Part A.

1(ABCD) 2 (AD) 3 (C) 4 (ABD) 5 (D)

6 (BC) 7(CD) 8 (ACD) 9 (ABC) 10 (BCD)

Part B.

1. Since both $\cos x$ and x^2 are even functions and since $\cos x \leq 1$ it is enough to show that $f(x) = \frac{x^2}{2} - 1 + \cos x \geq 0$ for $x \geq 0$.

We consider only $x \geq 0$. Note $f'(x) = x - \sin x$ and $f''(x) = 1 - \cos x$. Since f'' is positive we see f' is increasing. Also $f'(0) = 0$ so that f' is positive. Hence f is increasing. Since $f(0) = 0$ we see $f(x)$ is positive for all $x \geq 0$.

2. Let $\epsilon > 0$ be given. Need to find $\delta > 0$ so that $|f(x) - f(y)| < \epsilon$ for $x, y \in [0, 1]$ and $|x - y| < \delta$. Suppose there is no such δ. Thus for each integer $n \geq 1$, there are two points x_n and y_n in $[0, 1]$ with $|x_n - y_n| < 1/n$ but $|f(x_n) - f(y_n)| \geq \epsilon$. Since the interval $[0, 1]$ is closed and bounded there is a subsequence $\{x_{n_k}\}$ which converges to a point $x \in [0, 1]$. Since $|x_{n_k} - y_{n_k}| \leq 1/n_k$, we see y_{n_k} also converges to the same point x. For each k we have $|f(x_{n_k}) - f(y_{n_k})| \geq \epsilon$ where as by continuity of f we see $f(x_{n_k}) - f(y_{n_k}) \to f(x) - f(x) = 0$.

3. Take any $x \in R$. By hypothesis $\lim a_n(2x)^n = 0$ and hence this is a bounded sequence. Say $|a_n(2x)^n| \leq c$. Thus $|a_n x^n| \leq c/2^n$. Since $\sum(c/2^n)$ is convergent we conclude that $\sum |a_n x^n|$ is convergent. Thus $\sum a_n x^n$ is absolutely convergent and hence convergent.

4. (a) For every non-zero vector v we have $v^t(B-A)v > 0$ and $v^t(C-B)v > 0$. Add and conclude that $v^t(C - A)v > 0$.

(b) Since A is symmetric and strictly positive definite, it has diagonalization, say $A = P^tDP$ where P is orthogonal and D is diagonal with strictly positive entries. Let β be strictly larger than all diagonal entries of D. Then $\beta I - A = P^t(\beta I - D)P$. Since $\beta I - D$ is diagonal with strictly positive entries we conclude that $A << \beta I$. Similarly taking any number $\alpha > 0$ strictly smaller than all diagonal entries of D we conclude $\alpha I << A$.

5. The AM-GM inequality says $a < \sqrt{ab} < \frac{a+b}{2} < b$. Using this, by induction we see $a_1 < a_2 < \cdots < \cdots < b_2 < b_1$. Thus $\{a_n\}$ is increasing and bounded above (by any of the b_i) so converges to, say, c. Similarly $\{b_n\}$ is decreasing and bounded below (by any of the a_i), so converges to say C. Clearly $c \leq C$. Since $b_{n+1} = \frac{a_n+b_n}{2}$ we conclude, after taking limits, that $C = \frac{c+C}{2}$ showing $c = C$.

6. Since terms are positve, we can interchange the order of summation

$$\sum_{k=0}^{\infty}\sum_{j=k+1}^{\infty}\frac{\lambda^j}{j!}=\sum_{j=1}^{\infty}\sum_{k=0}^{j-1}\frac{\lambda^j}{j!}=\sum_{j=1}^{\infty}\frac{\lambda^j}{j!}j=\sum_{j=1}^{\infty}\frac{\lambda^j}{(j-1)!}=\lambda\sum_{j=1}^{\infty}\frac{\lambda^{j-1}}{(j-1)!}=\lambda e^{\lambda}$$

7. Let $\epsilon > 0$ be given. Choose $\delta_1 > 0$ so that $|f(x)-f(a)| < \epsilon/\{2(1+|g(a)|)\}$ whenever $|x-a| < \delta_1$. In particular for $|x-a| < \delta_1$ we have $|f(x) < |f(a)| + \epsilon/\{2(1+|g(a)|)\} = C$, say. Choose $\delta_2 > 0$ so that $|g(x)-g(a)| < \epsilon/\{2(C+1)\}$. Let $\delta = \min\{\delta_1, \delta_2\}$. Now let $|x-a| < \delta$. Then

$$|f(x)g(x)-f(a)g(a)| \le |f(x)||g(x)-g(a)| + |g(a)||f(x)-f(a)| \le \epsilon.$$

8. Let L be the max value of f. Claim: If $f(i,j) = L$, then at its four neighbours $(i \pm 1, j)$ and $(i, j \pm 1)$ the value of f must equal L. Indeed if f value is less than L at a neighbour, then the average would also be so.

Thus if $f(i,j) = L$ then f value must be L at $(i \pm 1, j), (i \pm 2, j)$ and finally at $(0,j)$ and then at $(0, j \pm 1), (0, j \pm 2)$ and finally at $(0,0)$. Now proceed to any (k,l) in the same manner to show f takes the value L at all points.