Draft solutions for CMI BSc entrance exam on May 19, 2024

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In the online exam the instruction page counted as question 1, so the 21 part A questions were numbered from 2 to 22. For any correspondence refer to the numbering below.

A test developed to detect Covid gives the correct diagnosis for 99% of people with Covid. It also gives the correct diagnosis for 99% of people without Covid. In a city $\frac{1}{1000}$ of the population has Covid. Answer questions (1) and (2) as per the instruction below.

Questions

(1) What is the probability that a randomly selected person tests positive? (We assume that in our random selection every person is equally likely to be chosen.) [2 points]

(2) Suppose that a randomly selected person tested positive. What is the probability that this person has Covid? [2 points]

Instruction for (1) and (2)

If the probability is $x\%$, then your answer should be the integer closest to x. E.g., for probability $\frac{1}{3} = 33.33\ldots\%$, you should type 33 as your answer. For probability $\frac{2}{3}$ you should type 67 as your answer.

Solution: Let C = event that the person has Covid, T = event that the test is positive. Question (1) asks for $P(T)$ and question (2) asks for $P[C|T]$.

$$P[T] = \frac{1}{1000} \times 0.99 + \frac{999}{1000} \times 0.01 = \frac{1}{1000}(0.99 + 9.99) = \frac{10.98}{1000} = \frac{549}{50000} = 1.098\% \approx 1\%.$$

$$P[C|T] = \frac{P[C\&T]}{P[T]} = \frac{\frac{1}{1000} \times 0.99}{\frac{10.98}{1000}} = \frac{0.99}{10.98} = \frac{11}{122} \approx 9.02\% \approx 9\%.$$

Note: Most of the positive tests are false positives coming from Covid-free people, which "explains" why the answer to (2) is so low. Because only 0.1% of the population has Covid, positive tests coming from this group are an order of magnitude fewer than false positives coming at 1% rate from Covid-free people who make up 99.9% of the population. This also explains the answer to (1): when rounded down, it is the "same" 1% rate.

Consider the polynomial

$$p(x) = x^6 + 10x^5 + 11x^4 + 12x^3 + 13x^2 - 12x - 11.$$

Questions

(3) Find the remainder when $p(x)$ is divided by $x + 1$. [1 point]

(4) Let $z_1, z_2, z_3, z_4, z_5, z_6$ be the six complex roots of $p(x)$. Evaluate $\sum_{i=1}^{6} z_i^2$. [2 points]

(5) Find an integer n with the least possible absolute value such that $p(x)$ has a real root between n and $n+1$. Write this number *along with your reason* as per the instruction below. [2 points]

Instruction for (5)

Write two numbers separated by a comma: value of n, number of the theorem below that justifies this answer. E.g., if you think that $n = 5$ because of the factor theorem, then type **5,1** as your answer with no space, full stop or any other punctuation.

1. Factor theorem
2. Mean value theorem
3. Intermediate value theorem
4. Fundamental theorem of algebra
5. Fundamental theorem of calculus

Solution: (3) $p(-1) = 4$

(4) $\sum_{i=1}^{6} z_i^2 = (\sum_i z_i)^2 - 2\sum_{i<j} z_i z_j = (-10)^2 - 2(11) = 78.$

(5) $p(0) = -11$ and $p(1) > 0$. By intermediate value theorem p has a root between 0 and 1.

Two mighty frogs jump *once per unit time* on the number line as described below.

Questions

(6) The first frog is at $x = 2^i$ at time $t = i$. How many numbers of the form $7n + 1$ (with n an integer) does the frog visit from $t = 0$ to $t = 99$ (both endpoints included)? [3 points]

(7) The second frog starts at $x = 0$ and jumps $i+1$ steps to the right just after $t = i$, so that at times $t = 0, 1, 2, 3, \ldots$ this frog is at positions $x = 0, 1, 3, 6, \ldots$ respectively. How many numbers of the form $7n+1$ (with n an integer) does the frog visit from $t = 0$ to $t = 99$ (both endpoints included)? [3 points]

Solution: (6) For $t = 0, 1, 2, 3, 4, 5, \ldots$ the positions mod 7 are 1,2,4,1,2,4,... by repeatedly multiplying by 2 modulo 7. So positions 1 mod 7 are occupied at $t =$ every multiple of 3, starting at 0. There are 34 multiples of 3 from 0 to 99 including both endpoints.

