CHENNAI MATHEMATICAL INSTITUTE

M.Sc. Data Science Entrance Examination 2025

Solutions

Part (A) - Multiple-choice questions

1. Let $X = \{1, 2, 3\}$ and $Y = \{a, b, c, d\}$ be two finite sets. Which of the following statements regarding functions from $X \to Y$ is/are true?

   (a) The number of one-to-one functions is 3^4.

   (b) There are as many one-to-one functions as onto functions.

   (c) The number of onto functions is strictly less than the number of one-to-one functions.

   (d) The number of one-to-one functions is 24.

   Solution: Options (c), (d). Y has more elements hence we can't have onto functions from X to Y. The number of one-to-one functions is the same as choosing 3 distinct elements from Y and then assigning them to the elements of X in 3! ways.

2. The two arguments to the function `foo(A, n)` in the code below are: (i) an integer array `A` indexed from `0`, and (ii) the number `n` of elements in `A`.

```
function foo(A, n) {
    count = 0;
    for i from 0 to (n-1) {
        for j from (i+1) to (n-1) {
            if (A[i] > 2 * A[j]) {
                count = count + 1;
            }
        }
    }
    return(count);
}
```

   Which of the following statements about the function `foo(A, n)` are correct?

   (a) `foo(A, n)` counts the number of index pairs (i, j) such that $i < j$ and $\texttt{A}[i] > 2 \times \texttt{A}[j]$.

   (b) For the input `A = [1, -4, 3, -5, -2], n = 5`, the function returns 7.

   (c) For the input `A = [1, 2, 3, 4, 5], n = 5`, the function returns 0.

   (d) For the input `A = [10, 5, 1], n = 3`, the function returns 2.

   Solutions: (a), (c), and (d) are correct.

   (a) Correct : The function counts the number of index pairs `i` $<$ `j` such that `A[i]` $> 2 \times$ `A[j]`.

   (b) Incorrect : Correct answer is 6.

   (c) Correct : in `[1,2,3,4,5]`, no element is more than twice any later element.

   (d) Correct : in `[10,5,1]`:

   - `(10,1): 10 > 2*1 → yes`
   - `(5,1): 5 > 2*1 → yes`

   So, 2 pairs.

3. What is the domain of the following real valued function?

$$f(x) = \log_2(x^2 - 5x + 6).$$

   (a) $(-\infty, 2)$

   (b) $(3, \infty)$

(c) $(-\infty, 2) \cup (3, \infty)$

(d) $(-\infty, \infty)$

Solution: Option (c), $(-\infty, 2) \cup (3, \infty)$.

The argument of a logarithmic function must be positive:

$$x^2 - 5x + 6 > 0 \iff (x-2)(x-3) > 0$$

The latter condition holds if and only if exactly one of the following holds:

- $(x-2) > 0$ **and** $(x-3) > 0$. This happens exactly when $x > 3$ holds.
- $(x-2) < 0$ **and** $(x-3) < 0$. This happens exactly when $x < 2$ holds.

So the function $f(x)$ is defined exactly when $x < 2$ or $x > 3$. This corresponds to the union of intervals $(-\infty, 2) \cup (3, \infty)$.

4. Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ be a real matrix which satisfies $A^{-1} = A$. Which of the following statements is/are *always* true?

(a) $a + d = 0$

(b) $ad - bc = 1$

(c) $ad - bc \neq 0$

(d) $a^2 + 2bc + d^2 = 2$

Solution: Options (c), (d)

- (a) is not always true. Consider $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.
- (b) is not always true. Consider $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$.
- (c) is true. Since A is invertible, $det(A) = ad - bc \neq 0$ holds.
- (d) is true. From $A^{-1} = A$ we get—by multiplying both sides with A—that $A^2 = I$ holds. That is:

$$A^2 = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a^2 + bc & ab + bd \\ ac + cd & d^2 + bc \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

  Thus $a^2 + bc = 1, d^2 + bc = 1$. Adding these equations, we get the statement.

5. Let $\mathbb{Z}$ be the set of all integers. Let $A = \{n \in \mathbb{Z} \mid n^2 + 10n + 21$ is divisible by $7\}$. Which of the following statements is/are true?

(a) $A = \{n \in \mathbb{Z} \mid (n \equiv 0 \mod 7)\}$

(b) $A = \{n \in \mathbb{Z} \mid (n \equiv 0 \mod 7) \text{ or } (n \equiv 4 \mod 7)\}$

(c) $A = \{n \in \mathbb{Z} \mid (n \equiv 1 \mod 7) \text{ or } (n \equiv 5 \mod 7)\}$

(d) $A = \{n \in \mathbb{Z} \mid (n \equiv 2 \mod 7) \text{ or } (n \equiv 6 \mod 7)\}$

Solution: Option (b), $A = \{n \in \mathbb{Z} \mid (n \equiv 0 \mod 7) \text{ or } (n \equiv 4 \mod 7)\}$.

