CHENNAI MATHEMATICAL INSTITUTE

M.Sc. / Ph.D. Programme in Computer Science

Entrance Examination, 18 May 2016

This question paper has 5 printed sides. Part A has 10 questions of 3 marks each. Part B has 7 questions of 10 marks each. The total marks are 100. Answers to Part A must be filled in the answer sheet provided.

Part A

1. In a connected undirected graph, the distance between two vertices is the number of edges in the shortest path between them. Suppose we denote by P the following property: *there exists a vertex that is a neighbour of all other vertices.* Consider the following statements:

   (i) If P is false, then there is a pair of vertices such that the distance between them is at least 4.

   (ii) If P is true, then the distance between any pair of vertices is at most 2.

   What can you say about these statements?

   (a) Only (i) is true. (b) Only (ii) is true.
   (c) Both (i) and (ii) are true. (d) Neither (i) nor (ii) is true.

   Answer:

   $\boxed{\text{(b) Only (ii) is true.}}$

   If P is true, there is a "central" vertex that is a neighbor of all others. We can start from any vertex and reach any other by going via the center. So (ii) is true. There are graphs where P is false but the distance is at most 3 between any pair of vertices, so (i) is false. $\dashv$

2. The symbol $\mid$ reads as "divides", and $\nmid$ as "does not divide". For instance, $2 \mid 6$ and $2 \nmid 5$ are both true. Consider the following statements:

   (i) There exists a positive integer a such that $(2 \mid (a^3 - 1))$ and $(2 \mid a)$.

   (ii) There exists a positive integer b such that $6 \nmid (b^3 - b)$.

   What can you say about these statements?

   (a) Only (i) is true. (b) Only (ii) is true.
   (c) Both (i) and (ii) are true. (d) Neither (i) nor (ii) is true.

Answer:

$\boxed{\text{(d) Neither (i) nor (ii) is true.}}$

(i) is false. Note that $(a^3 - 1) = (a - 1)(a^2 + a + 1)$, and hence $2 \mid a$ implies that both $a - 1$ and $a^2 + a + 1$ are odd, and hence so is their product. Thus $2 \nmid (a^3 - 1)$.

(ii) is false too. Note that $(b^3 - b) = (b - 1)b(b + 1)$, and at least one of these factors is even and one is divisible by 3. Hence $6 \mid (b^3 - b)$. $\dashv$

3. For a regular expression e, let $L(e)$ be the language generated by e. If e is an expression that has no Kleene star $*$ occurring in it, which of the following is true about e in general?

   (a) $L(e)$ is empty.

   (b) $L(e)$ is finite.

   (c) Complement of $L(e)$ is empty.

   (d) Both $L(e)$ and its complement are infinite.

   Answer:

   $\boxed{\text{(b) } L(e) \text{ is finite.}}$

   This is proved by a simple induction on the structure of the regular expression, using the fact that $L(a)$ is finite for each letter a, and that unions and concatenations of finite languages are also finite. $\dashv$

4. Consider a weighted undirected graph G with positive edge weights. Let (u, v) be an edge in the graph. It is known that the shortest path from a vertex s to u has weight 53 and the shortest path from s to v has weight 65. Which of the statements is always true?

   (a) Weight of $(u, v) \leq 12$.

   (b) Weight of $(u, v) = 12$.

   (c) Weight of $(u, v) \geq 12$.

   (d) Nothing can be said about the weight of (u, v).

   Answer:

   $\boxed{\text{(c) Weight of } (u, v) \geq 12.}$

   If the weight of (u, v) is strictly less than 12, then there is a path from s to v of weight at most $53 + 11 = 64$ (which goes from s to u and then takes the edge (u, v)). This contradicts the fact that the shortest path from s to v has weight 65. Thus the weight of (u, v) is at least 12. $\dashv$

5. A dodecahedron is a regular solid with 12 faces, each face being a regular pentagon. How many edges are there? And how many vertices?

   (a) 60 edges and 20 vertices. (b) 30 edges and 20 vertices.
   (c) 60 edges and 50 vertices. (d) 30 edges and 50 vertices.