(7) Here we repeatedly add the sequence $1, 2, 3, 4, 5, 6, 7 = 0$ mod 7 to previous position. For $t = 0, 1, 2, 3, 4, 5, 6, \ldots$ we get the repeating pattern $0, 1, 3, 6, 3, 1, 0$ of positions mod 7. Thus a 1 mod 7 position occurs precisely for $t = 1$ mod 7 and $t = 5$ mod 7. So till $98 = 14 \times 7$ we get $14 \times 2 = 28$ such occurrences. We also get an occurrence at 99, which is 1 mod 7. So the answer is 29.

Let $O = (0,0,0)$, $P = (19,5,2024)$ and $Q = (x,y,z)$ be points in 3-dimensional space where Q is an unknown point.

Consider vector $\mathbf{u} = \overrightarrow{OP} = 19\,\hat{i} + 5\,\hat{j} + 2024\,\hat{k}$ and unknown vector $\mathbf{v} = \overrightarrow{OQ} = x\,\hat{i} + y\,\hat{j} + z\,\hat{k}$.

Instruction: for each of the sets below choose the correct option describing it and enter the number of that option. E.g., if you think a given set is a line, enter **3** as your answer with no full stop or any other punctuation.

Questions

(8) $\{Q \mid \mathbf{u}\cdot\mathbf{v} = 2024\}$. [1 point]

(9) $\{Q \mid \mathbf{u}\cdot\mathbf{v} = -2024\sqrt{\mathbf{v}\cdot\mathbf{v}}\,\}$. [2 points]

(10) $\{Q \mid \mathbf{u}\cdot\mathbf{v} = 2024\,(\mathbf{v}\cdot\mathbf{v})\}$. [2 points]

Options

1. The empty set
2. A singleton set
3. A line
4. A pair of lines
5. A circle
6. A plane perpendicular to **u**
7. A plane parallel to **u**
8. An infinite cone
9. A finite cone
10. A sphere
11. None of the above

Solution: (8) A plane perpendicular to **u**.

(9) $\mathbf{u}\cdot\mathbf{v} = \sqrt{\mathbf{u}\cdot\mathbf{u}}\sqrt{\mathbf{v}\cdot\mathbf{v}}\cos(\theta)$ where θ is the angle between **u** and **v** (between 0 and 180°). So the given condition for nonzero **v** is $\cos(\theta) = \frac{-2024}{\sqrt{\mathbf{u}\cdot\mathbf{u}}}$, which is a constant between -1 and 0 as $\sqrt{\mathbf{u}\cdot\mathbf{u}} > 2024$. So the given set consists of endpoints of all vectors making a fixed obtuse angle with **u**, along with the origin. This is an infinite cone.

(10) This is the set of points satisfying $19x + 5y + 24z = 2024(x^2 + y^2 + z^2)$. Taking LHS to RHS and completing three squares, we get the equation of a sphere passing through the origin.

An integer d is called a factor of an integer n if there is an integer q such that $n = dq$. In particular the set of factors of n contains n and contains 1. You are given that $2024 = 8 \times 11 \times 23$.

Questions

(11) Write the number of *even positive integers* that are factors of 2024^2. [2 points]

(12) Write the number of ordered pairs (a, b) of *positive* integers such that $a^2 - b^2 = 2024^2$. If there are infinitely many such pairs, write the word **infinite** as your answer. [3 points]

Solution: (11) Answer: 54. As $2024^2 = 2^6 \times 11^2 \times 23^2$, the number of factors is $7 \times 3 \times 3 = 63$. Of these $6 \times 3 \times 3 = 54$ are even as the power of 2 in an even factor cannot be 0.

(12) Answer: 22. Observe that $2024^2 = (a+b)(a-b)$ is a factorization with unequal factors of the same parity which must be even. Conversely, given a factorization dq of 2024^2 into even unequal factors, we get a unique solution for a, b by solving $a + b =$ the larger factor and $a - b =$ the smaller factor. The number of factors d such that both d and $\frac{2024}{d}$ are even is $5 \times 3 \times 3 = 45$ as the power of 2 in d cannot be 0 or 6. Of these the equal factors case $d = q = 2024$ should be discarded (as that gives $b = 0$). The other 44 values of d break into 22 pairs, giving 22 solutions.