Factoring the expression, we get $n^2 + 10n + 21 = (n+3)(n+7)$. Since 7 is a prime number, this product is divisible by 7 exactly when at least one of the following holds:

- $n + 3$ is divisible by 7. This is equivalent to $n \equiv 4 \mod 7$
- $n + 7$ is divisible by 7. This is equivalent to $n \equiv 0 \mod 7$

Since the equivalence classes of $\mathbb{N}$ modulo 7 are disjoint, (b) is the only correct answer.

6. Let $n \geq 3$ be an integer, and let $x_1, x_2, \ldots, x_n$ be variables which take real values with $0 \leq x_i \leq 1$ for all $1 \leq i \leq n$. Let

$$\begin{aligned} A &= x_1 + x_2 + \ldots + x_n \\ B &= x_1x_2 + x_2x_3 + \ldots + x_{n-1}x_n + x_nx_1 \end{aligned}$$

Which of the following statements is/are true.

(a) $A \geq B$ is always true.

(b) $B > A$ is true for some values of the x_i's and $A > B$ is true for some values of the x_i's.

(c) $A = B$ has a finite number of solutions

(d) $A = B$ has an infinite number of solutions.

Solution: Options (a) and (c) are True. $A - B = x_1(1 - x_2) + x_2(1 - x_3) + \ldots + x_n(1 - x_1)$, so (a) is true. For equality. If x_1 is not zero, $x_2 = 1$, forcing $x_3 = 1$, and so on, forcing $x_1 = 1$. The same is true if any other x_i is non-zero. On the other hand, if $x_1 = 0$, then $x_n = 0$, forcing $x_{n-1} = 0$, and finally $x_2 = 0$, so (c) is also true

7. Let $B = ((b_{i,j}))$ be an $n \times n$ matrix. Let $p : \{1, 2, \ldots, n\} \mapsto \{1, 2, \ldots, n\}$ be a bijection (i.e. a one-to-one correspondence) and let a matrix $A = ((a_{i,j}))$ be defined by

$$a_{i,j} = b_{p(i),p(j)}, \quad 1 \leq i, j \leq n.$$

Which of the following statement(s) is/are true for all choices of B and P.

(a) A admits an inverse if and only if B admits an inverse.

(b) For any $x, y \in \mathbb{R}^n$, $Ax = y$ admits a solution if and only if $Bx = y$ admits a solution.

(c) A and B have the same trace.

(d) A and B have the same eigenvectors.

Solution: Options (a), (c). Option (a) is true since the absolute value of the determinant of A is same as that of B. Option (c) is true since the diagonal elements are preserved. Option (b) is false; for example, let $\begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix}, y \begin{bmatrix} 3 \\ 0 \end{bmatrix}$ and $A = \begin{bmatrix} 0 & 0 \\ 2 & 1 \end{bmatrix}$. Option (d) is false as the eigenvectors are permuted by the bijection p.

8. Let x be a variable that takes real values, and let $f(x) = x^3 - 3x$. Which of the following statements is/are true?

(a) $f(x)$ has a local maximum at $x = -\sqrt{3}$

(b) $f(x)$ has a local maximum at $x = -1$

(c) $f(x)$ has a local minimum at $x = \sqrt{3}$

(d) $f(x)$ has a global minimum at $x = 1$

Solution: Option (b).

Setting $f'(x) = 3x^2 - 3 = 0$ we get that the critical points of $f(x)$ are $x = \pm 1$. Now $f''(x) = 6x$, and so the only local maximum of $f(x)$ is at $x = -1$. Even $x = 1$ is a critical point and a local minima since $f''(x) > 0$. It is not a global minima since $-\infty$ is a global minimum.

9. A binary relation R defined on a set S is said to be **antisymmetric** if for $x, y \in S$, xRy **and** $yRx \implies x = y$. Let R_1, R_2 be two binary relations defined on a set S. The **union** of R_1, R_2 is the binary relation U defined on S as: For $x, y \in S$, $xUy \iff xR_1y$ **or** xR_2y. The **intersection** of R_1, R_2 is the binary relation I defined on S as: For $x, y \in S$, $xIy \iff xR_1y$ **and** xR_2y.

Which of the following statements is/are true?

(a) A binary relation cannot be both symmetric and antisymmetric.

(b) A binary relation can be both transitive and antisymmetric.

(c) The union of two equivalence relations is always an equivalence relation.

(d) The intersection of two equivalence relations is always an equivalence relation.

Solution: Options (b) and (d).

