“Basics of WQO theory, with some applications in computer science”

aka “WQOs for dummies”

Ph. Schnoebelen

LSV, CNRS, Cachan

CMI Silver Jubilee Lecture
Chennai, Feb. 23rd, 2015

INTRODUCTION

Well-quasi-orderings, or WQOs, are a generalization of well-orderings

They are to partial orderings what well-orderings are to linear orderings

The properties of WQOs have proved very useful in logic, combinatorics, graph theory, and computer science

WQOs, or their properties, have been rediscovered many times. It is certainly worthwhile to know their basic properties

Kříz & Thomas 1990 list four reasons to be interested in WQOs:

1.
2. excluded minor theorems
3. surprising algorithmic consequences
4. applications in logic and proof theory

INTRODUCTION

Well-quasi-orderings, or WQOs, are a generalization of well-orderings

They are to partial orderings what well-orderings are to linear orderings

The properties of WQOs have proved very useful in logic, combinatorics, graph theory, and computer science

WQOs, or their properties, have been rediscovered many times. It is certainly worthwhile to know their basic properties

Kříz & Thomas 1990 list four reasons to be interested in WQOs:

1. ?? (guess)
2. excluded minor theorems
3. surprising algorithmic consequences
4. applications in logic and proof theory

INTRODUCTION

Well-quasi-orderings, or WQOs, are a generalization of well-orderings

They are to partial orderings what well-orderings are to linear orderings

The properties of WQOs have proved very useful in logic, combinatorics, graph theory, and computer science

WQOs, or their properties, have been rediscovered many times. It is certainly worthwhile to know their basic properties

Kříz & Thomas 1990 list four reasons to be interested in WQOs:

1. it is fun!!!
2. excluded minor theorems
3. surprising algorithmic consequences
4. applications in logic and proof theory

OUTLINE

1. Basics and examples
2. Building more WQOs
3. From WQOs to BQOs
4. A hint of Graph Minor Theory

Basics and examples

(RECALLS) ORDERED SETS

Def. A non-empty $(X, \leqslant)$ is a quasi-ordering (QO) $\overset{\text{def}}{\Leftrightarrow}$ $\leqslant$ is a reflexive and transitive relation

- like partial ordering (PO) but not requiring antisymmetry
- QO technically simpler but essentially equivalent to PO

Examples.

- $(\mathbb{N}, \leqslant)$, also $(\mathbb{R}, \leqslant)$, $(\mathbb{N} \cup \{\omega\}, \leqslant)$, ...
- divisibility: $(\mathbb{Z}, \_ \mid \_)$ where $x \mid y \overset{\text{def}}{\Leftrightarrow} \exists a : a.x = y$
  also Gaussian integers: $(\mathbb{Z}[i], \_ \mid \_)$
- tuples: $(\mathbb{N}^3, \leqslant_\times)$, where $(0,1,2) <_\times (10,1,5)$ and $(1,2,3) \#_\times (3,1,2)$

Notation. $x \equiv y \overset{\text{def}}{\Leftrightarrow} x \leqslant y \leqslant x$

$x < y \overset{\text{def}}{\Leftrightarrow} x \leqslant y \land y \not\leqslant x$ $\quad$ $x \# y \overset{\text{def}}{\Leftrightarrow} x \not\leqslant y \land y \not\leqslant x$

(Recalls) Ordered Sets

Def. A non-empty $(X, \leqslant)$ is a quasi-ordering (QO) $\overset{\text{def}}{\Leftrightarrow}$ $\leqslant$ is a reflexive and transitive relation

- like partial ordering (PO) but not requiring antisymmetry
- QO technically simpler but essentially equivalent to PO

Examples.

