PART A:

1. A,B
2. A,C,D
3. A,D
4. A,B,D
5. A,B,C,D
(A,B,C is also graded as correct)
6. A,D
7. A,C
8. A,B,D
9. A,B,D
10. A,D

PART B:

1. For any two positive numbers a, b, binomial expansion shows

$$a^n + b^n \leq (a+b)^n$$

Take $a = (1-p), b = p$.

2. geometric mean is smaller than arithmetic mean

$$\frac{1}{n} \sum a_i \geq \sqrt[n]{a_1 \cdots a_n} \qquad \text{and} \qquad \frac{1}{n} \sum \frac{1}{a_i} \geq \sqrt[n]{\frac{1}{a_1 \cdots a_n}}$$

Multiply.

3. Continuous function on a closed bounded interval is bounded and attains its bounds. Let m and M be minimum and maximum of f on $[0, 1]$. Then

$$mx^2 \leq f(x)x^2 \leq Mx^2$$

Integrate. There is a number d, with $m \leq d \leq M$, such that

$$\int_0^1 f(x)x^2 dx = d\,\frac{1}{3}$$

Intermediate value theorem for f completes proof.

4. Using uniform continuity, get $\delta > 0$ such that

$$|x - y| \leq \delta, \quad x, y \geq 0 \to |f(x) - f(y)| < 1$$

Thus

$$x \in [0, \delta] \to |f(x) - f(0)| \leq 1 \to |f(x)| \leq |f(0)| + 1.$$

$$x \in [\delta, 2\delta] \to |f(x) - f(\delta)| \leq 1 \to |f(x)| \leq |f(0)| + 2.$$

In general for $k \geq 0$,

$$x \in [k\delta, (k+1)\delta] \to |f(x)| \leq |f(0)| + (k+1) \leq |f(0)| + 1 + \frac{1}{\delta}x.$$

(In this interval $k\delta \leq x$ so $k \leq x/\delta$) So $B = |f(0)| + 2$ and $A = 1/\delta$ will do.

5. Since $a_i \geq 0$ and $x \in [-5, 5]$, we see $|x - a_i| \leq |-5 - a_i| = 5 + a_i$ for each i. So required sum is at most $\sum_1^{17}(5+a_i)$. Since it is attained at $(x = -5)$ it is maximum.

6. Spectral theorem for real symmetric matrices: if A is a real symmetric matrix, then $A = UDU^t$ or $A = UDU^{-1}$ where D is a diagonal matrix with real entries and U is a (real) orthogonal matrix, $UU^t = I$. Diagonal entries of D are the eigen values of A and columns of U are the corresponding eigen vectors.

Since A is moreover non-negative definite, its eigen values are non-negative. Since its rank is one, only one diagonal entry of D is non-zero, because U has full rank. if the k-th diagonal entry is non-zero, say λ then denoting $\sqrt{D}$, the diagonal matrix with k-th entry $\sqrt{\lambda}$ and others zero we have

$$A = U\sqrt{D}\sqrt{D}U^t.$$

$U\sqrt{D}$ has all columns except k-th column zero. Denoting the k-th column by v we have

$$A = vv^t.$$

7. Note A is invertible iff its determinant $|A|$ is nonzero. So A^2 is invertible implies $|A^2| \neq 0$. Using $|AB| = |A||B|$ so that $|A^2| = |A|^2$, we conclude $|A| \neq 0$, so A is invertible.

8. Out of remaining 12 select any 4 and include A; out of the remaining 8 select any 4 and include B; with the remaining 4 include C. These are the three groups. Since the order is not important, the total number is

$$\binom{12}{4}\binom{8}{4} = \frac{12!}{4!4!4!} = 34,650$$