CHENNAI MATHEMATICAL INSTITUTE
M.Sc. Data Science Entrance Examination 2024
Solutions

Part (A) - Multiple-choice questions

1. Consider the following positive integers:

$$A = 1234679580, B = 1234789560.$$

Which of the following statements are true?

(a) Both A and B are divisible by 36.
(b) Both A and B are divisible by 72.
(c) Only B is divisible by 120.
(d) The number A is divisible by 80.

Solution: *(a), (c)*. We need to consider the divisibility by $4, 6, 8, 9, 10$. The number A is divisible by $2, 3, 4, 5, 6, 9$ and B is divisible by $2, 3, 4, 5, 6, 8, 9$.

2. Which of the following statements are true?

(a) Let $f(x) = x^3 + x - 1$ for $x \in \mathbb{R}$. Then the equation $f(x) = 0$ has at least one root in $[-1, 0)$.
(b) Let $f(x) = x^3 + x - 1$ for $x \in \mathbb{R}$. Then the equation $f(x) = 0$ has at least one root in $[0, 1)$.
(c) Let $f(x) = x^3 + x + 1$ for $x \in \mathbb{R}$. Then the equation $f(x) = 0$ has at least one root in $[0, 1)$.
(d) Let $f(x) = x^3 + x + 1$ for $x \in \mathbb{R}$. Then the equation $f(x) = 0$ has at least one root in $[-1, 0)$.

Solution: *(b), (d)*. First, (a) is false. Any value in $[-1, 0)$ would result a negative value of $f(x)$. Hence there is no root in $[-1, 0)$. Second, (b) is true. $f(0) = -1$ and $f(1) = 1$. Third, (c) is false as $f(x) > 0$ for all $x \in [0, 1)$. Finally, (d) is true. $f(-1) = -1$ and $f(0) = 1$.

3. 26 children participated in a chess tournament. A child got two points for winning, zero for losing and one point for a draw. Each child played against every other child. After the tournament was over no child had an odd score. There were no draws in the entire tournament and no two players had the same score. For convenience assume that the children are named $A, B, C, \ldots, Z$, by the 26 letters of the English alphabet in increasing order of scores, so A has lowest score, Z has highest score. Thus we have $A < B < C < \cdots < X < Y < Z$. Pick all statements which are true.

(a) K lost to Q
(b) K lost to B
(c) M lost to N
(d) If L lost to M then N lost to M.

Solution: *(a), (c)*. There are 26 children and no odd score. So the possible scores are 0, 2, 4, 6, ..., 50. Since no two had the same score, and there are exactly 26 even numbers here, all these numbers are attained. So Z won all games, Y all except against Z, and so on. So N defeated M and Q defeated K. Statement d is false because the premise is true but the conclusion is false.

4. 14 teams participate in a volleyball tournament. Each team plays the other exactly once. There are no draws. Assume the teams are labeled by j, $1 \leq j \leq 14$. Let x_j denote the number of games team j wins and y_j the number of games that team j lost. Pick the correct alternative(s):

(a) $\sum_j x_j^2 = \sum_j y_j^2$.
(b) $\sum_j x_j^2 > \sum_j y_j^2$.
(c) $\sum_j x_j = \sum_j y_j$.

(d) $\sum_j |x_j| = \sum_j |y_j|$.

Solution: *(a),(c), (d).* The logic works for any n. Note that $\sum x_j = \sum y_j = \binom{14}{2}$ since every game is won by somebody and lost by somebody. Also $x_j + y_j = 13$. Now $\sum_j x_j^2 = \sum_j (13 - y_j)^2$. The R.H.S. here works out to $13^2 * 14 - 2 * 13 * \sum_j y_j + \sum_j y_j^2$. The first two terms cancel out since $\sum_j y_j = \frac{14*13}{2}$.

5. Which of the following statements is/are true for real numbers x, y?

(a) If $x^2 = y^2$ then $x = y$.

(b) If $x^3 = y^3$ then $x = y$.

(c) If $x < y$ then $x^2 < y^2$.

(d) If $x < y$ then $x^3 < y^3$.

Solution: *(b), (d).*

- (a) is false. Consider $x = 1, y = -1$.
- (b) is true. From $x^3 = y^3$ we get $(x - y)(x^2 + xy + y^2) = 0$. This implies that either $x = y$, or $x = \frac{-y \pm i\sqrt{3}y}{2}$. Since the latter condition implies that x has an imaginary part, we get that the former condition must hold.
- (c) is false. Consider $x = -1, y = 1$.
- (d) is true. We have $x^3 - y^3 = (x - y)(x^2 + xy + y^2)$. Also, $x^2 + xy + y^2 = x^2 + xy + \frac{y^2}{4} + \frac{3y^2}{4} = (x + \frac{y}{2})^2 + (\frac{\sqrt{3}y}{2})^2$. Since $x^2 + xy + y^2$ is the sum of two squares, and since both of x, y cannot be zero, $x^2 + xy + y^2$ takes only positive values. And from $x < y$ we get $(x - y) < 0$. Thus $x^3 - y^3 = (x - y)(x^2 + xy + y^2) < 0$.