- $(\mathbb{N}, \leqslant)$, also $(\mathbb{R}, \leqslant)$, $(\mathbb{N} \cup \{\omega\}, \leqslant)$, ...
- divisibility: $(\mathbb{Z}, \_ \mid \_)$ where $x \mid y \overset{\text{def}}{\Leftrightarrow} \exists a : a.x = y$
  also Gaussian integers: $(\mathbb{Z}[i], \_ \mid \_)$
- tuples: $(\mathbb{N}^3, \leqslant_\times)$, where $(0,1,2) <_\times (10,1,5)$ and $(1,2,3) \#_\times (3,1,2)$

Notation. $x \equiv y \overset{\text{def}}{\Leftrightarrow} x \leqslant y \leqslant x$

$x < y \overset{\text{def}}{\Leftrightarrow} x \leqslant y \wedge y \not\leqslant x$ $\qquad$ $x \# y \overset{\text{def}}{\Leftrightarrow} x \not\leqslant y \wedge y \not\leqslant x$

(Recalls) Ordered Sets

Def. A non-empty $(X, \leqslant)$ is a quasi-ordering (QO) $\overset{\text{def}}{\Leftrightarrow}$ $\leqslant$ is a reflexive and transitive relation

- like partial ordering (PO) but not requiring antisymmetry
- QO technically simpler but essentially equivalent to PO

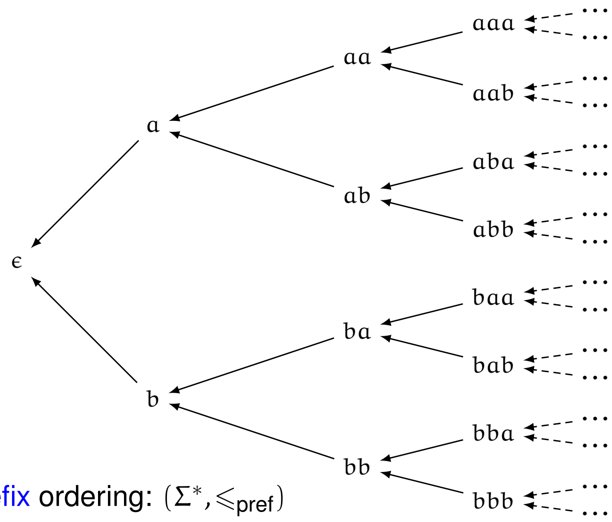
Examples.

- $(\mathbb{N}, \leqslant)$, also $(\mathbb{R}, \leqslant)$, $(\mathbb{N} \cup \{\omega\}, \leqslant)$, ...
- divisibility: $(\mathbb{Z}, \_ \mid \_)$ where $x \mid y \overset{\text{def}}{\Leftrightarrow} \exists a : a.x = y$
  also Gaussian integers: $(\mathbb{Z}[i], \_ \mid \_)$
- tuples: $(\mathbb{N}^3, \leqslant_\times)$, where $(0,1,2) <_\times (10,1,5)$ and $(1,2,3) \#_\times (3,1,2)$

Notation. $x \equiv y \overset{\text{def}}{\Leftrightarrow} x \leqslant y \leqslant x$

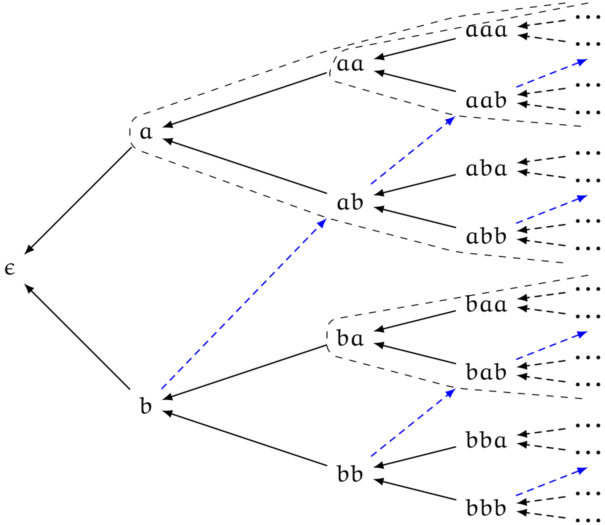
$x < y \overset{\text{def}}{\Leftrightarrow} x \leqslant y \wedge y \not\leqslant x$ $\qquad$ $x \# y \overset{\text{def}}{\Leftrightarrow} x \not\leqslant y \wedge y \not\leqslant x$