- (a) is false. E.g.: the identity relation is both symmetric and antisymmetric.
- (b) is true. E.g.: the "less than" relation on integers is both transitive and antisymmetric.
- (c) is false. E.g.: Consider the following two relations defined on the set $S = \{1, 2, 3, \ldots, 100\}$:
  - xR_1y: x and y are related if (i) x and y are both integral powers of 2, **or if** (ii) neither x nor y is an integral power of 2;
  - xR_2y if and only if $x \equiv y \mod 2$.

  Let U be the union of R_1 and R_2. Then: $2U4$ since $2R_14$ and $4U14$ since $4R_214$. But 2 is not related to 14 by U, since 2 and 14 are not related by either R_1 or R_2. Thus U is not transitive, and hence is not an equivalence relation.
- (d) is true. This follows from a straightforward application of the definitions of set intersection and an equivalence relation.

10. In how many ways can 10 identical chocolate bars be distributed among 5 children, in such a way that each child gets at least one chocolate bar?

(a) 50

(b) 126

(c) 252

(d) 3125

Solution: Option (b), 126.

Since the chocolate bars are identical, the only difference between two ways of distributing them to the children is in the counts of chocolate bars that the children get. Since every child must get at least one chocolate bar, this uses up 5 of the bars. The remaining $n = 5$ bars can be distributed in any manner among the $k = 5$ children. The number of ways of doing this is, using the textbook formula for distributing n identical balls into k distinct bins,

$$\binom{n+k-1}{k-1} = \binom{5+5-1}{5-1} = \binom{9}{4} = 126.$$

11. Which of the following is/are logically equivalent to $\neg(P \implies Q)$?

(a) $\neg P \vee Q$

(b) $\neg P \wedge Q$

(c) $Q \implies P$

(d) $P \wedge \neg Q$

Solution: Option (d), $P \wedge \neg Q$. This can be found, for instance, using truth tables.

12. Let $f(x) = \sqrt{x}$. We draw a tangent to the curve $y = f(x)$ at the point on the curve whose x coordinate is equal to 4. Where does this tangent intersect the X-axis?

(a) $x = 4$

(b) $x = -2$

(c) $x = -4$

(d) $x = 2$

Solution: Option (c), $x = -4$.

The derivative of $f(x) = \sqrt{x}$ is $f'(x) = \frac{1}{2\sqrt{x}}$. The slope of the tangent to $f(x)$ at $x = 4$ is then $f'(4) = \frac{1}{2\sqrt{4}} = \frac{1}{4}$. The equation of this tangent is thus $y = \frac{x}{4} + c$, where c is its Y-intercept. Since the point $(4, f(4)) = (4, 2)$ lies on this tangent, we get that $2 = \frac{4}{4} + c$ holds, from which we get $c = 1$. Thus the equation of the tangent is $y = \frac{x}{4} + 1$. Setting $y = 0$ gives us the X-intercept as $x = -4$.

13. Let n be an integer, $n \geq 4$. A is an $n \times n$ matrix with real entries. The matrix B is obtained by the following sequence of operations on A. First, multiply each entry of A by 2. Then add 3 times the second column to the third column. Finally, swap the first and the fourth columns. If $\det(A) = 5$, which of the following statements are true?

    (a) 10 divides $\det(B)$
    (b) $\det(B) = -5$
    (c) 100 divides $\det(B)$
    (d) $\det(B) = -2^n \cdot 5$

    Solution: Options (a) and (d) are True.

14. Let x, y, z be positive numbers such that $x^2 + y^2 = z^2$. Determine the value of the following expression:

$$\frac{\log_{y+z} x + \log_{z-y} x}{(\log_{y+z} x)(\log_{z-y} x)}.$$

    (a) Undefined.
    (b) $\frac{1}{2}$
    (c) $\frac{1}{4}$
    (d) 2

    Solution: Let $\log_{z+y} x = u$ and let $\log_{z-y} x = v$. We wish to compute $1/u + 1/v$. Then $(y+z)^u = x$ and $(z-y)^v = x$. So $y+z = x^{1/u}$ and $z-y = x^{1/v}$. So $z^2 - y^2 = x^2 = x^{1/u}x^{1/v} = x^{1/u+1/v}$. So $1/u + 1/v = 2$.

    Note: One can choose x, z such that z is odd and $x^2 = 2z - 1$. For example, $x = 3, z = 5$ or $x = 5, z = 13$ etc. In such cases, $z - y = 1$ and hence $\log_{z-y} x = \log_1 x$, an undefined quantity. The way out is to interchange the roles of x and y. Since, the question did not specifically mentions how x, y are chosen given z, Option (a) Undefined is also a valid answer. Candidates who picked either (a) or (d) or both (a) and (d) have received full marks for this question. We thank the candidate who pointed out this case.