6. Suppose that A is an $n \times n$ matrix with $n \geq 6$. For an $n \times 1$ vector b consider the equations

$$Ax = b, \text{ for an } n \times 1 \text{ vector } x \qquad (1)$$
$$A^T x = b \text{ for an } n \times 1 \text{ vector } x \qquad (2)$$

where A^T is the transpose of the matrix A. Which of the following statements are correct?

(a) If equation (1) admits a solution for all b, then A^{-1} exists.

(b) If equation (1) admits a solution for all b, then equation (2) also admits a solution for all b.

(c) If equation (1) admits a solution for some b, then A^{-1} exists.

(d) If equation (1) admits a solution for some b, then equation (2) also admits a solution for that b.

Solution: *(a), (b).* Easy to see that (a) and (b) are correct - uses the fact that if A admits a inverse, then so does A^T.

7. Which of the following inequalities are correct?

(a) $1 + x \leq e^x$ for all $x \in \mathbb{R}$

(b) $2^n \leq 2n!$ for all positive integers n

(c) $(1 + x^2)^n \leq (1 + x)^{2n}$ for all $x \in (0, \infty)$

(d) $\sin(x) \leq \tan(x)$ for all $x \in (-\frac{\pi}{2}, \frac{\pi}{2})$

Solution: *(a), (b), (c).* (a) is true : Using Taylor's theorem for $f(x) = e^x$ with Lagrange's form of remainder : $f(x) = f(0) + xf'(0) + \frac{x^2}{2}f''(\theta)$ for some θ between 0 and x. Here $f'(0) = 1$, $f''(0) = 1$ and $f''(\theta) > 0$ for all θ. (b) is true for $n = 1$ and $n = 2$ and for every $n > 2$, the L.H.S. at $(n - 1)$ gets multiplied by 2 and R.H.S. at $(n - 1)$ by n. Easy to see that (c) is true while (d) is false for x negative.

8. Which of the following statements are valid for all $n \times n$ matrices A, B:

(a) $(A^T A)^T = AA^T$.

(b) If A, B are invertible, then inverse of AB is $A^{-1}B^{-1}$.

(c) $(A+B)^T = A^T + B^T$

(d) $Ax = Bx$ for some $n \times 1$ vector x implies that $Ay = By$ for all $n \times 1$ vectors y.

Solution: *(c)*. Clear from the definitions.

9. Let x be a variable that takes real values, and let $f(x), g(x), h(x)$ be three polynomials with real-valued coefficients. Further, let $f(x)$ be a polynomial of degree 1, $g(x)$ a polynomial of degree 2, and $h(x)$ a polynomial of degree 3. Which of the following statements is/are **always true** for *any three* polynomials of these types?

   (a) The graphs of $f(x)$ and $g(x)$ intersect at one or more points.

   (b) The graphs of $f(x)$ and $h(x)$ intersect at one or more points.

   (c) The graphs of $g(x)$ and $h(x)$ intersect at one or more points.

   Solution: *(b), (c)*.

   - (a) is false. Consider $f(x) = x, g(x) = x^2 + 2$. These never intersect because $x^2 - x + 2 = 0$ has no real roots.
   - (b) is true. Let $f(x) = ax + b, h(x) = cx^3 + dx^2 + ex + f$. The curves of $f(x)$ and $h(x)$ intersect at all those values of x which are roots of the cubic polynomial $cx^3 + dx^2 + (e-a)x + (f-b)$. The cubic has three—perhaps repeated—solutions, of which at most two can have imaginary parts, since such roots occur in conjugate pairs. So the cubic has at least one real root, and hence the two graphs intersect at at least one point.
   - (c) is true for similar reasons as to why (b) is true.

10. A spherical ball of ice of radius $20m$ is dropped in a vat of hot water. The ice melts in such a way that (i) the shape of the ball remains spherical, and (ii) the radius of the ball decreases at a constant rate of $0.5ms^{-1}$. At what rate does the *volume* of the ice ball decrease, when the radius of the ball is $15m$?

