CHENNAI MATHEMATICAL INSTITUTE

M.Sc. / Ph.D. Programme in Computer Science

Entrance Examination, 15 May 2018

This question paper has 4 printed sides. Part A has 10 questions of 3 marks each. Part B has 7 questions of 10 marks each. The total marks are 100. Answers to Part A must be filled in the answer sheet provided.

Part A

1. Which of the words below matches the regular expression $a(a+b)^*b + b(a+b)^*a$?

   (a) aba (b) bab (c) $abba$ (d) $aabb$

2. Akash, Bharani, Chetan and Deepa are invited to a party. If Bharani and Chetan attend, then Deepa will attend too. If Bharani does not attend, then Akash will not attend. If Deepa does not attend, which of the following is true?

   (a) Chetan does not attend (b) Akash does not attend
   (c) either (a) or (b) (d) none of the above

3. In a running race, Geetha finishes ahead of Shalini and Vani finishes after Aparna. Divya finishes ahead of Aparna. Which of the following is a minimal set of additional information that can determine the winner?

   (a) Geetha finishes ahead of Divya and Vani finishes ahead of Shalini.

   (b) Aparna finishes ahead of Shalini.

   (c) Divya finishes ahead of Geetha.

   (d) None of the above.

4. Let $G = (V, E)$ be an undirected simple graph, and s be a designated vertex in G. For each $v \in V$, let $d(v)$ be the length of a shortest path between s and v. For an edge (u, v) in G, what *can not* be the value of $d(u) - d(v)$?

   (a) 2 (b) -1 (c) 0 (d) 1

5. How many paths are there in the plane from $(0, 0)$ to $(m, n) \in \mathbb{N} \times \mathbb{N}$, if the possible steps from (i, j) are either $(i + 1, j)$ or $(i, j + 1)$?

   (a) $\binom{2m}{n}$ (b) $\binom{m}{n}$ (c) $\binom{m+n}{n}$ (d) m^n

6. You are given two coins A and B that look identical. The probability that coin A turns up heads is $\frac{1}{4}$, while the probability that coin B turns up heads is $\frac{3}{4}$. You choose one of the coins at random and toss it twice. If both the outcomes are heads, what is the probability that you chose coin B?

   (a) $\frac{1}{16}$ (b) $\frac{1}{2}$ (c) $\frac{9}{16}$ (d) $\frac{9}{10}$

7. Let C_n be the number of strings w consisting of n X's and n Y's such that no initial segment of w has more Y's than X's. Now consider the following problem. A person stands on the edge of a swimming pool holding a bag of n red and n blue balls. He draws a ball out one at a time and discards it. If he draws a blue ball, he takes one

step back, if he draws a red ball, he moves one step forward. What is the probability that the person remains dry?

(a) $\dfrac{C_n}{2^{2n}}$ (b) $\dfrac{C_n}{\binom{2n}{n}}$ (c) $\dfrac{n \cdot C_n}{(2n)!}$ (d) $\dfrac{n \cdot C_n}{\binom{2n}{n}}$

8. There are 7 switches on a switchboard, some of which are *on* and some of which are *off*. In one move, you pick any 2 switches and toggle each of them—if the switch you pick is currently off, you turn it on, if it is on, you turn it off. Your aim is to execute a sequence of moves and turn all 7 switches on. For which of the following initial configurations is this *not* possible? Each configuration lists the initial positions of the 7 switches in sequence, from switch 1 to switch 7.

   (a) (off,on,off,on,off,off,on) (b) (off,on,on,on,on,on,off)
   (c) (on,off,on,on,on,on,on) (d) (off,off,off,off,off,on,off)

9. Your college has sent a contingent to take part in a cultural festival at a neighbouring institution. Several team events are part of the programme. Each event takes place through the day with many elimination rounds. Your contingent is multi-talented and each individual has the skills to take part in a subset of the events. However, the same individual cannot be part of the team for two different events because of a possible clash in timings. Your aim is to create teams to take part in as many events as possible.

   To do this, you decide to model the problem as a graph where the nodes are the events and edges represent pairs of events where the team that you plan to send shares a member. In this setting, the graph theoretic question to be answered is:

   (a) Find a maximum length simple cycle
   (b) Find a maximum size independent set
   (c) Find a maximum matching
   (d) Find a maximal connected component

10. What does the following function compute in terms of n and d, for integer values of n and d, $d > 1$? Note that `a//b` denotes the quotient (integer part) of `a` $\div$ `b`, for integers `a` and `b`. For instance `7//3` is `2`.

```
function foo(n,d){
  x := 0;
  while (n >= 1) {
    x := x+1;
    n := n//d;
  }
  return(x);
}
```

   (a) The number of ways of choosing d elements from a set of size n.
   (b) The number of ways of rearranging d elements from a set of size n.
   (c) The number of digits in the base d representation of n.
   (d) The number of ways of partitioning n elements into groups of size d.