   Answer:

   $\boxed{\text{(b) 30 edges and 20 vertices.}}$

   Each face has five edges, and there are 12 faces. But each edge is shared between two faces. Thus the number of edges is $(12 \times 5)/2 = 30$. The number of vertices is given by Euler's formula to be $|E| - |F| + 2 = 30 - 12 + 2 = 20$. ⊣

6. In the code fragment to the right, `start` and `end` are integer values and `prime(x)` is a function that returns `true` if `x` is a prime number and `false` otherwise.

```
i := 0; j := 0; k := 0;
for (m := start;
     m <= end;
     m := m+1){
  if (prime(m)){
    i := i + m;
    k := k - m;
  }else{
    j := j - m;
    k := k + m;
  }
}
```

   At the end of the loop:

   (a) `k == i-j`.

   (b) `k == j-i`.

   (c) `k == -j-i`.

   (d) Depends on `start` and `end`.

   Answer:

   $\boxed{\text{(c) k == -j-i.}}$

   The value of $i + j + k$ is 0 initially, and also at the end of each iteration of the loop. Thus $k = -j - i$ at the end of the loop. ⊣

7. Varsha lives alone and dislikes cooking, so she goes out for dinner every evening. She has two favourite restaurants, *Dosa Paradise* and *Kababs Unlimited*, to which she travels by local train. The train to *Dosa Paradise* runs every 10 minutes, at 0, 10, 20, 30, 40 and 50 minutes past the hour. The train to *Kababs Unlimited* runs every 20 minutes, at 8, 28 and 48 minutes past the hour. She reaches the station at a random time between 7:15 pm and 8:15 pm and chooses between the two restaurants based on the next available train. What is the probability that she ends up eating in *Kababs Unlimited*?

   (a) $\frac{1}{5}$ (b) $\frac{1}{3}$ (c) $\frac{2}{5}$ (d) $\frac{1}{2}$

Answer:

$\boxed{\text{(c) 2/5.}}$

If she arrives in the intervals 7:20–7:28, 7:40–7:48, or 8:00–8:08, the next train goes to *Kababs Unlimited*. This adds up to 24 minutes, out of a total of 60 minutes, so $\frac{24}{60} = \frac{2}{5}$. ⊣

8. An advertisement for a tennis magazine states, "If I'm not playing tennis, I'm watching tennis. And if I'm not watching tennis, I'm reading about tennis." We can assume that the speaker can do at most one of these activities at a time. What is the speaker doing?

   (a) Playing tennis. (b) Watching tennis.
   (c) Reading about tennis. (d) None of the above.

   Answer:

   $\boxed{\text{(b) Watching tennis.}}$

   If he is not watching tennis, then he is reading about tennis. Which means he is not playing tennis, which implies that he is watching tennis. Thus, the assumption that he is not watching tennis leads to a contradiction. Therefore he is watching tennis. ⊣

9. ScamTel has won a state government contract to connect 17 cities by high-speed fibre optic links. Each link will connect a pair of cities so that the entire network is connected—there is a path from each city to every other city. The contract requires the network to remain connected if *any* single link fails. What is the minimum number of links that ScamTel needs to set up?

   (a) 34 (b) 32 (c) 17 (d) 16

   Answer:

   $\boxed{\text{(c) 17}}$

   If we connect the cities in a loop with 17 links, then even if any single link fails, we can connect the cities by traversing the loop the other way. If we only had 16 links connecting 16 cities, that would constitute a tree which would have only a single path between any pair of cities (and thus get disconnected by a single link failure). ⊣

10. Which of the following relationships holds in general between the *scope* of a variable and the *lifetime* of a variable (in a language like C or Java)?

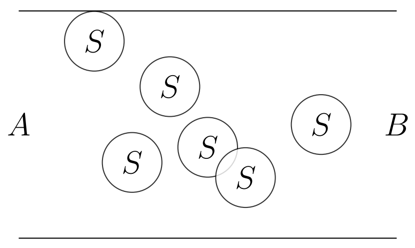
    (a) The scope of a variable is contained in the lifetime of the variable.
    (b) The scope of a variable is the same as the lifetime of the variable.
    (c) The lifetime of a variable is disjoint from the scope of the variable.
    (d) None of the above.