    (a) $100\pi m^3 s^{-1}$

    (b) $225\pi m^3 s^{-1}$

    (c) $450\pi m^3 s^{-1}$

    (d) $600\pi m^3 s^{-1}$

    Solution: *(c)*. Let r be the radius of the sphere at any instant. Then its volume at that instant is $V = \frac{4}{3}\pi r^3$. The rate at which this volume changes with respect to time t is $\frac{dV}{dt} = \frac{dV}{dr} \times \frac{dr}{dt} = 4\pi r^2 \times \frac{dr}{dt}$. Since $\frac{dr}{dt}$ is given to be $0.5ms^{-1}$, we get that $\frac{dV}{dt} = 2\pi r^2$. When $r = 15m$, this becomes $450\pi m^3 s^{-1}$.

11. Let x be a real-valued variable. What is the range of the function $f(x) = \sin^2 x - \sin x + 2$?

    (a) $[0, 2]$

    (b) $[1, 2]$

    (c) $[1, 4]$

    (d) $[\frac{7}{4}, 4]$

    Solution: *(d)*. The given expression can be written, by completing the square, as $(\sin x - \frac{1}{2})^2 + \frac{7}{4}$. The least value of the squared term is 0, attained when $\sin x = \frac{1}{2}$. Trying out the two extreme values of $\sin x$ we get that the squared term is maximum when $\sin x = -1$, and at such an x we get that $f(x) = (-\frac{3}{2})^2 + \frac{7}{4} = 4$.

12. In the following code, `A` is an array indexed from `0` whose elements are all positive integers, and `n` is the number of elements in `A`. It is given that `n` is at least 2. The operator `*` denotes multiplication.

```
function foo(A,n) {
  if A[0] > A[1] {
    first = A[0];
    second = A[1];
  } else {
    first = A[1];
    second = A[0];
  }

  for i from 2 to (n-1) {
    if A[i] > second {
      if A[i] > first {
          second = first;
          first = A[i];
      } else {
          second = A[i];
      }
    }
  }
  return(first * second);
}
```

If `A = [15, 7, 16, 12, 17, 14, 16, 4, 13, 12]`, what will `foo(A, 10)` return?

(a) 28
(b) 144
(c) 180
(d) 272

Solution: *(d)*. The function returns the product of the two largest numbers (possibly repeated) in the array given as its argument. For the given array these numbers are $16, 17$, and their product is 272.

13. In the following code, `A` is an array indexed from `0` whose elements are all positive integers, and `n` is the number of elements in `A`.

```
function foo(A,n) {
  max = 0;
  curr = 0;

  for i from 1 to (n-1) {
    if A[i] > A[i-1] {
      curr = curr + 1;
      if curr > max {
        max = curr;
      }
    } else {
      curr = 0;
    }
  }
  return(max+1);
}
```

If `A = [1, 3, 5, 2, 4, 7, 6, 8]`, what will `foo(A, 8)` return?

(a) 2
(b) 3
(c) 4

(d) 5

Solution: *(b)*. The function returns the length of a longest strictly increasing consecutive sub-sequence of the input array. For the given array there are two such sub-sequences—$[1, 3, 5]$ and $[2, 4, 7]$—each of length 3.

14. A prominent newspaper conducted a survey among a carefully chosen sample of 1,00,000 voters under the age of 25. A randomly chosen subset of 60% of this sample were asked the following question: *"Have you ever voted in a general election?"* The remaining 40% were asked the following, different question: *"Is it true that you have* never *voted in a general election?"*

    The editor in charge of collating the data from the survey, accidentally deleted some of the contents of the survey database. The only information that survived this mishap was that a total of 54,000 respondents replied YES to the question that they got. Specifically, it is not known how many of these YES answers correspond to the first question, and how many to the second.

    Assume that the sampling was done properly and that the respondents answered truthfully. What estimate can we derive from just the total number of YES answers, about the percentage of voters under the age of 25 that have voted in some general election?

    (a) At most 30%.
    (b) More than 30%, but less than 60%.
    (c) At least 60%, but less than 75%.
    (d) No such estimate can be drawn from just the total number of YES answers.

    Solution: *(c)*. Let p be the fraction of voters under the age of 25 that have ever voted in a general election. Then we have: $60000p + 40000(1 - p) = 54000$. Solving, we get $p = \frac{7}{10} = 70\%$.

15. Let f, g be two real valued functions defined on the set of real numbers. Which of the following statements are true?

    (a) If $f(x)$ is continuous and $g(x)$ discontinuous then the function $f(x)+g(x)$ is necessarily discontinuous.
    (b) If $f(x)$ is continuous and $g(x)$ discontinuous then the function $f(x)\cdot g(x)$ is necessarily discontinuous.
    (c) If the functions $f(x), g(x)$ are discontinuous at a point c in the domain the function $f(x) + g(x)$ is also discontinuous at c.
    (d) If the functions $f(x), g(x)$ are discontinuous at a point c in the domain the function $f(x) \cdot g(x)$ is also discontinuous at c.