SIMPLE ORDERINGS ON WORDS – 1



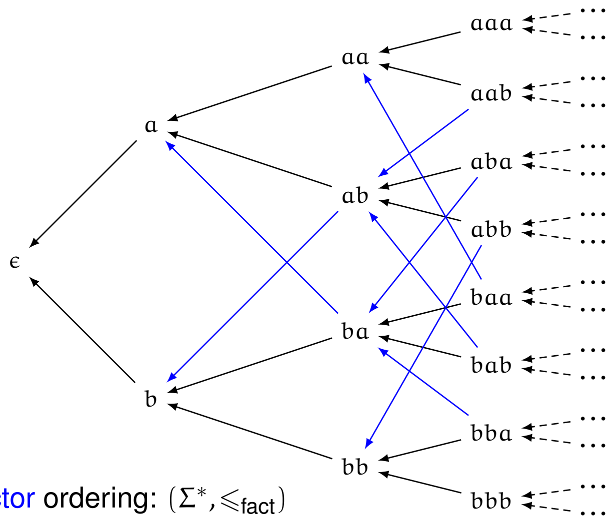
Prefix ordering: $(\Sigma^*, \leqslant_{\mathsf{pref}})$

SIMPLE ORDERINGS ON WORDS – 2



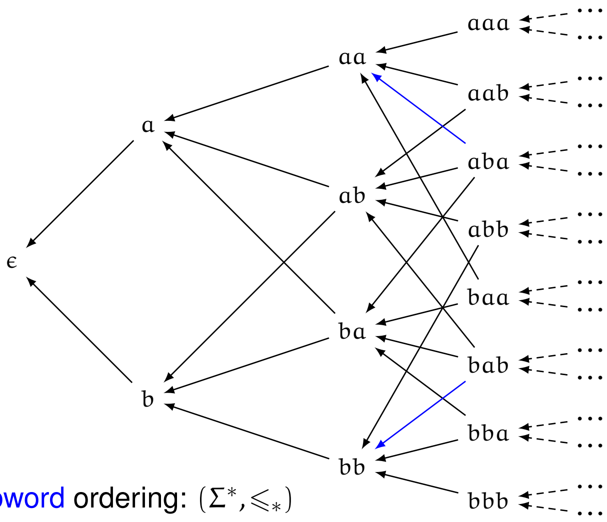
Lexicographic ordering: $(\Sigma^*, \leqslant_{\text{lex}})$

SIMPLE ORDERINGS ON WORDS – 3



Factor ordering: $(\Sigma^*, \leqslant_{\text{fact}})$

SIMPLE ORDERINGS ON WORDS – 4



Subword ordering: $(\Sigma^*, \leqslant_*)$

(Recalls) Ordered sets

Def. $(X, \leqslant)$ is linear if for any $x, y \in X$ either $x \leqslant y$ or $y \leqslant x$ (I.e., there is no $x \# y$)

Def. $(X, \leqslant)$ is well-founded (WF) if there is no infinite strictly decreasing sequence $x_0 > x_1 > x_2 > \cdots$

	linear?	well-founded?
$\mathbb{N}, \leqslant$		
$\mathbb{Z}, \_ \mid \_$		
$\mathbb{N} \cup \{\omega\}, \leqslant$		
$\mathbb{N}^3, \leqslant_\times$		
$\Sigma^*, \leqslant_{\text{pref}}$		
$\Sigma^*, \leqslant_{\text{lex}}$		
$\Sigma^*, \leqslant_*$		

(Recalls) Ordered sets

Def. $(X, \leqslant)$ is linear if for any $x, y \in X$ either $x \leqslant y$ or $y \leqslant x$ (I.e., there is no $x \# y$)