15. A game being offered in a casino consists of guessing the outcomes of two tosses of a fair coin. The gambler wins if she/he has correctly guessed at least one of the two tosses. To play a game, the gambler has to pay a fee of Rs. 80, and the winner gets a reward of Rs. 100 on winning the game (and nothing otherwise).

    Which of the following statements are correct?

    (a) In the first 10 minutes on a given day exactly three gamblers play the game, one after the other. The probability that the casino owner makes a profit in the first 10 minutes equals 1/4.
    (b) One gambler plays the game three times. The probability that she wins exactly two of the three games is 27/64.
    (c) Three friends go together and play the game with each playing once. The probability that all three win equals 27/64.
    (d) If 1200 players play the game on a given day, the expected profit of the casino owner for the day equals Rs. 6000.

    Solution: Options (b), (c), (d) are correct. The distribution of the number of wins of the casino owner in a sequence of 3 games is binomial with p = 1/4. Thus (b) and (c) are correct. The expected profit from each game is 80-100$\times$ 3/4 =5. (a) is false because the owner loses only when all three win, which is 27/64 thus the probability that owner makes a profit is 37/64.

16. In the following code the operator % denotes the remainder after integer division. That is: for positive integers a,b the value a % b is the remainder obtained when a is divided by b.

```
function fizzbuzz(n) {
   count = 0;
   for i from 0 to (n-1) {
       if ((i % 3) == 0) and ((i % 5) != 0) {
           count = count + 1;
     }
   }
   return(count);
}
```

What does fizzbuzz(100) return?

(a) 27
(b) 33
(c) 45
(d) 60

Solution: Option (a).

The function returns the number of non-negative integers less than n which are both (i) multiples of 3, and (ii) not multiples of 5. There are 27 such integers: $[3, 6, 9, 12, 18, 21, 24, 27, 33, 36, 39, 42, 48, 51, 54, 57, 63, 66, 69, 72,$

17. Alpha and Beta are inhabitants of an island of knights and knaves, where knights always tell the truth and knaves always lie. Alpha and Beta are alone at a beach when Alpha says: "At least one of us is a knave." And Beta says: "We are both knaves." Which of the following is/are true?

(a) Alpha and Beta are both knights
(b) Alpha and Beta are both knaves
(c) Alpha is a knight and Beta is a knave
(d) Alpha is a knave and Beta is a knight

Solution: Option (c).

If Beta is a knight then both are knaves, a contradiction. So Beta has to be a knave. Thus Beta's statement must be false. This means Alpha has to be a knight. And Alpha being a knight and Beta being a knave is consistent with both their statements.

Questions 18 to 20 are based on the following code.

The two arguments to the function Mystery(A, n) in the code below are: (i) an integer array A indexed from 0, and (ii) the number n of elements in A. Each element of A is an integer from the set $\{1, 2, \ldots, n\}$. The expression [0] * (n+1) creates an array, indexed from 0, that contains $n + 1$ zeroes.

Note: There is a typo in the following code, which results in the code referring to the non-existent array location A[n]. As a result of this typo, the three Part A questions 18, 19 and 20 based on this pseudocode were not considered for evaluation.

```
function Mystery(A, n) {
    found = False;
    value = None;
    B = [0] * (n+1);

    for i from 1 to n {
        B[A[i]] = B[A[i]] + 1;
    }

    for i from 1 to n {
```

```
        if (found == False) {
            if (B[A[i]] == 1) {
                found = True;
                value = A[i];
            }
        }
    }

    if (found == True) {
        return(value);
    } else {
        return(None);
    }
}
```

Answer the next three questions about this function.

18. What does the function call `Mystery([1, 2, 3, 3, 2, 1], 6)` return?

    (a) `None`
    (b) `1`
    (c) `2`
    (d) `3`

19. What does the function call `Mystery([1, 2, 4, 3, 2, 1], 6)` return?

    (a) `None`
    (b) `2`
    (c) `3`
    (d) `4`

20. What does the function call `Mystery([6, 5, 4, 3, 2, 5], 6)` return?

    (a) `3`
    (b) `4`
    (c) `5`
    (d) `6`

Part (B) - Short-answer questions

For questions in part (B), you have to write your answer with a short explanation in the space provided for the question in your answer sheet. If you need more space, you may continue on the pages provided for rough work. Any such overflows must be clearly labeled.

1. A girl writes five consecutive positive integers on a blackboard. She then erases one of them. The sum of the remaining four numbers is 2025. What number did she erase?

   Solution:

   We assume the five consecutive integers are $n, n+1, n+2, n+3, n+4$. Their total sum is:

   $$S = 5n + 10$$

   One number is erased and the sum of the remaining four numbers is given as 2025.