    Solution: *(a)*. First part is straightforward. For the second part consider the functions $1/x$ and x. For the last two parts consider the function $f(x) = 1$ when $x \geq 0$ and $f(x) = -1$ when $x < 0$, it is discontinuous at 0. Let $g(x) := -f(x)$, which is also discontinuous at 0. However, both $f + g$ and the product fg are continuous everywhere.

16. In the following code, `A` is an array indexed from `0` whose elements are all positive integers, `n` is the number of elements in `A`, and `x` is a positive integer. The call `abs(z)` returns the absolute value of integer `z`.

```
function foo(A,n,x) {
  max = 0;

  for i from 0 to (n-1) {
    for j from (i+1) to (n-1) {
      diff = abs(A[i]-A[j]);
      if (diff >= x) and (diff >= max) {
        max = diff;
      }
    }
  }

  return(max);
}
```

If `A = [10, 8, 10, 4, 10, 7, 1, 2, 2, 9]`, what will `foo(A, 10, 5)` return?

(a) 2
(b) 4
(c) 6
(d) 9

Solution: *(d).* The function call returns the maximum absolute difference between two elements of `A` if this difference is at least 5, and zero otherwise. Since the maximum absolute difference between two elements of the given array is 9—due to the elements $10, 1$—this call returns 9.

17. One day, Captain Haddock receives a mysterious letter with a confusing paragraph. Captain Haddock and Tintin are investigating the matter. There are two possible suspects: Professor Calculus and Thomson & Thompson. Based on their past experience:

- The probability that Professor Calculus sends a letter is 60%, while the probability that Thomson & Thompson send a letter is 40%.
- When Professor Calculus sends letters, there is an 80% probability that the letter contains a confusing paragraph.
- When Thomson & Thompson send letters, there is a 5% probability that the letter contains a confusing paragraph.

What is the probability that the letter was sent by Professor Calculus?

(a) 0.96
(b) 0.80
(c) 0.50
(d) 0.48

Solution: *(a).* Let A be the event that Professor Calculus sent the letter. Let B be the event that the letter contains a confusing paragraph. We are looking for $P(A|B)$. We have $P(A) = 0.6$ and $P(A^c) = 0.4$ and $P(B|A) = 0.8$ (probability that the letter contains a confusing paragraph when we know that it was sent by Professor Calculus). $P(B|A^c) = 0.05$ (probability that the letter contains a confusing paragraph when we know that it was sent by Thompson & Thompson). By the theorem of total probability we find the probability that the letter contains a confusing paragraph is:

$$\begin{aligned} P(B) &= P(B|A)P(A) + P(B|A^c)P(A) \\ &= 0.8 \times 0.6 + 0.05 \times 0.4 \\ &= 0.5 \end{aligned}$$

Now we apply Bayes' theorem:

$$\begin{aligned} P(A|B) &= \frac{P(B|A)P(A)}{P(B)} \\ &= \frac{0.8 \times 0.6}{0.5} \\ &= 0.96 \end{aligned}$$

18. Let f be a function on the positive real numbers such that $f(xy) = f(x) + f(y)$. If $f(2024) = 2$ then which of the following statement(s) is/ are true?

(a) $f(\frac{1}{2024}) = 1$
(b) $f(\frac{1}{2024}) = -1$
(c) $f(\frac{1}{2024}) = -2$
(d) $f(\frac{1}{2024}) = 2$

Solution: *(c).* We have, $f(1) = 2f(1)$, so $f(1) = 0$. So $f(1) = f(2024) + f(1/2024)$, hence the answer.

The following description is for questions 19 and 20.

A perfect shuffle of a deck of cards divides the deck into two equal parts and then interleaves the cards from each half, starting with the first card of the first half.

For instance, if we shuffle a deck of cards containing 10 cards arranged $[1, 2, 3, 4, 5, 6, 7, 8, 9, 10]$ we first create two equal decks with cards $[1, 2, 3, 4, 5]$ and $[6, 7, 8, 9, 10]$ and then interleave them to get a new deck $[1, 6, 2, 7, 3, 8, 4, 9, 5, 10]$.

19. We shuffle the deck $[3, 6, 11, 4, 7, 9, 2, 8, 5, 10, 12, 1]$. What are the neighbours of 4 after the shuffle?

    (a) 1
    (b) 5
    (c) 10
    (d) 12

    Solution: *(b), (c)*. Since the shuffled deck is $[3, 2, 6, 8, 11, 5, 4, 10, 7, 12, 9, 1]$.