A good path is a sequence of points in the XY plane such that in each step exactly one of the coordinates increases by 1 and the other stays the same. E.g.,

$$(0,0),(1,0),(2,0),(2,1),(3,1),(3,2),(3,3)$$

is good path from the origin to (3,3). It is a fact that there are exactly 924 good paths from the origin to (6,6).

Questions

(13) Find the number of good paths from $(0,0)$ to $(6,6)$ that pass through both the points $(1,4)$ and $(2,3)$. [1 point]

(14) Find the number of good paths from $(0,0)$ to $(6,6)$ that pass through both the points $(1,2)$ and $(3,4)$. [2 points]

(15) Find the number of good paths from $(0,0)$ to $(6,6)$ such that *neither* of the two points $(1,2)$ and $(3,4)$ occurs on the path, i.e., the path must *miss both* of the points $(1,2)$ and $(3,4)$. [3 points]

Solution: (13) 0. Going to one of $(1,4)$ and $(2,3)$ precludes going to the other as for that to happen one coordinate would have to decrease and that is not possible in a good path.

(14) $\binom{2+1}{1} \times \binom{3+4-(1+2)}{3-1} \times \binom{6+6-(3+4)}{6-3} = \binom{3}{1} \times \binom{4}{2} \times \binom{5}{3} = 3 \times 6 \times 10 = 180.$

(15) We use basic set theory.

Number of paths through $(1,2)$ is $\binom{3}{1} \times \binom{9}{5} = 3 \times 126 = 378.$

Number of paths through $(3,4)$ is $\binom{7}{3} \times \binom{5}{3} = 35 \times 10 = 350.$

Number of paths through $(1,2)$ or $(3,4)$ is $350 + 378 - 180 = 548.$

Number of paths missing both $(1,2)$ and $(3,4)$ is $924 - 548 = 376.$

Suppose f is a function whose domain is X and codomain is Y. It is given that $|X| > 1$ and $|Y| > 1$. No other information is known about X, Y and f. **Instruction:** for each question below *write the number of a single correct option* for the given statement S.

Questions [1 point each]

(16) S = "For each x in X, there exists y in Y such that $f(x) = y$."

(17) S = "For each y in Y, there exists x in X such that $f(x) = y$."

(18) S = "There exists a unique x in X such that for each y in Y it is true that $f(x) = y$."

(19) S = "There exists a unique y in Y such that for each x in X it is true that $f(x) = y$."

Options (with each question's number written next to its matching option)

1. S is always true. (16)
2. S is always false. (18)
3. S is true if and only if f is one-to-one.
4. If S is true then f is one-to-one but the converse is false.
5. If f is one-to-one then S is true but the converse is false.
6. S is true if and only if f is onto. (17)
7. If S is true then f is onto but the converse is false.
8. If f is onto then S is true but the converse is false.
9. S is true if and only if f is a constant function. (19)
10. If S is true then f is a constant function but the converse is false.
11. If f is a constant function then S is true but the converse is false.
12. None of the above.

Solution: This is a matter of careful reading and interpretation of precise language.

Suppose a differentiable function f from $\mathbb{R}$ to $\mathbb{R}$ has a local maximum at $(a, f(a))$ (This means there are numbers m and M such that (i) $m < a < M$ and (ii) $f(a) \geq f(x)$ for any $x \in [m, M]$.) The proof of a standard result is sketched below. Complete it as instructed.

Proof: For sufficiently ___1___ $h > 0$, it is given that $f(a + h)$ ___2______3___.

Therefore for such h the quantity ___4___ must be ___5______6___.

By taking the limit of this quantity as $h \to 0$ from the right, we get that ___7___ must be ___8______9___.

A parallel argument for suitable negative values of h gives that ___10___ must be ___11______12___.

Combining both conclusions gives the desired result: ___13______14______15___. Note that the mentioned limits exist because ___16___.

Questions

(20) Write a sequence of 9 letters indicating the correct options to fill in the numbered blanks 1 to 9. Do not use any spaces, full stop or other punctuation. E.g., **ABACDIJKB** is in the correct format. [3 points]

(21) Write a sequence of 7 letters indicating the correct options to fill in the numbered blanks 10 to 16. [2 points]

Options

A. small	B. large
C. $\geq$	D. $>$
E. $\leq$	F. $<$
G. $=$	H. $\neq$
I. 0	J. $f(a)$
K. $\frac{f(a+h)-f(a)}{h}$	L. $f'(a)$
M. f is differentiable	N. f is continuous

Solution: For sufficiently small $h > 0$, it is given that $f(a + h) \leq f(a)$.

