* each column is arranged in ascending order. We want to check whether a number k appears in A. If k is present, we should report its position—that is, the row i and column j such that $A(i, j) = k$. Otherwise, we should declare that k is not present in A.

   (a) Describe an algorithm that solves this problem in time $O(n \log n)$. Justify the complexity of your algorithm.

   (b) Describe an algorithm that solves this problem by examining at most $2n$ values in A. Justify the complexity of your algorithm.

   (c) For both algorithms, describe a worst-case input where k is present in A.

7. A college professor gives several quizzes during the semester, with negative marking. He has become bored of the usual *"Best M out of N quizzes"* formula to award marks for internal assessment. Instead, each student will be evaluated based on the sum of the best contiguous segment (i.e., no gaps) of marks in the overall sequence of quizzes. However, the student is allowed to drop upto K quizzes before calculating this sum.

   Suppose a student has scored the following marks in 10 quizzes during the semester.

Quiz	1	2	3	4	5	6	7	8	9	10
Marks	6	-5	3	-7	6	-1	10	-8	-8	8

   Without dropping any quizzes, the best segment is quiz 5-7, which yields a total of 15 marks. If the student is allowed to drop upto 2 quizzes in a segment, the best segment is quiz 1-7, which yields a total of 24 marks after dropping quizzes 2 and 4. If the student is allowed to drop upto 6 quizzes in a segment, the best total is obtained by taking the entire list and dropping all 5 negative entries, yielding 33 marks.

   For $1 \leq i < N$, $0 \leq j \leq K$, let $B[i][j]$ denote the maximum sum segment ending at position i with at most j marks dropped.

   (a) Write a recursive formula for $B[i][j]$.

   (b) Explain how to calculate, using dynamic programming, the score the professor needs to award each student.

   (c) Describe the space and time complexity of your dynamic programming algorithm.