CHENNAI MATHEMATICAL INSTITUTE

MSc Applications of Mathematics Entrance Examination

18 May 2016

Instructions:

- Enter your *Registration Number* here **CMI PG–**

- Enter the name of the city where you write this test:

- The allowed time is 3 hours.
- This examination has two parts. Part A has multiple-choice questions, while questions in Part B require detailed answers.
- **The score in Part A will be used for screening purposes. Your answers to the questions in Part B will be marked only if your score in Part A places you over the cut-off. However, the scores in both the sections will be taken into account to decide whether you qualify for the interview.**
- **Answers to questions in Part A must be recorded on the sheet provided for the purpose.**
- You may use the blank pages at the end for your rough-work.

For office use only

Part B

Qno	1	2	3	4	5	6	7	8	9	10	11	12
Marks												

.	Part A	Part B	Total
Score			

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18 May 2015

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You must record your answers to Part A here by filling in the appropriate circles:
For example, if your answer to question number 7 is (A) and (D), record it as follows:

7. ● (B) (C) ●

Part A

1. (A) (B) (C) (D)

2. (A) (B) (C) (D)

3. (A) (B) (C) (D)

4. (A) (B) (C) (D)

5. (A) (B) (C) (D)

6. (A) (B) (C) (D)

7. (A) (B) (C) (D)

8. (A) (B) (C) (D)

9. (A) (B) (C) (D)

10. (A) (B) (C) (D)

Important

The score in Part A will be used for screening purposes. Your answers to the questions in Part B will be marked only if your score in Part A places you over the cut-off. However, the scores in both the sections will be taken into account to decide whether you qualify for the interview.

Part A

This section consists of Ten (10) multiple-choice questions, each with one or more correct answers. Record your answers on the attached sheet by filling in the appropriate circles. *Each question carries 5 marks. A solution receives credit if and only if all the correct answers are chosen, and no incorrect answer is chosen.* Throughout, R denotes the set of real numbers.

1. Which of the following functions are uniformly continuous on their respective domains of definition?

   (A) $f(x) = \exp(-x), \quad x \in [0, \infty)$.

   (B) $g(x) = x^2 \exp(-x), \quad x \in [0, \infty)$.

   (C) $h(x) = \frac{\sin(x)}{x}, \quad x \in (0, \infty)$.

   (D) $k(x) = \frac{x}{1+x^2}, \quad x \in (0, \infty)$.

2. Let $\{a_n : n \geq 1\}$ be a sequence of real numbers, $s_n = \sum_{k=1}^{n} a_k$ and $m_n = \frac{s_n}{n}$. Which of the following statements is/are true?

   (A) If $\{s_n\}$ converges to a real number s, then $\{a_n\}$ converges to 0.

   (B) If $\{a_n\}$ converges to 0 then $\{s_n\}$ converges to a real number s.

   (C) If $\{m_n\}$ converges to a real number m, then $\{a_n\}$ converges to m.

   (D) If $\{a_n\}$ converges to a real number a, then $\{m_n\}$ converges to a.

3. Let $\mathbf{A}$ be a real symmetric $d \times d$ matrix, let D denote its determinant and T denote its trace. Which of the following statements is/are true?

   (A) If $D > 0$ then $\mathbf{A}$ is strictly positive definite.

   (B) If $T > 0$ then $\mathbf{A}$ is invertible.

   (C) If $\mathbf{A}$ is strictly positive definite then $D > 0$.

   (D) If $D > 0$ then $T > 0$.

4. Let M denote the number of 5−tuples $(a_1, \cdots, a_5)$ where each $a_i \geq 1$ is an integer and $\sum\limits_{i=1}^{5} a_i = 7$. Then

   (A) $M \geq 14$.

   (B) $M \leq 20$.

   (C) $M \in \{14, 21, 28\}$.

   (D) $M \in \{10, 15, 20\}$.

5. Let V be the vector space of polynomials in one variable with real coefficients. Which of the following subsets is/are subspaces of V?

   (A) The set U consisting of all polynomials with integer coefficients.

   (B) The set W consisting of all polynomials of degree at least 6.

   (C) The set X consisting of all polynomials having at least one real root.

   (D) The set Y consisting of all polynomials only with even powers of the variable.

6. Consider the sequence of functions f_n defined on the interval $[0, \infty)$ by

$$f_n(x) = \frac{nx}{1 + n^2x^2} \quad , \quad x \in [0, \infty) \quad n \geq 1.$$

Then the sequence $\{f_n(x)\}$

(A) does not converge for some $x \in [0, \infty)$.

(B) converges for every $x \in [0, \infty)$.