Therefore for such h the quantity $\frac{f(a+h)-f(a)}{h}$ must be ≤ 0.

By taking the limit of this quantity as $h \to 0$ from the right, we get that $f'(a)$ must be ≤ 0.

A parallel argument for suitable negative values of h gives that $f'(a)$ must be ≥ 0.

Combining both conclusions gives the desired result: $f'(a) = 0$.

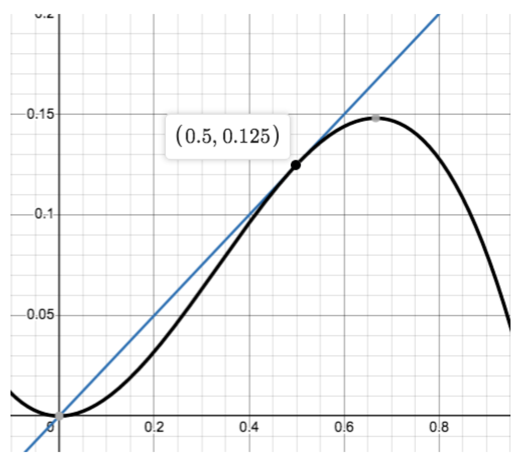
Note that the mentioned limits exist because f is differentiable.

Solutions to Part B Problems

B1. [10 points] (a) Draw a qualitatively accurate sketch of the unique bounded region R in the *first quadrant* that has *maximum possible* finite area with boundary described as follows. R is bounded below by the graph of $y = x^2 - x^3$, bounded above by the graph of an equation of the form $y = kx$ (where k is some constant), and R is *entirely enclosed by the two given graphs*, i.e., the boundary of the region R must be *a subset* of the union of the given two graphs (so R does not have any points on its boundary that are not on these two graphs). Clearly mark the relevant point(s) on the boundary where the two given graphs meet and **write the coordinates of every such point**.

(b) Consider the solid obtained by rotating the above region R around Y-axis. Show how to find the volume of this solid by doing the following: Carefully set up the calculation with justification. Do enough work with the resulting expression to reach a stage where the final numerical answer can be found mechanically by using standard symbolic formulas of algebra and/or calculus and substituting known values in them. Do not carry out the mechanical work to get the final numerical answer.

Solution to B1: (a) Increasing k from 0 rotates the line counterclockwise and increases the desired area until the line stops intersecting the graph of $y = x^2 - x^3$ in the first quadrant. So bounded R with maximum possible area is obtained when the line is tangent to the graph of $y = x^2 - x^3$ in the first quadrant. See the picture. At the point of tangency the slope is $k = 2x - 3x^2$ and the y coordinate is $kx = x^2 - x^3$. Solving gives the point of tangency $= (0.5, 0.125)$ and slope $k = 0.25$.



(b) This can be done by either the "washer" method or the "shell" method. The washer method divides R into horizontal slices of "tiny height dy" and integrates along Y-axis the volumes of the resulting thin washers of revolution around Y-axis. This gives

$$\text{desired volume} = \int_0^{0.125} \pi x^2 dy - (\text{volume of a cone of radius 0.5 and height 0.125}),$$

where x and y in the integral are related by $y = x^2 - x^3$. So $dy = (2x - 3x^2)dx$. After substitution the integral becomes $\pi \int_0^{0.5} x^2(2x - 3x^2)dx$. Cone volume $= \frac{\pi}{3}(0.5)^2(0.125)$ (or $\int_0^{0.125} \pi(4y)^2 dy$ or $\int_0^{0.5} (0.25)\pi x^2 dx$). Overall answer (not asked for) is $\frac{\pi}{480}$.

The shell method divides R into vertical slices of "tiny width dx" and integrates along X-axis the volumes of resulting shells of revolution around Y-axis. This gives the following integral and the same numerical answer.

$$\text{desired volume} \ = \int_0^{0.5} 2\pi x(y_1 - y_2)dx = 2\pi \int_0^{0.5} x(0.25x - (x^2 - x^3))dx.$$

B2. [15 points] (a) Find the domain of the function $g(x)$ defined by the following formula.

$$g(x) = \int_{10}^{x} \log_{10}\left(\log_{10}(t^2 - 1000t + 10^{1000})\right)dt.$$

Calculate the quantities below. You may give an approximate answer where necessary, but clearly state which answers are exact and which are approximations.