20. After shuffling once we obtain a deck with cards $[15, 6, 4, 1, 13, 9, 8, 7, 12, 11, 5, 10, 3, 14, 2, 16]$. What were the neighbours of 2 before the deck was shuffled?

    (a) 3
    (b) 6
    (c) 7
    (d) 14

    Solution: *(a), (b)*. Write down the shuffled deck, then the answer is straightforward.

Part (B) - Short-answer questions

For questions in part (B), you have to write your answer with a short explanation in the space provided for the question in your answer sheet. If you need more space, you may continue on the pages provided for rough work. Any such overflows must be clearly labeled.

Questions 1 and 2 are based on the following description.

Each round of a TV game show consists of ten questions. Before each round the host takes ten boxes and places prizes in nine of them, leaving one empty. The host then shuffles these boxes and labels them from 1 to 10. When the guest answers a question correctly, the host opens the corresponding box. If the box has a prize, the guest earns the prize. If the box is empty, the round ends and the guest gets to keep their earnings so far.

1. In the Easy Round, all questions are easy, and each prize is ₹1000. Guest Chatur knows all the answers. What is the expected earnings for Chatur in this round? Explain your answer.

   Solution: ₹4500. Explanation: For each $1 \leq i \leq 10$ the probability that the empty box is labelled i, is $1/10$. The *cumulative* earnings for each value i of the label of the empty box are: $1 : 0$, $2 : 1000$, $3 : 2000$, $4 : 3000$, and so on, up till $10 : 9000$. Note that, the expected earning if the round ends after 2nd questions is $(1000)(\frac{9}{10})(\frac{1}{9})$, after 3rd question is $(2000)(\frac{9}{10})(\frac{8}{9})(\frac{1}{8})$ etc. So the expected earnings for Chatur from this round is $(1000 + 2000 + \cdots + 9000)/10 =$ ₹4500.

2. To proceed to the next round, a guest must earn at least ₹7000 in the Easy Round. What is the probability that guest Chatur, who knows all the answers of the Easy Round, progresses to the second round? Explain your answer.

   Solution: 3/10. Explanation: For each $1 \leq i \leq 10$, the probability that the empty box is labelled i, is $1/10$. Chatur *fails* to move on to the second round exactly when this label takes one of the values from 1 to 7. The probability of this happening is $7/10$, since for $i \neq j$ the two events "the empty box is labelled i" and "the empty box is labelled j" are disjoint. So Chatur progresses to the second round with probability $3/10$.

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3. Find values of x and y that satisfy both the following equations:

$$\sqrt{\frac{x}{y}} + \sqrt{\frac{y}{x}} = \frac{5}{2}$$
$$\frac{x}{\sqrt{y}} + \frac{y}{\sqrt{x}} = \frac{9}{2}.$$

   Solution: Either $x = 1, y = 4$, or $x = 4, y = 1$.

   The key is to realize that the first equation is of the form $z + (1/z) = c$, and hence is a quadratic equation that can be solved to find the ratio x/y.

   Set $z = \sqrt{\frac{x}{y}}$. Then the first equation becomes $z + (1/z) = 5/2$. Simplifying and solving the resulting quadratic equation, we get $z \in \{2, 1/2\}$. Suppose $z = 2$. Then $\sqrt{\frac{x}{y}} = 2$, and so $x = 4y$. Substituting this in the second equation we get $y = 1$, and thence $x = 4$. Suppose $z = 1/2$. Then $\sqrt{\frac{x}{y}} = 1/2$, and so $y = 4x$. Substituting this in the second equation we get $x = 1$, and thence $y = 4$.

Questions 4 and 5 are based on the following description.

The following question appeared in a quiz:

> "*Write the code for a function* `SecondBest(A,n)` *that takes an array* `A` *and a positive integer* `n` *as arguments. The elements of* `A` *are all integers, and* `n` *is the number of elements in* `A`*. The call* `SecondBest(A,n)` *should return the second largest element in* `A`*. If* `A` *has no second largest element, then the function should return the special value* `None`*.*"

A student submitted the code below as the answer to this question. In the code the array `A` is indexed from `0`.

```
function SecondBest(A, n) {
  if n == 1 {
    return(None);
  }

  first = A[0];
  second = A[1];

  for i from 2 to (n-1) {
    if (A[i] >= first) and (A[i] >= second) {
      second = first;
      first = A[i];
    } else {
      if A[i] >= second {
        second = A[i];
      }
    }
  }

  if first != second {
    return(second);
  } else {
    return(None);
  }
}
```

This answer turned out to be wrong; this function gives the correct answer for some valid inputs, and wrong answers for other valid inputs. Answer the next two questions about this function.

4. What do the following function calls return? Briefly explain your answer.

   (a) `SecondBest([2,2,3], 3)`

   (b) `SecondBest([3,2,2], 3)`

   Solution: For both these inputs the function returns the correct answer, namely 2.