Part B

1. Consider the following non-deterministic finite automata (NFA) $\mathcal{A}_1$ and $\mathcal{A}_2$:

   (a) Give an example of a word which is accepted by both $\mathcal{A}_1$ and $\mathcal{A}_2$.

   (b) Give an example of a word which is accepted by $\mathcal{A}_1$, but not by $\mathcal{A}_2$.

   (c) Draw the deterministic finite automaton (DFA) equivalent to $\mathcal{A}_1$.

2. A student requests a recommendation letter from a professor. The professor gives three sealed envelopes. Each envelope contains either a good recommendation letter or a bad recommendation letter.

   (a) Make a list of all the possible scenarios.

   (b) Suppose now the professor tells the student that exactly one envelope contains a good recommendation letter and the other two contain bad recommendation letters. In the list of scenarios you prepared above, mark the ones that are still possible.

   (c) On envelope 1 is written the clue “This contains a bad recommendation letter”. On envelope 2 is written the clue “This contains a bad recommendation letter”. On envelope 3 is written the clue “Envelope 2 contains a good recommendation letter”. Suppose now the professor gives the additional information that exactly one of these three clues are true and the other two are false. Can the student find out the contents of the envelopes without breaking their seals?

3. Let G be a simple graph on n vertices.

   (a) Prove that if G has more than $\binom{n-1}{2}$ edges then G is connected.

   (b) For every $n > 2$, find a graph G_n which has exactly n vertices and $\binom{n-1}{2}$ edges, and is not connected.

4. You are given a sorted array of n elements which has been *circularly shifted.* For example, $\{35, 42, 5, 12, 23, 26\}$ is a sorted array that has been circularly shifted by 2 positions.

   Give an $\mathcal{O}(\log n)$ time algorithm to find the largest element in a circularly shifted array. (The number of positions through which it has been shifted is unknown to you.)

5. Let $G = (V, E)$ be an undirected graph and $V = \{1, 2, \ldots, n\}$. The input graph is given to you by a $0 - 1$ matrix A of size $n \times n$ as follows. For any $1 \leq i, j \leq n$, the entry $A[i, j] = 1$ if and only if (i, j) is an edge in G.

   A connected component in G is a subgraph in which any two vertices are connected to each other by paths. Give a simple algorithm to find the number of connected components in G. Analyze the time taken by your procedure.

6. You are playing an old-style video game in which you have to shoot down alien spacecships as they fly across the screen from left to right. Each spaceship flies across the screen at a specified height. You have an antiaircraft gun set to shoot down all spaceships at a certain height. Spaceships fly one at a time, so if your gun is set to fire at the correct height, it will shoot down the spaceship currently flying across the screen.

   You can set the initial height at which the gun fires. As the game progresses, you can reset the height, but only to a lower value. You are given in advance the height at which each spaceship flies. There are N spaceships numbered $1, 2, \ldots, N$ in the order in which they fly across the screen. For $1 \leq i \leq N$, $h[i]$ denotes the height at which spaceship i flies.

   (a) Let $V[i]$ denote the maximum number of spaceships from $i, i{+}1, \ldots, N$ that you can shoot down with a single gun. Write a recurrence for $V[i]$ and describe a strategy to compute $V[i]$ using dynamic programming. What is the space and time complexity of your solution?

   (b) Describe an algorithm to compute the minimum number of guns required to shoot down all the space ships. Each gun can be initialized separately to a firing height and each gun can be separately reset to a lower value.

7. A First In First Out queue is a data structure supporting the operations *Enque*, *Deque*, *Print*. *Enque*(x) adds the item x to the tail of the queue. *Deque* removes the element at the head of the queue and returns its value. *Print* prints the head of the queue.

   (a) You are given a queue containing 5 elements. Using a single additional temporary variable X, write down a sequence of statements each being one of *Enque*, *Deque*, *Print* so that the output of the program results in the 5 elements of the queue being printed in reverse order.

   (b) If the queue had n elements to begin with, how many statements would you need to print the queue in reverse order?