(b) $g(1000)$.

(c) x in [10,1000] where $g(x)$ has the maximum possible slope.

(d) x in [10,1000] where $g(x)$ has the least possible slope.

(e) $\lim_{x\to\infty} \frac{\ln(x)}{g(x)}$ if it exists.

Solution to B2: (a) The parabola $t^2 - 1000t$ takes lowest value -250000 (at $t = 500$), which is absolutely dwarfed by 10^{1000} = (1 followed by 1000 zeros). So $\log_{10}(t^2 - 1000t + 10^{1000})$ is defined for any t and is always greater than 999. So $\log_{10}\left(\log_{10}(t^2 - 1000t + 10^{1000})\right)$ is defined for any t. The integrand is a continuous function, so $g(x)$ is defined for all real x.

(b) The log function increases very slowly, so log log increases extremely slowly. In the interval [10,1000], the values of $t^2 - 1000t$ are in $[-250000, 0]$ and have no practical effect on the order of magnitude of the far larger 10^{1000}. So throughout the interval of integration [10,1000], $\log_{10}(t^2 - 1000t + 10^{1000})$ is extremely close to 1000 and $\log_{10}$ of that is extremely close to 3. So we are essentially integrating the constant function 3. The answer is $\approx 3 \times (1000 - 10) = 2970$. This answer is correct to a very high degree of accuracy and a rigorous calculation bounding the error can be given, but for this exam the above simple qualitative answer was enough. (Idea taken from "Lucky numbers" in *Surely You're Joking, Mr. Feynman!* "People started giving me problems they thought were difficult such as integrating a function ... which hardly changed over the range they gave me". Also worth reading are Feynman's thoughts on education near the end of "O Americana, Outra Vez!", starting with "In regard to education in Brazil, I had a very interesting experience ...")

(c,d) Slope of g is $g'(x)$, which is $\log_{10}\left(\log_{10}(x^2 - 1000x + 10^{1000})\right)$ by the fundamental theorem of calculus. As $\log_{10}$ is an increasing function, we may simply discard the $\log_{10}\log_{10}$ in the front and find extrema of $x^2 - 1000x + 10^{1000}$ over the interval [10,1000]. This parabola takes minimum value at its vertex which is at $x = 500$. By symmetry of the parabola, the maximum value occurs at the endpoint farther from the vertex of the parabola, namely at $x = 1000$. (Note that while these are theoretically exact answers, $g'(x)$ sees extremely tiny variation thanks to flattening due to log log.)

(e) Both $\ln(x)$ and $g(x)$ go to ∞ as $x \to \infty$ but the denominator dominates. (Why?) Proof: By L'Hôpital's rule the required limit can be checked via $\lim_{x\to\infty} \frac{1/x}{g'(x)}$, which is 0 as $1/x \to 0$ and $g'(x) = \log_{10}\left(\log_{10}(x^2 - 1000x + 10^{1000})\right) \to \infty$. (Note: calculating $\lim_{x\to\infty} \frac{x\ln(x)}{g(x)}$ is a bit more interesting. What is the answer?)

B3. [15 points] (a) For non-negative numbers a, b, c and any positive real number r prove the following inequality *and* state precisely when equality is achieved.

$$a^r(a-b)(a-c) + b^r(b-a)(b-c) + c^r(c-a)(c-b) \geq 0$$

Hint: Assuming $a \geq b \geq c$ do algebra with just the first two terms. What about the third term? What if the assumption is not true?

(b) As a special case obtain an inequality with $a^4 + b^4 + c^4 + abc(a+b+c)$ on one side.

(c) Show that if $abc = 1$ for positive numbers a, b, c, then

$$a^4 + b^4 + c^4 + a^3 + b^3 + c^3 + a + b + c \geq \frac{a^2+b^2}{c} + \frac{b^2+c^2}{a} + \frac{c^2+a^2}{b} + 3.$$

Solution to B3: (a) This is known as Schur's inequality. Here is the standard argument.

We may assume $a \geq b \geq c$ because the inequality is completely symmetric in a, b, c, i.e., any interchange of two letters gives the same inequality (check this), and therefore so does any permutation of a, b, c. (Alternatively, if $a \geq b \geq c$ is not true, then arrange a, b, c in a decreasing order and use similar reasoning as below by separating the term with three occurrences a least one among of a, b, c. Check this in two cases: $b \geq a \geq c$ and $b \geq c \geq a$.)