   Let a be an integer between 0 to 4. Then we have

   $$\begin{aligned} 5n + 10 &= 2025 + n + a \\ 4n &= 2025 + (a - 10) \end{aligned}$$

   The RHS is divisible by 4 only when $a = 1$. Hence, $n = 504$ and the erased number is 505. Verification: the numbers are $504, 505, 506, 507, 508$, they add up to 2530 and subtracting 505 gives us 2025.

2. A toy company currently sells 1,000 toys each month at a price of ₹500 per toy. To increase their sales, the company is considering to lower the price. Market research shows that for every ₹10 decrease in the price, the number of toys sold increases by 100. However, the price cut applies to all the toys sold.

   (a) By how much should the company reduce the price, so as to maximize its monthly revenue?

   (b) What would be the maximum revenue per month that the company can achieve?

   Assume that the decrease in price is an integer multiple of ₹1.

   Solution using Derivative Technique:

   Let the company reduce the price by ₹x.

   Then,

   - New price per toy = ₹(500 - x)
   - Increase in toys sold $= \frac{x}{10} \times 100 = 10x$
   - Total toys sold $= 1000 + 10x$

   Revenue Function:

   Revenue, $R(x) =$ Price per toy $\times$ Quantity sold

   $$\begin{aligned} R(x) &= (500 - x)(1000 + 10x) \\ &= 500000 + 5000x - 1000x - 10x^2 \\ &= 500000 + 4000x - 10x^2 \end{aligned}$$

   Take Derivative

   $$R'(x) = \frac{d}{dx}(500000 + 4000x - 10x^2) = 4000 - 20x$$

   Critical Point

$$R'(x) = 0 \Rightarrow 4000 - 20x = 0 \Rightarrow x = 200$$

Check Maximum

$$R''(200) = -20 < 0$$

So, the function attains a maximum at $x = 200$.

Final Answer:

The company should reduce the price by ₹200 to maximise revenue.

New price = ₹300

Number of toys sold $= 1000 + 10 \times 200 = 3000$

Maximum revenue = ₹300 × 3000 = ₹9,00,000

3. 15 balls are placed independently and uniformly at random into 15 bins numbered from 1 to 15. The probability that a ball ends up in a particular bin is $\frac{1}{15}$.

   (a) What is the probability that the first 5 balls go into different bins, conditioned on the event that first four balls are in different bins?

   (b) What are the possible values of the expected number of balls in bin 1 conditioned on the event that the first 13 balls are in different bins?

Solution: Problem (a) 11/15. Problem (b) Conditioned on X_{13}, the first bin may have 0 or 1 balls. If the first bin is empty, then it may have 0, 1 or 2 balls finally. 1 ball is with probability $2 * 1/15 * (14/15)$. 2 balls is with probability 1/225. So the expected value of Y_1 is then $\frac{30}{225}$.

On the other hand, if it already has 1, then the final number can be 1 (w.p 196/225), 2 (w.p $2*1/15*14/15$ and 3 w.p 1/225. So the expected value is $196/225 + 56/225 + 3/225$ which is 255/225.

4. There are 18 chocolates in a bag, of which 7 are green, 6 are blue, and 5 are red. We pick chocolates one at a time from the bag without replacement.

   (a) What is the probability that the first and the third chocolate are green?

   (b) What is the probability that after picking twelve chocolates, only chocolates of one colour remain in the bag?

Solution:

- $\frac{16!}{5!6!5!}$ divided by $\frac{18}{7!6!5!}$ which turns out to be $\frac{7*6}{17*18}$. Another way to see this. If the first three are green it happens with probability $\frac{7}{18}\frac{6}{17}\frac{5}{16}$. If the second is not green the probability is $\frac{7}{18}\frac{11}{17}\frac{6}{16}$. These are disjoint events and the probabilities add to give the same answer.
- We can have all blue or all green left in the bag. If the chocolates remaining are all blue, this can happen in $x := \frac{12!}{7!5!}$ ways of picking the first 12. If the remaining chocolates are all green, then in the first twelve we have picked all red, one green and all blue. Those can be picked in $y := \frac{12!}{5!6!}$ ways. So it is $\frac{x+y}{N}$.

5. Let M be an $n \times n$ matrix. We define an elementary row operation on M to be one of the following:

   i Interchanging some two rows of M.

   ii Multiplying a row in M by a non-zero scalar.

   iii Adding a scalar multiple of one row of M to another row of M.

An $n \times n$ matrix is said to be elementary if it is the result of a single elementary row operation performed on the $n \times n$ identity matrix.

Which of the following statements is/are true? Justify your answer with a short proof, if the statement is true, otherwise, provide a counterexample.

a Every elementary operation on a $n \times n$ matrix A can be performed by multiplying A by an elementary $n \times n$ matrix on the right.

b An elementary row operation on a $n \times n$ matrix A results in a matrix with the same determinant as that of A.