Def. $(X, \leqslant)$ is well-founded (WF) if there is no infinite strictly decreasing sequence $x_0 > x_1 > x_2 > \cdots$

	linear?	well-founded?
$\mathbb{N}, \leqslant$	✓	
$\mathbb{Z}, \_ \mid \_$	×	
$\mathbb{N} \cup \{\omega\}, \leqslant$	✓	
$\mathbb{N}^3, \leqslant_\times$	×	
$\Sigma^*, \leqslant_{\text{pref}}$	×	
$\Sigma^*, \leqslant_{\text{lex}}$	✓	
$\Sigma^*, \leqslant_*$	×	

(Recalls) Ordered sets

Def. $(X, \leqslant)$ is linear if for any $x, y \in X$ either $x \leqslant y$ or $y \leqslant x$ (I.e., there is no $x \# y$)

Def. $(X, \leqslant)$ is well-founded (WF) if there is no infinite strictly decreasing sequence $x_0 > x_1 > x_2 > \cdots$

	linear?	well-founded?
$\mathbb{N}, \leqslant$	✓	✓
$\mathbb{Z}, \_ \mid \_$	×	✓
$\mathbb{N} \cup \{\omega\}, \leqslant$	✓	✓
$\mathbb{N}^3, \leqslant_\times$	×	✓
$\Sigma^*, \leqslant_{\text{pref}}$	×	✓
$\Sigma^*, \leqslant_{\text{lex}}$	✓	×
$\Sigma^*, \leqslant_*$	×	✓

Well-quasi-ordering (WQO)

Def1. $(X, \leqslant)$ is a WQO $\overset{\text{def}}{\Leftrightarrow}$ any infinite sequence $x_0, x_1, x_2, \ldots$ contains an increasing pair: $x_i \leqslant x_j$ for some $i < j$

Well-quasi-ordering (WQO)

Def1. $(X, \leqslant)$ is a WQO $\overset{\text{def}}{\Leftrightarrow}$ any infinite sequence $x_0, x_1, x_2, \ldots$ contains an increasing pair: $x_i \leqslant x_j$ for some $i < j$

Def2. $(X, \leqslant)$ is a WQO $\overset{\text{def}}{\Leftrightarrow}$ any infinite sequence $x_0, x_1, x_2, \ldots$ contains an infinite increasing subsequence: $x_{n_0} \leqslant x_{n_1} \leqslant x_{n_2} \leqslant \cdots$

WELL-QUASI-ORDERING (WQO)

Def1. $(X,\leqslant)$ is a WQO $\overset{\text{def}}{\Leftrightarrow}$ any infinite sequence $x_0,x_1,x_2,\ldots$ contains an increasing pair: $x_i \leqslant x_j$ for some $i < j$

Def2. $(X,\leqslant)$ is a WQO $\overset{\text{def}}{\Leftrightarrow}$ any infinite sequence $x_0,x_1,x_2,\ldots$ contains an infinite increasing subsequence: $x_{n_0} \leqslant x_{n_1} \leqslant x_{n_2} \leqslant \cdots$

Def3. $(X,\leqslant)$ is a WQO $\overset{\text{def}}{\Leftrightarrow}$ there is no infinite strictly decreasing sequence $x_0 > x_1 > x_2 > \cdots$ —i.e., $(X,\leqslant)$ is well-founded (WF)— and no infinite set $\{x_0,x_1,x_2,\ldots\}$ of mutually incomparable elements $x_i \# x_j$ when $i \neq j$ —we say that $(X,\leqslant)$ has no infinite antichain (FAC)—

WELL-QUASI-ORDERING (WQO)

Def1. $(X,\leqslant)$ is a WQO $\overset{\text{def}}{\Leftrightarrow}$ any infinite sequence $x_0, x_1, x_2, \ldots$ contains an increasing pair: $x_i \leqslant x_j$ for some $i < j$

Def2. $(X,\leqslant)$ is a WQO $\overset{\text{def}}{\Leftrightarrow}$ any infinite sequence $x_0, x_1, x_2, \ldots$ contains an infinite increasing subsequence: $x_{n_0} \leqslant x_{n_1} \leqslant x_{n_2} \leqslant \cdots$