5. Give one example of an input array `A` with exactly 3 elements for which the call `SecondBest(A, 3)` returns a **wrong** answer. What is this wrong answer? What is the correct answer?

   Solution: Some examples of such inputs are:

   - $A = [1, 3, 2]$. The function returns 3. The correct value is 2.
   - $A = [1, 1, 0]$. The function returns `None`. The correct value is 0.

   In each case, an explanation would involve a description of how the code sets the values `first` and `second`, and how the comparisons in the final `if` block using these values, results in a wrong answer.

6. Let F be the set of points on the plane with coordinates (x, y) such that $||x| - |y|| + |x| + |y| = 4$. What is the number of points in F with integer coordinates?

   Solution: Suppose $|x| \geq |y|$, then we have $|x|-|y|+|x|+|y| = 4$. Or $2|x| = 4$, so $|x| = 2$. So if $x = 2$, then y can take values from $[-2, 2]$. On the other hand if $|x| \leq |y|$ then we get $2|y| = 4$, so $|y| = 2$ and x takes the values in the range $[-2, 2]$. So the points are $(2, -2), (2, -1), (2, 0), (2, 1), (2, 2), (-2, -2), (-2, -1), (-2, 0), (-2, 1), (-2, 2), (1, 2), (0, 2), (1, 2), (-1, -2), (0, -2), (1, -2)$, a total of 16 points.

7. We select three points at random on the circumference of a circle. What is the probability that ΔABC contains the centre O in the interior?

   Solution: Assume the circumference is 1 unit. Take the first point. The diameter joining this point and the centre divides the circumference into two halves. Choose the second point on any of the two halves. The calculation doesn't change. Now the third has to be on the other half for the center to be in the triangle. If the second point is at a distance x from the first, the third has to be choosen in an arc of length x (out of 1/2) so the probability of success is $2x$. Integrating this over 0 to 1/2 we get $2x^2/2$ which is 1/4. Note that with probability 1/2 the second was on either side, so the final probability is 1/4.

8. Let α be a fixed real number and let f be a function from $\mathbb{R}^2$ to $\mathbb{R}^2$ defined as

$$f(x, y) = (x \cos \alpha - y \sin \alpha, x \sin \alpha + y \cos \alpha).$$

   Write down an expression for $f^{10}(x, y)$, where f^{10} denotes the function obtained by composing f with itself 10 times.

   Solution: One may realize that this map is the rotation about the origin by α. So, applying it ten times we get the rotation by 10α, i.e., in the expression above α is to be replaced by 10α. However, one need not know this fact; by explicitly composing f with it self and doing the calculations it is clear what the pattern is. (Note - some may consider 10 compositions as applying the functions 11 times. In that case the answer is 11α. Since this is not a problem about convention either of the answer is considered correct. The important thing is to be able to understand the concept).

9. "*I've defeated more than* 10 *FIDE masters,*" a chess player boasted. "*Surely, it must be fewer than* 10," said the referee. "*Well, I suppose it was at least one,*" said the Grand Master. If exactly one of them spoke the truth, how many FIDE masters did the chess player actually defeated?

   Solution: The answer is that the player defeated not a single FIDE master. Since exactly one statement is true, there are 3 choices as follows: If the statement 1 is true then the statement 3 can't be false, a contradiction. If the statement 3 is true and the statement 2 is false then the statement 2 can't be false. However, if one considers that there is strict inequality in statements 1 and 2 and that the player has defeated exactly 10 players then both these statements are false and only the Grand Master is speaking the truth. Finally, if the statement 2 is true then the statement 1 false and the statement 3 will be false if the player did not defeat any FIDE master. Hence, the player has defeated either 0 or 10 FIDE masters.

10. A supplier of art material has four reams of handmade paper, three boxes of acrylic colors and two printing blocks. The two artists in the shop want to buy one item each, but insist on having the same kind of art material. How many items does the supplier have to take out to be sure that the artists' demand is met?

    Solution: The supplier has to take out at least 4 items; in order to have at least two of those of one type. Note that this is a pigeonhole principal problem, with 9 pigeons and 3 holes. The shopkeeper can, first, take out 1 item each of the 3 types and then the 4th item of taken out such that the demand is met.

11. In a farmer's stable three animals, a donkey, a cow and a horse had to share two types of feed bags, one containing hay and the other containing grain, as follows:

    - If the donkey ate grain then the horse ate what the cow ate.
    - If the horse ate grain then then donkey ate what the cow did not eat.
    - If the cow ate hay, then the donkey ate what the horse ate.

Among all the assignments of feedbags that satisfy the above condition which animal always gets to eat from the same feedbag?