Solution: Both are false. Example $\begin{bmatrix} 1 & 0 \\ 2 & 0 \end{bmatrix}$ and interchanging the two rows. Scaling changes determinant.

6. Consider the following polynomial of positive degree n

$$P_n(x) = 1 + 2x + 3x^2 + \ldots + (n+1)x^n.$$

Show that there is no real number r such that $P_n(r) = 0$ when n is even.

Solution:

$$(1-x)P_n(x) = \frac{1 - x^{n+1}}{1-x} - (n+1)x^{n+1}$$

So

$$P_n(x) = \frac{1 - (n+2)x^{n+1} + (n+1)x^{n+2}}{(1-x)^2}$$

From the given expression of $P_n(x)$ there are no zeros when $x > 0$. From the above expression, when n is even, and $x < 0$, the numerator is positive, so there are no roots.

7. How many positive integers less than 1000 are **neither** divisible by 3 **nor** divisible by 5? Explain how you arrived at your answer.

Solution: The answer is 533.

Let $U = \{1, 2, \ldots, 999\}$. Let A be the set of all elements of U which are divisible by 3, and let B be the set of all elements of U which are divisible by 5. Then $|A| = \lfloor \frac{999}{3} \rfloor = 333, |B| = \lfloor \frac{999}{5} \rfloor = 199$, and $|A \cap B| = \lfloor \frac{999}{15} \rfloor = 66$. The number of elements of U which are divisible by at least one of $\{3, 5\}$ is $|A \cup B| = |A| + |B| - |A \cap B| = 333 + 199 - 66 = 466$. The number of elements of U which are divisible by neither 3 nor 5 is thus $|U \setminus (A \cup B)| = |U| - |A \cup B| = 999 - 466 = 533$.

8. Three hostel friends Amar, Prem and Raj are suspected of breaking a window. They made the following statements when questioned by the warden:

- Amar: *I did not break it. Prem is lying.*
- Prem: *Amar is telling the truth. Raj broke the window.*
- Raj: *I did not break it. Either Amar is telling the truth or Prem is telling the truth.*

You know that exactly one of them lied and the other two told the truth. Then, who broke the window? Justify.

Solution: Prem broke the window. In addition, he is the one who lied.

Justification: since we are told that exactly one of them lied, so we can assume each of them lied to see which assumptions lead to contradiction. If Amar lied then Prem is speaking the truth. Which implies Amar is speaking the truth. A clear contradiction.

Suppose Raj lied. This implies either Raj broke the window or both Amar and Prem are lying, again a contradiction.

Finally, suppose Prem lied, which means either Amar is telling the truth is false or Raj broke the window is false. Raj did not break the window doesn't contradict anything. However, Amar lying certainly contradicts Prem lying. However, both Amar and Raj are speaking the truth do not contradict anything. Hence, Prem is lying and his "Raj broke the window" statement is false, and more importantly, Prem broke the window.

9. A cloth bag labeled X contains two apples, bag Y contains two oranges and bag Z one apple and one orange. You pick a bag at random and then remove one fruit from that bag at random. Suppose you removed an apple. What is the probability that the fruit remaining in the bag is also an apple? Justify.

Solution: The answer is $\frac{2}{3}$.

Justification: Let $p(B_X)$ denote the probability of picking bag X and $p(A)$ be the probability of removing an apple on the first draw. Hence, we want to calculate

$$p(B_X|A) = \frac{p(A|B_X) \cdot p(B_X)}{p(A)}.$$

Note that $p(A) = \frac{2}{2} \cdot \frac{1}{3} + \frac{0}{2} \cdot \frac{1}{3} + \frac{1}{2} \cdot \frac{1}{3}$. Hence the answer is

$$p(B_X|A) = \frac{1 \cdot 1/3}{1/2} = \frac{2}{3}.$$

Questions 10 and 11 are based on the following description.

The following question appeared in a quiz:

> "*Write the pseudocode for a function* `Closest(A,n,x)` *that takes an array* `A`*, a positive integer* `n`*, and an integer* `x` *as arguments. The elements of* `A` *are all integers less than* 2^{64}*, and* `n` *is the number of elements in* `A`*. The call* `Closest(A,n,x)` *should return an integer* `y` *such that: (i)* $y \neq x$*, (ii)* `y` *is present in* `A`*, and (iii) there is* no $z \neq x$ *in* A *where* $|z - x| < |y - x|$ *holds. If* `A` *has no such element* `y`*, then the function should return the special value* `None`*.*"

A student submitted the code below as the answer to this question. In the code the array `A` is indexed from `0`, and `MAXINT` $= 2^{64} - 1$. The call `abs(z)` returns the absolute value |`z`| of integer `z`.

```
function Closest(A, n, x) {
    minVal = MAXINT;

    for i from 0 to (n-1) {
        absDiff = abs(x - A[i]);
        if (absDiff < minVal) {
            minVal = absDiff;
            y = A[i];
        }
    }

    if (minVal != MAXINT) {
        return(y);
    } else {
        return(None);
    }
}
```

This answer turned out to be wrong; this function gives the correct answer for some valid inputs, and wrong answers for other valid inputs. Answer the next two questions about this function.

