CHENNAI MATHEMATICAL INSTITUTE

M.Sc. / Ph.D. Programme in Computer Science

Entrance Examination, 18 May 2016

This question paper has 5 printed sides. Part A has 10 questions of 3 marks each. Part B has 7 questions of 10 marks each. The total marks are 100. Answers to Part A must be filled in the answer sheet provided.

Part A

1. In a connected undirected graph, the distance between two vertices is the number of edges in the shortest path between them. Suppose we denote by P the following property: *there exists a vertex that is a neighbour of all other vertices.* Consider the following statements:

   (i) If P is false, then there is a pair of vertices such that the distance between them is at least 4.

   (ii) If P is true, then the distance between any pair of vertices is at most 2.

   What can you say about these statements?

   (a) Only (i) is true. (b) Only (ii) is true.
   (c) Both (i) and (ii) are true. (d) Neither (i) nor (ii) is true.

2. The symbol $\mid$ reads as "divides", and $\nmid$ as "does not divide". For instance, $2 \mid 6$ and $2 \nmid 5$ are both true. Consider the following statements:

   (i) There exists a positive integer a such that $(2 \mid (a^3 - 1))$ and $(2 \mid a)$.

   (ii) There exists a positive integer b such that $6 \nmid (b^3 - b)$.

   What can you say about these statements?

   (a) Only (i) is true. (b) Only (ii) is true.
   (c) Both (i) and (ii) are true. (d) Neither (i) nor (ii) is true.

3. For a regular expression e, let $L(e)$ be the language generated by e. If e is an expression that has no Kleene star $*$ occurring in it, which of the following is true about e in general?

   (a) $L(e)$ is empty.

   (b) $L(e)$ is finite.

   (c) Complement of $L(e)$ is empty.

   (d) Both $L(e)$ and its complement are infinite.

4. Consider a weighted undirected graph G with positive edge weights. Let (u, v) be an edge in the graph. It is known that the shortest path from a vertex s to u has weight 53 and the shortest path from s to v has weight 65. Which of the statements is always true?

   (a) Weight of $(u, v) \leq 12$.
   (b) Weight of $(u, v) = 12$.
   (c) Weight of $(u, v) \geq 12$.
   (d) Nothing can be said about the weight of (u, v).

5. A dodecahedron is a regular solid with 12 faces, each face being a regular pentagon. How many edges are there? And how many vertices?

   (a) 60 edges and 20 vertices. (b) 30 edges and 20 vertices.
   (c) 60 edges and 50 vertices. (d) 30 edges and 50 vertices.

6. In the code fragment to the right, `start` and `end` are integer values and `prime(x)` is a function that returns `true` if `x` is a prime number and `false` otherwise.

   At the end of the loop:

   (a) `k == i-j`.
   (b) `k == j-i`.
   (c) `k == -j-i`.
   (d) Depends on `start` and `end`.

```
i := 0; j := 0; k := 0;
for (m := start;
     m <= end;
     m := m+1){
  if (prime(m)){
    i := i + m;
    k := k - m;
  }else{
    j := j - m;
    k := k + m;
  }
}
```

7. Varsha lives alone and dislikes cooking, so she goes out for dinner every evening. She has two favourite restaurants, *Dosa Paradise* and *Kababs Unlimited*, to which she travels by local train. The train to *Dosa Paradise* runs every 10 minutes, at 0, 10, 20, 30, 40 and 50 minutes past the hour. The train to *Kababs Unlimited* runs every 20 minutes, at 8, 28 and 48 minutes past the hour. She reaches the station at a random time between 7:15 pm and 8:15 pm and chooses between the two restaurants based on the next available train. What is the probability that she ends up eating in *Kababs Unlimited*?

   (a) $\frac{1}{5}$ (b) $\frac{1}{3}$ (c) $\frac{2}{5}$ (d) $\frac{1}{2}$

8. An advertisement for a tennis magazine states, "If I'm not playing tennis, I'm watching tennis. And if I'm not watching tennis, I'm reading about tennis." We can assume that the speaker can do at most one of these activities at a time. What is the speaker doing?

   (a) Playing tennis. (b) Watching tennis.
   (c) Reading about tennis. (d) None of the above.

9. ScamTel has won a state government contract to connect 17 cities by high-speed fibre optic links. Each link will connect a pair of cities so that the entire network is connected—there is a path from each city to every other city. The contract requires the network to remain connected if *any* single link fails. What is the minimum number of links that ScamTel needs to set up?

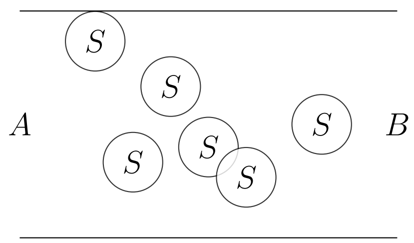
   (a) 34 (b) 32 (c) 17 (d) 16

10. Which of the following relationships holds in general between the *scope* of a variable and the *lifetime* of a variable (in a language like C or Java)?

    (a) The scope of a variable is contained in the lifetime of the variable.
    (b) The scope of a variable is the same as the lifetime of the variable.
    (c) The lifetime of a variable is disjoint from the scope of the variable.
    (d) None of the above.