Solution: When all the 3 conditions are satisfied, only the donkey always eats from the feed bag containing hay. There are 2 feed bags - grains and hay, so we can start with 8 possibilities of assigning them. If the donkey eats grains then the horse and the cow have to share the same food. However, all of them can't eat grains since it contradicts condition 2. The combination grains, hay, hay contradicts the last condition. Concluding that the donkey can never eat grains. The assignment where the horse eats grains and the donkey and the cow eat hay contradicts condition 2. Hence, there are exactly 3 possibilities that satisfy the given three conditions and in each, the donkey eats from the hay feed bag.

12. There are 7 elevators in a large shopping mall, each stopping at ground floor and at most six other floors. If at least 3 elevators stop at each floor and if it is possible to go from any floor to any other floor without changing elevators, what is the maximum number of floors in the mall?

    Solution: There are at most 14 floors = 7 (elevators) $\times$ 6 (stops per elevator)/3 (stops per floor). Here is a sample elevator-floor assignment that serves 14 floors. For each elevator we give 6 floors at which it should stop.

    (a) Elevator 1: floors $1, 2, 3, 4, 5, 6$.
    (b) Elevator 2: floors $1, 2, 7, 8, 11, 12$.
    (c) Elevator 3: floors $1, 2, 9, 10, 13, 14$.
    (d) Elevator 4: floors $3, 4, 7, 8, 13, 14$.
    (e) Elevator 5: floors $5, 6, 7, 8, 9, 10$.
    (f) Elevator 6: floors $3, 4, 9, 10, 11, 12$.
    (g) Elevator 7: floors $5, 6, 11, 12, 13, 14$.

    For example, if someone wants to go from 6th floor to 10th then they should use Elevator 5. The fourth floor is served by exactly 3 elevators, namely, $1, 4$ and 6.

13. Consider the following expression

$$\sqrt{x + 2\sqrt{x-1}} + \sqrt{x - 2\sqrt{x-1}}.$$

    Find two real numbers a, b such that the above expression is constant for $a \leq x \leq b$.

    Solution: One does the following simplification -

$$\begin{aligned} f(x) &= \sqrt{(x-1) + 2\sqrt{x-1} + 1} + \sqrt{(x-1) - 2\sqrt{x-1} + 1} \\ &= \sqrt{(\sqrt{x-1}+1)^2} + \sqrt{(\sqrt{x-1}-1)^2} \\ &= \sqrt{x-1} + 1 + |\sqrt{x-1} - 1|. \end{aligned}$$

    Hence, the expression is undefined for $x < 1$ and it is $2\sqrt{x-1}$ for $x > 2$. However, for $1 \leq x \leq 2$ the expression is 2.

14. You are given a strange, analogue wall clock whose hour and minute hands are identical. Both the hands move continuously and there is no second hand. How many times are there from noon to midnight when it is not possible to tell what time is it by looking at the clock at that instant? For example, at 06 : 00 pm one can be sure that the upper hand is the minute hand, for otherwise at 12 : 30 pm the top hand is between 12 and 1. However, a bit after 1 : 15 pm and 3 : 06 pm the clock looks identical and you wouldn't be able to tell the exact time.

    Solution: The answer is 132. Here is an explanation - when the hour hand has moved x degrees around the clock from the top, the minute hand has moved $y = 12x$ degrees. If the time is still a valid configuration when the hands are switched around then $x = 12y$ as well. Hence there 144 configurations that are indistinguishable, for example, the position at 1 : 51 is identical to that at 10 : 09 if the hands are identical. Same is true when the time is 4 : 47 or 9 : 24 etc. However, of these 144 configurations there are exactly 12 configurations in which the hour and the minute hand are in the exact same position. In such a case it doesn't matter which hand is which because we can still tell the time. So, there are exactly $144 - 12 = 132$ ambiguous configurations.

15. Does there exist a polynomial $q(x)$ with integer coefficients such that $q(1) = 2$ and $q(3) = 5$? Given an example if there is one. Justify, if there is not.

Solution: Such a polynomial can't exist because

$$\begin{aligned} 3 &= q(3) - q(1) \\ &= \sum_i a_i(3^i - 1^i) \end{aligned}$$

The left hand side is odd and the right hand side is even, a contradiction.

