CHENNAI MATHEMATICAL INSTITUTE

MSc Applications of Mathematics Entrance Examination

18 May 2017

Instructions:

- Enter your Admit Card Number: **A** – ☐☐☐ – ☐☐☐☐
- The allowed time is 3 hours.
- This examination has two parts. Part A has multiple-choice questions, while questions in Part B require detailed answers.
- **The score in Part A will be used for screening purposes. Your answers to the questions in Part B will be marked only if your score in Part A places you over the cut-off. However, the scores in both the sections will be taken into account to decide whether you qualify for the interview.**
- **Answers to questions in Part A must be recorded on the sheet provided for the purpose.**
- You may use the blank pages at the end for your rough-work.

For office use only

Part B

Qno	1	2	3	4	5	6	7	8
Marks								

.	Part A	Part B	Total
Score			

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- Enter your Admit Card Number: **A** – ☐☐☐ – ☐☐☐☐

You must record your answers to Part A here by filling in the appropriate circles:
For example, if your answer to question number 7 is (A) and (D), record it as follows:

7. ● (B) (C) ●

Part A

1. (A) (B) (C) (D)

2. (A) (B) (C) (D)

3. (A) (B) (C) (D)

4. (A) (B) (C) (D)

5. (A) (B) (C) (D)

6. (A) (B) (C) (D)

7. (A) (B) (C) (D)

8. (A) (B) (C) (D)

9. (A) (B) (C) (D)

10. (A) (B) (C) (D)

Important

The score in Part A will be used for screening purposes. Your answers to the questions in Part B will be marked only if your score in Part A places you over the cut-off. However, the scores in both the sections will be taken into account to decide whether you qualify for the interview.

Part A

This section consists of Ten (10) multiple-choice questions, each with one or more correct answers. Record your answers on the attached sheet by filling in the appropriate circles. *Each question carries 5 marks. A solution receives credit if and only if all the correct answers are chosen, and no incorrect answer is chosen.* Throughout, R denotes the set of real numbers.

1. For subsets U, V of a non-empty set Ω, let

$$U\Delta V = (U \cap V^c) \cup (U^c \cap V).$$

Which of the following statements is/are true for subsets A, B, C of Ω:

(A) $(A\Delta B)\Delta C = A\Delta(B\Delta C)$.

(B) $(A\Delta B) \cap C = (A \cap C)\Delta(B \cap C)$.

(C) $(A\Delta B) \cup C = (A \cup C)\Delta(B \cup C)$.

(D) $(A\Delta B) = (A^c\Delta B^c)^c$.

2. Which of the following statements is/are true?

(A) $(n+1)^n \leq n^{(n+1)}$ for all $n \geq 100$.

(B) $2^n \leq n^2$ for all $n \geq 100$.

(C) $(n!)^n \leq n^{(n!)}$ for all $n \geq 100$.

(D) $n^{\log(n)} \leq (\log(n))^n$ for all $n \geq 100$.

3. Let $\{a_n : n \geq 1\}$ be a sequence such that $a_n \geq 0$ for all $n \geq 1$. Which of the following sets of conditions implies that the sequence $\{a_n : n \geq 1\}$ is convergent:

(A) The subsequences $\{a_{2n} : n \geq 1\}$, $\{a^2_{3n} : n \geq 1\}$ and $\{a^3_{2n+1} : n \geq 1\}$ are convergent.

(B) The subsequences $\{a_{3n} : n \geq 1\}$, $\{a^2_{5n} : n \geq 1\}$ and $\{a^3_{2n+1} : n \geq 1\}$ are convergent.

(C) The subsequences $\{a_{5n} : n \geq 1\}$, $\{a^2_{7n} : n \geq 1\}$ and $\{a^3_{2n+1} : n \geq 1\}$ are convergent.

(D) The subsequences $\{a_{7n} : n \geq 1\}$, $\{a^2_{2n} : n \geq 1\}$ and $\{a^3_{2n+1} : n \geq 1\}$ are convergent.

4. For $n \geq 1$ and $x \in [0, 1]$ let

$$f_n(x) = \frac{1}{nx+1}$$
$$g_n(x) = \frac{x}{nx+1}.$$

Which of the following statements is/are true?

(A) The sequence of functions $\{f_n\}$ converges pointwise on $[0, 1]$.

(B) The sequence of functions $\{g_n\}$ converges pointwise on $[0, 1]$.

(C) The sequence of functions $\{f_n\}$ converges uniformly on $[0, 1]$.