Part B

1. A group of war prisoners are trying to escape from a prison. They have thoroughly planned the escape from the prison itself, and after that they hope to find shelter in a nearby village. However, the village (marked as B, see picture below) and the prison (marked as A) are separated by a canyon which is also guarded by soldiers (marked as S). These soldiers sit in their pickets and rarely walk; the range of view of each soldier is limited to exactly 100 meters. Thus, depending on the locations of soldiers, it may be possible to pass the canyon safely, keeping the distance to the closest soldier strictly larger than 100 meters at any moment. The situation is depicted in the following picture, where the circles around S indicate range of view.



   Provide an algorithm to determine if the prisoners can pass the canyon unnoticed, given the width and the length of the canyon and the coordinates of every soldier in the canyon, and assuming that soldiers do not change their locations. (*Hint*: Model this as a graph, with soldiers represented by the vertices.)

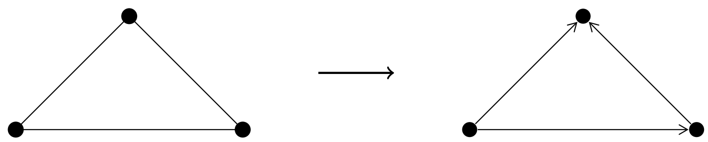
2. A *simple path* (respectively cycle) in a graph is a path (respectively cycle) in which no edge or vertex is repeated. The *length* of such a path (respectively cycle) is the number of edges in the path (respectively cycle).

   Let G be an undirected graph with minimum degree $k \geq 2$.

   (a) Show that G contains a simple path of length at least k.

   (b) Show that G contains a simple cycle of length at least $k+1$.

3. An undirected graph can be converted into a directed graph by choosing a direction for every edge. Here is an example:



   Show that for every undirected graph, there is a way of choosing directions for its edges so that the resulting directed graph has no directed cycles.

4. Let $\Sigma = \{0, 1\}$. Let A, B be arbitrary subsets of Σ^*. We define the following operations on such sets:

$$\begin{aligned} A + B &:= \{ w \in \Sigma^* \mid w \in A \text{ or } w \in B \} \\ A \cdot B &:= \{ uv \in \Sigma^* \mid u \in A \text{ and } v \in B \} \\ 2A &:= \{ ww \in \Sigma^* \mid w \in A \} \end{aligned}$$

   Is it true that $(A+B)\cdot(A+B) = A\cdot A + B\cdot B + 2(A\cdot B)$ for all choices of A and B? If yes, give a proof. If not, provide suitable A and B for which this equation fails.

5. For a string $x = a_0 a_1 \cdots a_{n-1}$ over the alphabet $\{0, 1, 2\}$, define $val(x)$ to be the value of x interpreted as a ternary number, where a_0 is the most significant digit. More formally, $val(x)$ is given by

$$\sum_{0 \leq i < n} 3^{n-1-i} \cdot a_i.$$

   Design a finite automaton that accepts exactly the set of strings $x \in \{0, 1, 2\}^*$ such that $val(x)$ is divisible by 4.

6. An automatic spelling checker works as follows. Given a word w, first check if w is found in the dictionary. If w is not in the dictionary, compute a dictionary entry that is close to w. For instance if the user types `ocurrance`, the spelling checker should suggest `occurrence`, which belongs to the dictionary. Similarity between words such as `occurrence` and `occurance` is quantified in terms of *alignment*.

   An alignment between two strings w_1 and w_2 (over the alphabet $\{\mathtt{a}, \mathtt{b}, \mathtt{c}, \ldots, \mathtt{z}\}$) is obtained by inserting hyphens in the two strings such that the modified strings *align* (i.e., the modified strings are of equal length, and at each position, either both strings have the same letter or one of the strings has a hyphen).

   Here are three examples of alignments. The first is between `ocurrance` and `occurrence` and the second and third are between `ctatg` and `ttaagc`.

```
(1) oc-urr-ance     (2) ct-at-g-     (3) ctat---g-
    occurre-nce         -tta-agc         ---ttaagc
```

A *mismatch* in an alignment is a position where one of modified strings has a hyphen and the other does not. There are three mismatches in the first alignment given above, five mismatches in the second, and seven mismatches in the third.

Use dynamic programming to give an efficient algorithm that takes two strings w_1 and w_2 (over the alphabet $\{\mathtt{a}, \mathtt{b}, \mathtt{c}, \ldots, \mathtt{z}\}$) as its input, and computes the minimum number of mismatches among all alignments of w_1 and w_2. What is the running time of your algorithm (in terms of the lengths of w_1 and w_2)?

7. Consider the function M defined as follows:

$$M(n) = \begin{cases} n - 10 & \text{if } n > 100 \\ M(M(n+11)) & \text{if } n \leq 100 \end{cases}$$

   (a) Compute the following.
      i. $M(101)$
      ii. $M(99)$
      iii. $M(87)$
   (b) Give a constant-time algorithm that computes $M(n)$ on input n. (A constant-time algorithm is one whose running time is independent of the input n.)