16. Find the number of all 4-digit natural numbers formed with exactly two distinct digits.

Solution: The answer is 567. First choose 2 distinct digits can be chosen from 10 available digits in 45 ways. With the selected 2 digits one can form 2^4 different 4-digit positive integers. However, there are two integers with all the 4 digits same. Finally, we have to discard those 'integers' that start with 0. Hence the final answer is

$$45(2^4 - 2) - \binom{9}{1}(2^3 - 1) = 630 - 63 = 567.$$

17. An electronic card shuffling machine always rearranges the cards in the same way relative to the order in which they are placed in it. One iteration of shuffling means that the cards are placed in the machine and they are taken out after the rearrangement. Two iterations means that you place the cards in the machine immediately after the first iteration. The ace through the king of hearts are arranged in order with the ace on top and the king at the bottom. After 2 iterations the order of the cards, from top to bottom, is

$$10, 9, Q, 8, K, 3, 4, A, 5, J, 6, 2, 7.$$

What will be the order after 13 iterations?

Solution: The cards will return to their original, sorted position, Ace, all the way down to the King. Note that the shuffling process can be considered as a bijection from the set $\{1, \ldots, 13\}$ onto itself. Let f denote the bijection induced by shuffling these cards once. Hence the given information tells us - $f^2(1) = 10, f^2(2) = 9$ etc. Let's express this information in the cycle notation, i.e., for every i, look at it's inverse image as well the image -

$$1 \to 8 \to 4 \to 7 \to 13 \to 5 \to 9 \to 2 \to 12 \to 3 \to 6 \to 11 \to 10 \to 1.$$

We see that there are no smaller cycles in f^2, which means they can't be there in f as well. Hence we have to start with every number and then move to right, 13 places. You will see that the order returns.

Questions 18–20 are based on the following description.

In June 2017, a cyberattack named NotPetya spread all over the world. Big companies like Marex, Merck, and FedEx's TNT Express lost a lot of money because of this attack.

Company	Pre-attack Average Monthly Revenue (USD million)	Post-Attack Average Monthly Revenue (USD million)
Marex	1000	700
Merck	800	480
TNT Expanse	500	250

Table 1: Financial impact of NotPetya attack on three major companies

18. Which company experienced the smallest percentage decrease in total revenue over a one month period relative to its estimated total revenue over the same period?

Solution: First, intermediate calculations. The average monthly revenue (loss) for each company:

- Marex $1000 - 700 = 300$ million USD.
- Merc $800 - 480 = 320$ million USD.
- TNT $500 - 250 = 250$ million USD

The percentage decrease in total revenue relative to the total revenue for each company:

- Marex: $\frac{300}{1000} \times 100 = 30\%$
- Merc: $\frac{320}{800} \times 100 = 40\%$
- TNT Expanse: $\frac{250}{500} \times 100 = 50\%$

Answer: Marex experienced the smallest percentage decrease in total revenue, with a decrease of 30%. This calculation reflects that, even though all companies faced significant losses, Marex's total revenue impact was relatively smaller in percentage terms compared to the others.

19. Calculate the overall percentage decrease in revenue across all three companies in one month period following the NotPetya attack.

    Solution: First, intermediate calculations. The total monthly revenue before and after the attack for all companies:

    - Before attack: Total = Maersk (1000) + Merck (800) + TNT Express (500) = 2300 million USD
    - After attack: Total = Maersk (700) + Merck (480) + TNT Express (250) = 1430 million USD

    Total revenue lost = 2300 - 1430 = 870 million USD

    The overall percentage decrease in revenue = $\left(\frac{870}{2300}\right) \times 100\% \approx 37.83\%$

    Answer: The overall percentage decrease in revenue across all three companies in the month following the NotPetya attack is approximately 37.83%.

20. Assuming the monthly revenue of Marex follows a Gaussian distribution with a standard deviation of 75, (i) what is the probability that Marex's revenue will be less than \$850 million in the pre-attack scenario, and (ii) what is the probability that Marex's revenue will be more than \$850 million in the post-attack scenario?

z	$P(Z \leq z)$
-2	0.0227
-1	0.1586
0	0.5000
1	0.8413
2	0.9772

Table 2: Note that $Z \sim N(0, 1)$

Solution: The pre-attack scenario. Suppose R is the monthly revenue and $R \sim N(\mu = 1000, \sigma = 75)$

$$\begin{aligned} P(R \leq 850) &= P\left(\frac{R - 1000}{75} \leq \frac{850 - 1000}{75}\right) \\ &= P(Z \leq -2), \quad where \quad Z \sim N(0,1) \\ &= 0.0227 \end{aligned}$$

The post-attack scenario. Suppose R is the monthly revenue and $R \sim N(\mu = 700, \sigma = 75)$

$$\begin{aligned} P(R \geq 850) &= P\left(\frac{R - 700}{75} \geq \frac{850 - 700}{75}\right) \\ &= P(Z \geq 2), \quad where \quad Z \sim N(0,1) \\ &= 1 - P(Z \leq 2) \\ &= 1 - 0.9772 \\ &= 0.0227 \end{aligned}$$