So let $a \geq b \geq c$. First note $a^r(a-b)(a-c) + b^r(b-a)(b-c) = (a-b)\big(a^r(a-c) - b^r(b-c)\big)$. Now $a^r \geq b^r \geq 0$ and $a - c \geq b - c \geq 0$. Multiplying the left sides and the middle terms we get $a^r(a-c) \geq b^r(b-c)$, so $a^r(a-c) - b^r(b-c) \geq 0$. Now multiplying by $(a-b)$ gives

$$(a-b)\big(a^r(a-c) - b^r(b-c)\big) = a^r(a-b)(a-c) + b^r(b-a)(b-c) \geq 0.$$

Adding $c^r(c-a)(c-b) \geq 0$ (which is true when $a \geq b \geq c$), we get the desired inequality.

To have equality, trace all used inequalities to deduce the necessary and sufficient condition

$$\Big((a=b) \text{ OR } \big((a^r = b^r) \text{ and } (a-c = b-c)\big)\Big) \text{ AND } c^r(c-a)(c-b) = 0.$$

This happens exactly when $(a = b)$ AND ($c = 0$ or $c = a$ or $c = b$). Using symmetry we get equality precisely when $a = b = c$ or when one of a, b, c is zero and the other two are equal.

(b) Substitute $r = 2$ and do algebra to get

$$a^4 + b^4 + c^4 + abc(a + b + c) \geq ab(a^2 + b^2) + bc(b^2 + c^2) + ca(c^2 + a^2).$$

(c) (Problem by Dan Sitaru, this solution by Imad Zak, source www.cut-the-knot.org) Use $abc = 1$ in part (b). Cancel abc from the left and divide by abc on the right to get

$$a^4 + b^4 + c^4 + a + b + c \geq \frac{a^2 + b^2}{c} + \frac{b^2 + c^2}{a} + \frac{c^2 + a^2}{b}.$$

We will be done by showing $a^3 + b^3 + c^3 \geq 3 = 3abc$, which is true by $AM \geq GM$.

B4. **[10 points]** Find all solutions of the following equation where it is required that x, k, y, n are *positive integers* with the exponents k and n both > 1.

$$20x^k + 24y^n = 2024.$$

Solution to B4: First solve $20a + 24b = 2024$ in integers. For any solution (a, b), the pair $(a - 6k, b + 5k)$ is also a solution for any integer k.

Moreover any solution (a', b') is obtainable this way from a single known solution (a, b). Proof: we have $20(a - a') + 24(b - b') = 0$, so $5(a - a') + 6(b - b') = 0$. As $gcd(5, 6) = 1$, we see that 6 must be a factor of $a - a'$ (and 5 must be a factor of $b - b'$). Letting $a - a' = 6k$, the equation $5(a - a') + 6(b - b') = 0$ implies $b - b' = -5k$, so $(a', b') = (a - 6k, b + 5k)$ as claimed.

Observe that $a = 100, b = 1$ is a solution. As subtracting 5 from $b = 1$ gives a negative number, we can take only positive values of k to generate the remaining positive integer solutions. One gets 17 solutions in positive integers, the last one being $a = 100 - 16 \times 6 = 4$ paired with $b = 1 + 16 \times 5 = 81$. Listing all 17 solutions shows that (100,1) and (4,81) are the only ones in which a, b are both perfect powers: $100 = 10^2, 1 = 1^{\text{any } n}, 4 = 2^2, 81 = 3^4 = 9^2$.

B5. **[15 points]** (a) Find all complex solutions of $z^6 = z + \overline{z}$.

(b) For an integer $n > 1$, how many complex solutions does $z^n = z + \overline{z}$ have?

Solution to B5: (b) $z = 0$ is a solution. From now on let $z \neq 0$. Suppose $z = re^{\theta i}$. So $z^n = r^n e^{n\theta i} = z + \overline{z} = 2r\cos\theta$, which is real. So the equation is true if and only if

$$\left(e^{n\theta i} = 1 \text{ and } r^n = 2r\cos\theta\right) \text{ OR } \left(e^{n\theta i} = -1 \text{ and } r^n = -2r\cos\theta\right).$$

Case 1. In the first case $e^{\theta i}$ must be an n^{th} root of 1 and its real part $\cos\theta$ must be positive because $\cos\theta = \frac{r^{n-1}}{2}$ is positive. Conversely for any one of such values of θ, the preceding equation uniquely determines r, thus giving a solution of the given equation. The number of such θ can be calculated by cases mod 4.