Answer:

$\boxed{\text{(a) The scope of a variable is contained in the lifetime of the variable.}}$ ⊣

Part B

1. A group of war prisoners are trying to escape from a prison. They have thoroughly planned the escape from the prison itself, and after that they hope to find shelter in a nearby village. However, the village (marked as B, see picture below) and the prison (marked as A) are separated by a canyon which is also guarded by soldiers (marked as S). These soldiers sit in their pickets and rarely walk; the range of view of each soldier is limited to exactly 100 meters. Thus, depending on the locations of soldiers, it may be possible to pass the canyon safely, keeping the distance to the closest soldier strictly larger than 100 meters at any moment. The situation is depicted in the following picture, where the circles around S indicate range of view.



Provide an algorithm to determine if the prisoners can pass the canyon unnoticed, given the width and the length of the canyon and the coordinates of every soldier in the canyon, and assuming that soldiers do not change their locations. (*Hint*: Model this as a graph, with soldiers represented by the vertices.)

Answer:

Consider a graph $G = (V \cup \{N, S\}, E)$ where $V = \{v_s \mid s \text{ is a soldier}\}$. N and S are two special vertices representing the north and south boundary of the canyon, respectively. For two soldiers s and s', $(v_s, v_{s'}) \in E$ iff s and s' are at most 200 meters apart. For a soldier s, $(v_s, N) \in E$ iff s is at most 100 meters from the north boundary, and similarly $(v_s, S) \in E$ iff s is at most 100 meters from the south boundary. It is clear that the prisoners cannot pass the canyon unnoticed iff there is a path between N and S. Thus it suffices to test reachability of S from N using any standard algorithm (for instance, one based on breadth-first search). ⊣

2. A *simple path* (respectively cycle) in a graph is a path (respectively cycle) in which no edge or vertex is repeated. The *length* of such a path (respectively cycle) is the number of edges in the path (respectively cycle).

   Let G be an undirected graph with minimum degree $k \geq 2$.

   (a) Show that G contains a simple path of length at least k.

   (b) Show that G contains a simple cycle of length at least $k + 1$.

Answer:

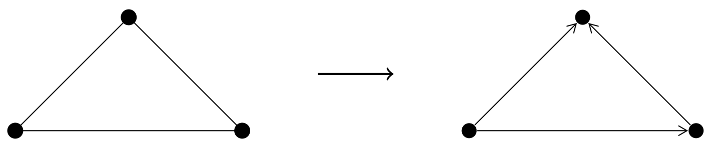
Build a path in the graph as follows:

> Pick an arbitrary vertex v_1 in the first step. At each subsequent step, let v_t be the vertex picked last. If all neighbours of v_t have already been picked, stop. Otherwise v_{t+1} is chosen to be an arbitrary neighbour of v_t hitherto unpicked.

Since the graph is finite, the above procedure terminates eventually. Let $\pi = v_1 \cdots v_t$ be the path produced at the end. By construction, it is a simple path. It follows that all neighbours of v_t have already been picked and they occur among $\{v_1, \ldots, v_{t-1}\}$. Since the minimum degree of the graph is k, it follows that $k \leq t - 1$. Thus there are at least $k + 1$ vertices and hence at least k edges in π.

Let i be the least index such that v_i is adjacent to v_t. It is clear that the path from v_i to v_t is of length at least k. Adding the edge (v_i, v_t) yields a simple cycle of length at least $k + 1$. $\dashv$

3. An undirected graph can be converted into a directed graph by choosing a direction for every edge. Here is an example:



Show that for every undirected graph, there is a way of choosing directions for its edges so that the resulting directed graph has no directed cycles.

Answer:

Assign a distinct natural number $N(v)$ to each vertex v of the graph. For an edge between v and w, choose the direction $v \to w$ provided $N(v) < N(w)$. It is easy to prove that if there is a directed path from v to w in this scheme, then $N(v) < N(w)$. Suppose there is a directed cycle involving v. This means that there is a directed path from v to itself, whence $N(v) < N(v)$, which is a contradiction. $\dashv$

4. Let $\Sigma = \{0, 1\}$. Let A, B be arbitrary subsets of Σ^*. We define the following operations on such sets:

$$\begin{aligned} A + B &:= \{\, w \in \Sigma^* \mid w \in A \text{ or } w \in B \,\} \\ A \cdot B &:= \{\, uv \in \Sigma^* \mid u \in A \text{ and } v \in B \,\} \\ 2A &:= \{\, ww \in \Sigma^* \mid w \in A \,\} \end{aligned}$$

Is it true that $(A + B) \cdot (A + B) = A \cdot A + B \cdot B + 2(A \cdot B)$ for all choices of A and B? If yes, give a proof. If not, provide suitable A and B for which this equation fails.