10. What do the following function calls return?

    (a) `Closest([-10,2,10], 3, 8)`

    (b) `Closest([0,-5,4], 3, 7)`

    Solution: For both these inputs the function returns the correct answer, namely: 10 for part (a) and 4 for part (b). A complete solution will also explain (e.g., using dry runs) as to why the function returns these values.

11. Give one example of (i) an input array `A` with exactly 3 elements and (ii) an integer `x` for which the call `Closest(A, 3, x)` returns a **wrong** answer. What is this wrong answer? What is the correct answer?

    Solution: Some examples of such inputs are:

    - $A = [1, 2, 3], x = 1$. The function returns 1. The correct value is 2.

- $A = [5, 10, 3], x = 5$. The function returns 5. The correct value is 3.

If x is present in A then this pseudocode always outputs x, even if a different number is there in A.

A different kind of input for which this pseudocode gives a wrong answer, is one where the magnitude of of `x` is so large that `absDiff` is larger than `MAXINT`. For such inputs the function may return `None`, even if there is an element of `A` that satisfies the requirements for a closest element to `x`.

12. A sequence of five natural numbers $s_1 \leq s_2 \leq s_3 \leq s_4 \leq s_5$ satisfy the following conditions:

- $\sum_{i=1}^{5} s_i = 35$
- $\sum_{i=1}^{3} s_i = 15$
- $\sum_{i=3}^{5} s_i = 27$
- s_2 is even
- $s_4 - s_2 = 2$

Find all such sequences that satisfy the above conditions.

Solution: The answer is $2, 6, 7, 8, 12$. Step 1 -

$$\sum_{i=1}^{3} + s_i \sum_{i=3}^{5} s_i - \sum_{i=1}^{5} s_i = s_3 = 7.$$

Using this we get, $s_1 + s_2 = 8$ and $s_4 + s_5 = 20$. Now we have $s_3 < s_4 \implies s_3 < s_2 + 2$, which gives us

$$5 < s_2 < 7.$$

Hence $s_2 = 6$, and we can use it to find other values.

13. Five executives of a company namely CEO (chief executive officer), CFO (chief financial officer), COO (chief operating officer), CTO (chief technology officer), CMO (chief marketing officer) are to be seated around a circular table.

- The CEO must sit next to the CFO.
- The COO must not sit next to the CTO.
- The CTO must not sit next to the CEO.

In how many distinct ways can they be seated? (Rotations of the same arrangement are considered the same).

Solution: There are 2 distinct ways. There are total 5 people so, without restrictions there will be 24 circular permutations. One can begin by considering CEO and CFO has a single unit. Hence, there are 6 ways to arrange (CEO, CFO), COO, CTO, CMO. The unit (CEO, CFO) can be arranged in 2 ways, hence, in total there are 12 ways in which CEO is seated next to CFO. Now, proceed by adding one restriction at a time. We get the following 2 valid seating arrangements:

- CEO-CFO-CTO-CMO-COO-CEO
- CEO-COO-CMO-CTO-CFO-CEO

14. Evaluate the following limit:

$$\lim_{x \to 0} \frac{e^{2x} - 2x - \cos x}{x \sin x}.$$

Solution: One has to use the L'Hospital rule in order to get the answer $\frac{5}{2}$.

15. Find the determinant of the following matrix:

$$A = \begin{bmatrix} 1 & 1 & 4 & 3 \\ 5 & 8 & 26 & 21 \\ 2 & 3 & 10 & 8 \\ 3 & 5 & 16 & 13 \end{bmatrix}$$

Solution: $det(A) = 0$; the 3rd column is twice the sum of first and second column.

16. Let $f : \mathbb{R} \to \mathbb{R}$ be a function which is twice differentiable everywhere and suppose that $f'(q) = 0$ for every rational number q. Find the value of $f(\pi^2) - f(\pi)$.

    Solution: 0. Since f' exists everywhere and is continuous, its value can't be nonzero on irrationals. As the derivative is identically zero the function f is constant.

17. Consider the following 2×2 matrix

$$B = \begin{bmatrix} 0 & r \\ -1 & 0 \end{bmatrix},$$

    where r is a nonzero real number. Find B^{2025}, i.e., the matrix obtained by multiplying B with itself 2025 times.