- For $n = 4k$ we get $2k - 1$ valid values of θ as two of the n^{th} roots of 1 are imaginary and half of the remaining $4k - 2$ have positive real part.

- For $n = 4k + 1, 4k + 2, 4k + 3$, we get $2k + 1$ valid values of θ, namely $\theta = 0$ and k values each in the first and fourth quadrant because $\frac{2k\pi}{n} < \frac{\pi}{2} < \frac{2(k+1)\pi}{n}$.

Case 2. Otherwise $e^{\theta i}$ must be an n^{th} root of -1 and its real part $\cos\theta$ must be negative because $\cos\theta = -\frac{r^{n-1}}{2}$ is negative. Conversely for any one of such values of θ, the preceding equation uniquely determines r, giving a solution of the given equation. The number of valid θ can again be calculated by cases mod 4. Let $S =$ (set of angles of n^{th} roots of -1) $= \{\frac{(2j+1)\pi}{n} \mid j = 1, \ldots, n\}$, i.e., the set of angles obtained by rotating each n^{th} root of 1 halfway to the next root. We need to find angles θ in S with $\frac{\pi}{2} < \theta < \frac{3\pi}{2}$.

- For $n = 4k$ we get $2k$ valid values of θ as exactly half the angles in S are on each side of the Y axis.

- For $n = 4k + 2$ we get $2k$ valid values of θ as two of the angles in S are on the Y axis.

For odd n, note that if z is a root, then so is $-z$. So for odd n, (number of roots in case 2) $=$ (number of roots in case 1), giving $2k + 1$ case 2 roots for $n = 4k + 1, 4k + 3$. We can also get this by continuing brute force analysis.

- For $n = 4k + 1$ we get $2k + 1$ valid values of θ, namely $\theta = \frac{(2j+1)\pi}{4k+1}$ for $j = k, \ldots, 3k$.

- For $n = 4k+3$, we get $2k+1$ valid values of θ, namely $\theta = \frac{(2j+1)\pi}{4k+3}$ for $j = k+1, \ldots, 3k+1$.

Adding up the number of solutions of two types and counting $z = 0$, we get the following result. The total number of solutions of the equation $z^n = z + \overline{z}$ (for integer $n > 1$) is $n + 2$ when $n = 1 \bmod 4$ and n otherwise. One can check this for several n on a numerical solver, e.g., at https://www.wolframalpha.com.

Aside: if z is a root, then so is $\overline{z}$. (Why?) This symmetry does not seem helpful to reduce work, but it does explain some of the numerology, e.g., case 1 always gives an odd number of roots (because $\theta = 0$ always gives a root) while case 2 gives odd or even number of roots depending on whether $\theta = \pi$ gives a root or not. But $\theta = \pi$ means z is a negative real number and such a root exists exactly when n is odd.

(a) Applying the general analysis above gives five nonzero solutions: three with positive real part (case 1 above), and two with negative real part (case 2 above). Working out the details gives the following list of six solutions.

$$0 \qquad \sqrt[5]{2} \qquad e^{\pm\frac{\pi i}{3}} = \frac{1 \pm \sqrt{3}i}{2} \qquad \sqrt[10]{3}\, e^{\pm\frac{5\pi i}{6}} = \sqrt[10]{3}\left(\frac{-\sqrt{3} \pm i}{2}\right)$$

B6. **[15 points]** A list of k elements, possibly with repeats, is given. The goal is to find if there is a *majority element*. This is defined to be an element x such that the number of times x occurs in the list is *strictly* greater than $\frac{k}{2}$. (Note that there need not be such an element, but if it is there, it must be unique.) A celebrated efficient way to do this task uses two functions f and m with domain $\{1, 2, \ldots, k\}$. The functions are defined inductively as follows.

Define $f(1)$ = first element of the list, $m(1) = 1$.