Answer:

The equation fails for $A = \{0\}$ and $B = \{1\}$. In that case $(A + B) \cdot (A + B) = \{00, 01, 10, 11\}$, whereas $A \cdot A + B \cdot B + 2(A \cdot B) = \{00, 11, 0101\}$. $\dashv$

5. For a string $x = a_0 a_1 \cdots a_{n-1}$ over the alphabet $\{0, 1, 2\}$, define $val(x)$ to be the value of x interpreted as a ternary number, where a_0 is the most significant digit. More formally, $val(x)$ is given by
$$\sum_{0 \le i < n} 3^{n-1-i} \cdot a_i.$$
Design a finite automaton that accepts exactly the set of strings $x \in \{0, 1, 2\}^*$ such that $val(x)$ is divisible by 4.

Answer:

Let $L = \{x \in \{0, 1, 2\}^* \mid val(x) \text{ is divisible by } 4\}$. For any string x and $b \in \{0, 1, 2\}$:

(a) $x \in L$ iff $val(x) \mod 4 = 0$.

(b) if $val(x) = i \mod 4$, $val(xb) \equiv (3i + b) \mod 4$, for $b = 0, 1, 2$.

So to construct a DFA for L, it suffices to take 4 states q_0, q_1, q_2, q_3, with q_i recording the fact that the value of the string read so far is $i \mod 4$. Under this interpretation, q_0 will be the initial and (only) final state, and the automaton will transition from state q_i to q_j on reading letter $b \in \{0, 1, 2\}$, where $j = (3i + b) \mod 4$.

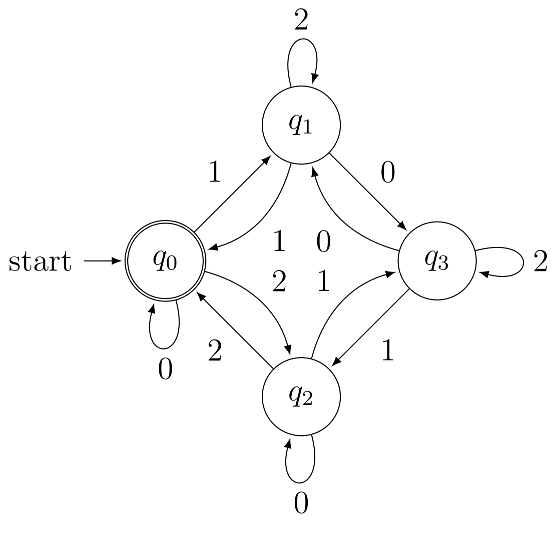


Figure 1: Automaton for strings whose value is divisible by 4.

$\dashv$

6. An automatic spelling checker works as follows. Given a word w, first check if w is found in the dictionary. If w is not in the dictionary, compute a dictionary entry that is close to w. For instance if the user types `ocurrance`, the spelling checker should

suggest `occurrence`, which belongs to the dictionary. Similarity between words such as `occurrence` and `occurance` is quantified in terms of *alignment.*

An alignment between two strings w_1 and w_2 (over the alphabet $\{\mathtt{a}, \mathtt{b}, \mathtt{c}, \ldots, \mathtt{z}\}$) is obtained by inserting hyphens in the two strings such that the modified strings *align* (i.e., the modified strings are of equal length, and at each position, either both strings have the same letter or one of the strings has a hyphen).

Here are three examples of alignments. The first is between `ocurrance` and `occurrence` and the second and third are between `ctatg` and `ttaagc`.

```
(1)  oc-urr-ance     (2)  ct-at-g-     (3)  ctat---g-
     occurre-nce          -tta-agc          ---ttaagc
```

A *mismatch* in an alignment is a position where one of modified strings has a hyphen and the other does not. There are three mismatches in the first alignment given above, five mismatches in the second, and seven mismatches in the third.

Use dynamic programming to give an efficient algorithm that takes two strings x and y (over the alphabet $\{\mathtt{a}, \mathtt{b}, \mathtt{c}, \ldots, \mathtt{z}\}$) as its input, and computes the minimum number of mismatches among all alignments of x and y. What is the running time of your algorithm (in terms of the lengths of x and y)?