(D) The sequence of functions $\{g_n\}$ converges uniformly on $[0, 1]$.

5. Let A, B be $n \times n$ symmetric real matrices. Which of the following statements is/are true?

(A) $AB = 0$ and B invertible implies $A = 0$.

(B) $A^2 = 0$ implies $A = 0$.

(C) $AB = 0$ implies $BA = 0$.

(D) $AC = 0$ for all singular $n \times n$ matrices C implies $A = 0$.

6. Which of the following sets are Vector spaces under usual operations?

(A) $\{(x_1, x_2, x_3) \in \mathsf{R}^3 \,:\, x_1 + x_3 = 2x_2\}$.

(B) $\{(x_1, x_2, x_3) \in \mathsf{R}^3 \,:\, x_1 + x_3 = 2 + x_2\}$.

(C) The set of all $n \times n$ matrices A such that $\det(A) = 0$.

(D) The set of all $n \times n$ matrices A such that $\mathrm{tr}(A) = 0$.

(*Here det*(A) *denotes the determinant and tr*(A) *denotes the trace of the matrix* A.)

7. Which of the following sets of vectors in R^3 forms a basis of R^3?

(A) $\{(1, 1, 0), (0, 1, 1), (1, 0, 1)\}$.

(B) $\{(1, -1, 0), (0, 1, -1), (-1, 0, 1)\}$.

(C) $\{(1, 2, 3), (2, 3, 1), (3, 1, 2)\}$.

(D) $\{(1, 2, 3), (4, 5, 6), (7, 8, 9)\}$.

8. Let $f : (0, 1) \mapsto (0, 1)$ be a continuously differentiable function. Then we can conclude that

(A) $g = \frac{1}{f}$ is a continuous function on $(0, 1)$.

(B) $g = \frac{1}{f}$ is a continuously differentiable function on $(0, 1)$.

(C) $g = \frac{1}{f}$ is a uniformly continuous function on $(0, 1)$.

(D) h defined by $h(x) = x(1 - x)f(x)$ for $x \in (0, 1)$ is uniformly continuous.

9. Let $f : \mathsf{R} \mapsto \mathsf{R}$ be function such that

$$|f(x) - f(y)| \leq 100|x - y| \quad \forall x, y \in \mathsf{R}.$$

Which of the following statements is/are always true?

(A) f is uniformly continuous on $[0, 1]$.

(B) f is uniformly continuous on R.

(C) f is a differentiable function on $[0, 1]$.

(D) If f is differentiable at $x = x_0$ then $|f'(x_0)| \leq 100$ where f' denotes the derivative of f.

10. Let $p(x)$ be an odd degree polynomial and let $q(x) = (p(x))^2 + 2p(x) + 2$.

(A) The equation $q(x) = 5p(x)$ admits at least two distinct real solutions.

(B) The equation $q(x) = 4p(x)$ admits at least one real solution.

(C) The equation $p(x)q(x) = -4$ admits at least two distinct real solutions.

(D) The equation $q(x) = 3$ admits at least two distinct real solutions.

Part B

Answer all questions. Each question carries 10 marks.

1. Show that
$$(1-p)^n \le 1 - p^n \quad \text{for} \quad 0 \le p \le 1.$$

2. Let $a_1, a_2, \cdots, a_n$ be strictly positive real numbers. Show
$$\left(\sum_1^n a_i\right)\left(\sum_1^n \frac{1}{a_i}\right) \ge n^2.$$

3. Let f be a real valued continuous function on $[0,1]$. Prove that there is a number $c \in [0,1]$ such that
$$\int_0^1 f(x)x^2 dx = \frac{f(c)}{3}.$$

4. Let f be a real valued uniformly continuous function on $[0,\infty)$. Show that there are numbers $A > 0$ and $B > 0$ such that
$$|f(x)| < Ax + B \quad \text{for all} \quad x \ge 0$$

5. Let $a_1, a_2, \cdots, a_{17}$ be numbers each in $[0,1]$. Find
$$\max\left\{\sum_1^{17} |x - a_i| : -5 \le x \le 5\right\}$$

6. State the spectral decomposition theorem for real symmetric matrices. If A is a real symmetric non-negative definite matrix of rank one, prove that there is a (column) vector v such that $A = vv^t$ where v^t is transpose of v.

7. Let A be a square matrix such that A^2 is invertible. Show that A is invertible.

8. There are 15 people in a party, which includes three people A, B and C. They are to be divided into three groups of five each. In how many ways can it be done if no two of A, B, C should be in the same group.