Assuming f and m are defined for all inputs from 1 to i, define

$$f(i+1) = \begin{cases} f(i) & \text{if } m(i) > 0 \\ (i+1)^{th} \text{ element of the list} & \text{if } m(i) = 0 \end{cases}$$

$$m(i+1) = \begin{cases} m(i) - 1 & \text{if } m(i) > 0 \text{ and } (i+1)^{th} \text{ element of the list is other than } f(i) \\ m(i) + 1 & \text{otherwise} \end{cases}$$

(a) For the example of length 15 given below, write a sequence of 15 letters showing the values of $f(i)$ and a sequence of 15 numbers directly underneath showing the values of $m(i)$ for $i = 1, 2, \ldots, 15$.

a a b a b c c b b b a b b c b

(b) Prove that in general the list can be divided into two disjoint parts A and B such that

- Part A contains $m(k)$ elements of the list each of which is $f(k)$.
- Part B contains the remaining $k - m(k)$ elements of the list and B can be written as disjoint union of pairs such that the two elements in each pair are distinct.

(c) If there is a majority element, show that it must be $f(k)$. You may assume part (b) even if you did not do it.

(d) Assuming $f(k)$ is the majority element, answer the following two questions. Show by examples that the number of occurrences of $f(k)$ in the list does not determine the value of $m(k)$. Can the value of $m(k)$ be anything in $\{0, \ldots, k\}$? Find constraints if any on the possible values of $m(k)$.

(e) Now assume instead that an element occurs exactly $\frac{k}{2}$ times in the list. Is it necessary that $f(k)$ is such an element?

Solution to B6: This is the well known Boyer-Moore majority vote algorithm. Note some features of this clever procedure. To calculate $f(i+1)$ and $m(i+1)$ one needs only $f(i)$ and $m(i)$. No need to "remember" previous values! The inductive nature means a single pass of the list gives a unique candidate $f(k)$ for the majority element. (To verify that $f(k)$ indeed has majority, one pass won't do. One can count occurrences of $f(k)$ in a second pass.)

(a) One follows the procedure to get the following table of values of f and m for $i = 1, \ldots, 15$.

i	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
list	a	a	b	a	b	c	c	b	b	b	a	b	b	c	b
$f(i)$	a	a	a	a	a	a	c	c	b	b	b	b	b	b	b
$m(i)$	1	2	1	2	1	0	1	0	1	2	1	2	3	2	3

(b) Use induction. Initially part A contains $f(1)$ and part B is empty, so the claim is true. Assuming it is true at stage i we make three cases at stage $i + 1$.

- If $m(i) = 0$, by induction part A is empty. We just put the $(i + 1)^{th}$ element in part A and leave part B untouched.
- If $m(i) > 0$ and the incoming element is the same as $f(i)$, again put the $(i+1)^{th}$ element in A and leave B untouched. This case can also be combined with the previous one.
- If $m(i) > 0$ and the incoming element is different from $f(i)$, we remove one copy of $f(i)$ from part A (which is there by induction as $m(i) > 0$), pair it with the incoming element and place this pair of distinct elements in part B.

In all cases the claim stays true by using induction and the definition of $f(i+1)$ and $m(i+1)$.

(c) By the claim proved in (b), the number of occurrences of any element in part B can be at most half of the size of B. So the majority element has to occur in part A (otherwise it cannot occur more that $\frac{k}{2}$ times). But all elements in part A are $f(k)$, so $f(k)$ must be the majority element.

(e) No. The list *cabc* of length 4 has c occurring twice but $f(4) = b$.

(d) The lists *abccc* and *aaccc* both have the majority element c occurring thrice but different $m(5)$, namely 3 and 1 respectively.

Some constraints on possible values of $m(k)$ are as follows. First, the answer to (b) shows that if there is a majority element, it has to occur in part A, so $m(k) =$ size of part A cannot be 0. Second, by part (b) we also have that $m(k) \leq$ number of occurrences of $f(k)$. Finally as $m(1) = 1$ and since the value of m changes by 1 at each step, $m(k)$ must have the same parity as k. The second and third properties are true for all lists regardless of whether there is a majority element.

Conversely, given the list length k, the number of occurrences n of the majority element (say x) and any nonzero number $p \leq n$ with p of the same parity as k, one can easily construct a list with $m(k) = p$ as follows. Put p occurrences of x at the end of the list and ensure that every element in an even numbered slot among the first $k - p$ elements differs from the preceding element. This means that values of $m(i)$ follow the pattern $1, 0, 1, 0, \ldots, 1, 0$ until index $i = k - p$, which is even. (This simpleminded scheme can realize a desired value p of $m(k)$ of the correct parity regardless of existence of the majority element if one is given an element x with pre-specified frequency $\geq p$. If there are more constraints, e.g., if frequencies of *all* list elements are pre-specified, then possible values of m can have more restrictions.)