Def3. $(X,\leqslant)$ is a WQO $\overset{\text{def}}{\Leftrightarrow}$ there is no infinite strictly decreasing sequence $x_0 > x_1 > x_2 > \cdots$ —i.e., $(X,\leqslant)$ is well-founded (WF)— and no infinite set $\{x_0, x_1, x_2, \ldots\}$ of mutually incomparable elements $x_i \# x_j$ when $i \neq j$ —we say that $(X,\leqslant)$ has no infinite antichain (FAC)—

Fact. These three definitions are equivalent

Well-quasi-ordering (WQO)

Def1. $(X,\leqslant)$ is a WQO $\overset{\text{def}}{\Leftrightarrow}$ any infinite sequence $x_0,x_1,x_2,\ldots$ contains an increasing pair: $x_i \leqslant x_j$ for some $i < j$

Def2. $(X,\leqslant)$ is a WQO $\overset{\text{def}}{\Leftrightarrow}$ any infinite sequence $x_0,x_1,x_2,\ldots$ contains an infinite increasing subsequence: $x_{n_0} \leqslant x_{n_1} \leqslant x_{n_2} \leqslant \cdots$

Def3. $(X,\leqslant)$ is a WQO $\overset{\text{def}}{\Leftrightarrow}$ there is no infinite strictly decreasing sequence $x_0 > x_1 > x_2 > \cdots$ —i.e., $(X,\leqslant)$ is well-founded (WF)— and no infinite set $\{x_0,x_1,x_2,\ldots\}$ of mutually incomparable elements $x_i \# x_j$ when $i \neq j$ —we say that $(X,\leqslant)$ has no infinite antichain (FAC)—

Fact. These three definitions are equivalent

- Clearly, Def2 $\Rightarrow$ Def1 and Def1 $\Rightarrow$ Def3
  But the reverse implications are non-trivial

- In fact proving Def3 $\Rightarrow$ Def1 or Def1 $\Rightarrow$ Def2 for a specific structure has been a key lemma in many works (both before and after the introduction of the concept of WQOs)

NB. For finite X, it is the Pigeonhole Principle

WELL-QUASI-ORDERING (WQO)

Def1. $(X,\leqslant)$ is a WQO $\overset{\text{def}}{\Leftrightarrow}$ any infinite sequence $x_0, x_1, x_2, \ldots$ contains an increasing pair: $x_i \leqslant x_j$ for some $i < j$

Def2. $(X,\leqslant)$ is a WQO $\overset{\text{def}}{\Leftrightarrow}$ any infinite sequence $x_0, x_1, x_2, \ldots$ contains an infinite increasing subsequence: $x_{n_0} \leqslant x_{n_1} \leqslant x_{n_2} \leqslant \cdots$

Def3. $(X,\leqslant)$ is a WQO $\overset{\text{def}}{\Leftrightarrow}$ there is no infinite strictly decreasing sequence $x_0 > x_1 > x_2 > \cdots$ —i.e., $(X,\leqslant)$ is well-founded (WF)— and no infinite set $\{x_0, x_1, x_2, \ldots\}$ of mutually incomparable elements $x_i \# x_j$ when $i \neq j$ —we say that $(X,\leqslant)$ has no infinite antichain (FAC)—

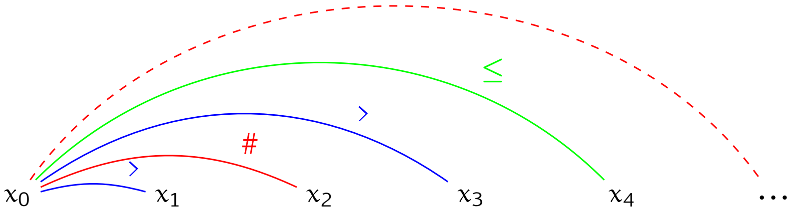
Fact. These three definitions are equivalent