Answer:

Let $x = a_1 a_2 \cdots a_m$ and $y = b_1 b_2 \cdots b_n$. For any string w and any $i \in \{0, \ldots, |w|\}$, we let $w[\ldots i]$ denote the prefix of w of length i. For $i \in \{0, \ldots, m\}$ and $j \in \{0, \ldots, n\}$, we let $\textsc{opt}(i, j)$ denote the minimum number of mismatches among all alignments of $x[\ldots i]$ and $y[\ldots j]$. It is easy to see that $\textsc{opt}(0, j) = j$ and $\textsc{opt}(i, 0) = i$. When $i, j > 0$, an optimal alignment of $x[\ldots i]$ and $y[\ldots j]$ either aligns $x[\ldots i-1]$ with $y[\ldots j-1]$ and tries to match a_i and b_j (if possible), or aligns $y[\ldots j]$ with $x[\ldots i-1]$, leaving a_i unmatched, or aligns $x[\ldots i]$ with $y[\ldots j-1]$, leaving b_j unmatched.

This is reflected by the following recurrence :

$$\textsc{opt}(i, j) = \min\{\textsc{opt}(i-1, j-1) + c_{ij}, \textsc{opt}(i-1, j) + 1, \textsc{opt}(i, j-1) + 1\}.$$

where c_{ij} denotes the cost of aligning a_i (the string containing just the one letter) with b_j. It is easy to see that

$$c_{ij} = \begin{cases} 0 & \text{if } a_i = b_j \\ 2 & \text{otherwise} \end{cases}$$

The following algorithm (which runs in $O(mn)$ time) computes the minimum number of mismatches among alignments of x and y.

```
array A[0...m, 0...n]
array c[0...m, 0...n]
initialize A[i,0] = i for each i
initialize A[0,j] = j for each j
for j = 1,...,n
```

```
        for i = 1,...,m
            if a[i] == b[j] then c[i,j] = 0 else c[i,j] = 2
        endfor
    endfor
    for j = 1,...,n
        for i = 1,...,m
            A[i,j] = min(A[i-1,j-1]+c[i,j], A[i-1,j]+1, A[i,j-1]+1)
        endfor
    endfor
    return A[m,n]
```

$\dashv$

7. Consider the function M defined as follows:

$$M(n) = \begin{cases} n - 10 & \text{if } n > 100 \\ M(M(n+11)) & \text{if } n \leq 100 \end{cases}$$

   (a) Compute the following.
      i. $M(101)$
      ii. $M(99)$
      iii. $M(87)$

   (b) Give a constant-time algorithm that computes $M(n)$ on input n. (A constant-time algorithm is one whose running time is independent of the input n.)

Answer:

(a) $M(101) = M(99) = M(87) = 91$

(b) The following is a simple algorithm for computing M. It just compares n with 100, and either subtracts 10 from n or returns a constant, all of which are constant-time operations.

```
if (n> 100) return (n-10) else return 91
```

To prove that the algorithm is correct, we need to show that $M(n) = 91$ for $n < 101$. Notice first that $M(101) = 91$. From the following claim, it follows that $M(n) = 91$ for all $n \leq 100$

Claim: For all $k \geq 1$ and all n such that $101 - 11k \leq n < 101 - 11(k-1)$, $M(n) = 91$.

The proof of the claim is by induction on k.

Suppose $k = 1$ and $90 \leq n < 101$.

$$\begin{aligned} M(n) &= M(M(n+11)) \\ &= M(n+11-10) \quad (\text{ since } n+11 \geq 101) \\ &= M(n+1) \end{aligned}$$

Therefore, $M(90) = M(91) = \cdots = M(100) = M(101) = 91$.
Assume the claim holds for $k \leq m$. Consider n such that $101 - 11k \leq n < 101 - 11(k-1)$ for $k = m + 1$.

$$\begin{aligned} M(n) &= M(M(n+11)) \\ &= M(91) && (M(n+11) = 91, \text{ by IH }) \\ &= 91 && \text{by base case} \end{aligned}$$

Thus the claim holds for all k, and hence $M(n) = 91$ for all $n < 101$.

$\dashv$