    Solution: One has to realize that $B^{4k} = r^{2k} I$, by explicitly computing B^2, B^3 and B^4. Hence, $B^{2025} = r^{1012} B$ as 2025 is congruent to 1 mod 4.

Instructions for Questions 18,19, and 20:

Read the following description carefully and answer the questions that follow. Use all the information provided. Clearly show all your calculations.

Description:

NorthCool Beverages Pvt. Ltd. is a leading beverage company that operates across six northern states of India. During the past summer the company launched an aggressive marketing campaign to promote its range of flavoured drinks. The charts below summarise the sales data collected during this campaign.

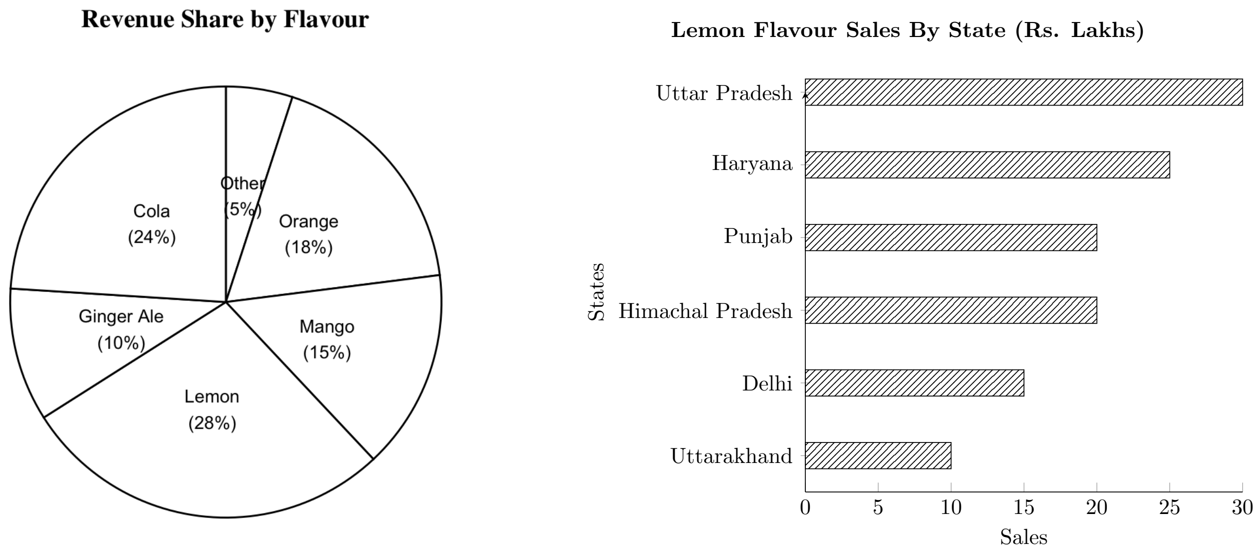

Figure 1: Revenue share by flavour, and sales of lemon-flavoured drink by state

18. What is the revenue from the sales of Mango flavour drinks.

    Solution: Let total revenue be R. Given:

    - Lemon revenue = ₹120 lakhs
    - Lemon accounts for 28% of total revenue

$$\frac{28}{100} \times R = 120 \Rightarrow R = \frac{120 \times 100}{28} = \frac{12000}{28} \approx 428.57 \text{ lakhs}$$

Total revenue = ₹428.57 lakhs. Now, the share of mango flavoured drink is 15%, which amounts to 64.28 lakhs.

19. NorthCool Beverages plans to launch a new variant, "Lemon Max", in the two states with the highest sales of the lemon flavour. The company expects Lemon Max to generate additional revenue equal to 20% of the current lemon flavour sales in those two states. What would be the percentage revenue share of lemon flavour drinks if their expectations are met?

    Solutions:

    Top 2 states by lemon sales:

    - Uttar Pradesh: ₹30 lakhs
    - Haryana: ₹25 lakhs
    - Total = ₹55 lakhs

    Lemon Max will generate 20% additional revenue over this; which is 11 lakhs. So, if the expectations are met then the total revenue generated by lemon flavoured drinks is 131 lakhs. The total expected revenue is 439.57 lakhs. Hence, the increased share of lemon flavoured drink is approximately 30% (29.8% to be precise).

20. On average, for every ₹1 lakh in total revenue from the lemon-flavoured drink, 10,000 litres of lemon-flavoured drink are sold across the six states. Estimate the total volume (in litres) of lemon-flavoured drink sold by NorthCool Beverages across all six states.

    solutions: Given:

    - ₹1 lakh revenue = 10,000 litres
    - Total lemon revenue = ₹120 lakhs

    $$\text{Total volume} = 120 \times 10{,}000 = 1{,}200{,}000 \text{ litres}$$

    Total volume of lemon beverage sold = 12,00,000 litres.