(C) converges for every $x \in [0, \infty)$ and $f(x) = \lim_{n\to\infty} f_n(x)$ is a continuous function.

(D) converges uniformly in $x \in [0, \infty)$.

7. Consider the matrix $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & t \\ 1 & 4 & t^2 \end{bmatrix}$. Which of the following is/are true?

(A) A is invertible for all real values of t.

(B) A is invertible for all positive t.

(C) A is invertible for all negative integers t.

(D) A is invertible for $t = 3$ and $t = 4$.

8. The dimension d_n of the vector space W of all $n \times n$ real symmetric matrices satisfies

(A) $d_n \geq \frac{n^2}{2}$ for all $n \geq 2$.

(B) $d_n \geq \frac{2n^2}{3}$ for all $n \geq 2$.

(C) $d_n \leq \frac{3n^2}{4}$ for all $n \geq 2$.

(D) $d_n < (n + \frac{n^2}{2})$ for all $n \geq 2$.

9. Let A and B be symmetric strictly positive definite $d \times d$ matrices with real entries. Then we can conclude that

(A) $A + B$ is a non-singular matrix.

(B) AB is a non-singular matrix.

(C) $A + B$ is a symmetric strictly positive definite matrix.

(D) AB is a symmetric strictly positive definite matrix.

10. Let $f : (0, 1] \mapsto [-1, 1]$ and $g : [-1, 1] \mapsto (0, 1]$ be continuous functions. Which of the following statements is/are always true?

(A) f is uniformly continuous.

(B) g is uniformly continuous.

(C) $f \circ g$ is uniformly continuous.

(D) The function $h(x) = xf(x)$ for $x \in (0, 1]$ is uniformly continuous.

(Here $f \circ g(x) = f(g(x))$.).

Part B

1. Show that
$$| 1 - \cos(x) | \leq \frac{1}{2}x^2, \quad \forall x \in \mathsf{R}.$$

2. Show that every continuous function $f : [0, 1] \mapsto \mathsf{R}$ is uniformly continuous.

3. Let $\{a_n : n \geq 1\}$ be a sequence of numbers such that $\lim_{n\to\infty} a_n x^n = 0$ for all $x > 0$. Show that the radius of convergence R of the power series
$$\sum_{n=1}^{\infty} a_n x^n$$
is infinite.

4. Let $\mathcal{M}$ denote the class of all $d \times d$ real symmetric matrices that are strictly positive definite. We define a relation $\ll$ on M as follows: For $\mathbf{A}, \mathbf{B} \in \mathcal{M}$, $\mathbf{A} \ll \mathbf{B}$ if $\mathbf{B} - \mathbf{A} \in \mathcal{M}$. Show that

   (a) For $\mathbf{A}, \mathbf{B}, \mathbf{C} \in \mathcal{M}$
   $$\mathbf{A} \ll \mathbf{B} \text{ and } \mathbf{B} \ll \mathbf{C} \Rightarrow \mathbf{A} \ll \mathbf{C}.$$

   (b) For $\mathbf{A} \in \mathcal{M}$ show that $\exists 0 < \alpha < \beta < \infty$ such that
   $$\alpha \mathbf{I_d} \ll \mathbf{A} \ll \beta \mathbf{I_d}$$
   where $\mathbf{I_d}$ is the $d \times d$ identity matrix.

5. Let $0 < a < b < \infty$ be two real numbers. Define two sequences of real numbers as follows. $a_1 = a$; $\quad b_1 = b$;
$$a_{n+1} = \sqrt{a_n b_n}\,, \quad b_{n+1} = \frac{a_n + b_n}{2}; \quad \text{for} \quad n \geq 1.$$
Show that the sequence $\{a_n\}$ is increasing and the the sequence $\{b_n\}$ is decreasing. Show that $\lim_n a_n = \lim_n b_n$.

6. For $\lambda > 0$ show that
$$\sum_{k=0}^{\infty} \sum_{j=k+1}^{\infty} \frac{\lambda^j}{j!} = \lambda e^{\lambda}.$$

7. Recall that a function $f : R \to R$ is continuous at a point $a \in R$ if, given $\epsilon > 0$ there is a $\delta > 0$ such that $|f(x) - f(a)| < \epsilon$ whenever $|x - a| < \delta$. Using this definition, show that if f and g are continuous at a then their product $h = fg$ is also continuous at a.

8. Let L be the integer lattice, that is, the set of points (i, j) in the plane where i and j are integers. Let f be a real valued function defined on L. Suppose that for all $(i, j) \in L$,
$$f(i, j) = \frac{f(i + 1, j) + f(i - 1, j) + f(i, j + 1) + f(i, j - 1)}{4}.$$
If f attains its maximum at some point in L, show that f is a constant function.