Recall Infinite Ramsey Theorem: "Let X be some countably infinite set and colour the elements of $X^{(n)}$ (the subsets of X of size n) in c different colours. Then there exists some infinite subset M of X s.t. the size n subsets of M all have the same colour"

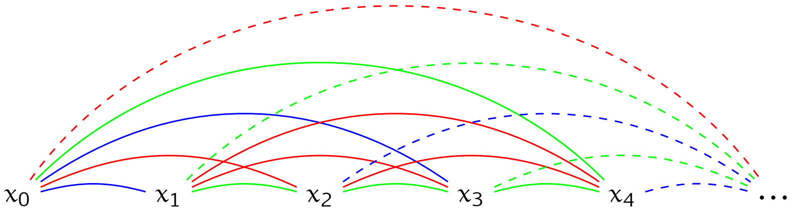
PROVING DEF3 ⇒ DEF2

$$x_0 \qquad x_1 \qquad x_2 \qquad x_3 \qquad x_4 \qquad \cdots$$

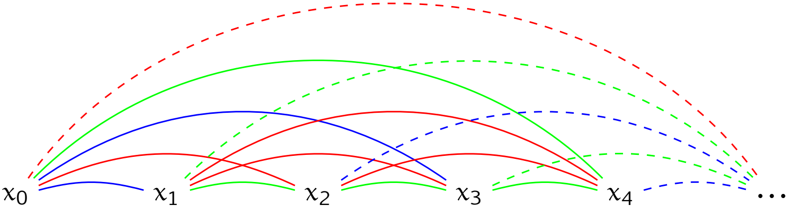
Proving Def3 ⇒ Def2



Proving Def3 ⇒ Def2



Proving Def3 ⇒ Def2



Infinite Ramsey Theorem:

there is an infinite subset $\{x_i\}_{i \in I}$ that is monochromatic

PROVING DEF3 $\Rightarrow$ DEF2

Infinite Ramsey Theorem:

there is an infinite subset $\{x_i\}_{i \in I}$ that is monochromatic

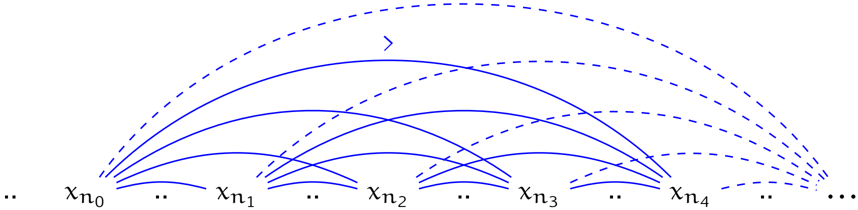
$$.. \quad x_{n_0} \quad .. \quad x_{n_1} \quad .. \quad x_{n_2} \quad .. \quad x_{n_3} \quad .. \quad x_{n_4} \quad .. \quad \ldots$$

What color?

PROVING DEF3 $\Rightarrow$ DEF2

Infinite Ramsey Theorem:

there is an infinite subset $\{x_i\}_{i \in I}$ that is monochromatic

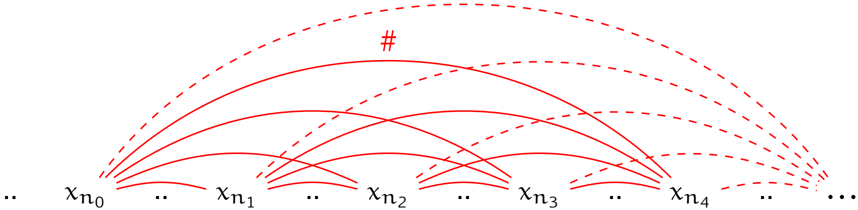


Blue $\Rightarrow$ infinite strictly decreasing sequence, contradicts WF

PROVING DEF3 ⇒ DEF2

Infinite Ramsey Theorem:

there is an infinite subset $\{x_i\}_{i \in I}$ that is monochromatic

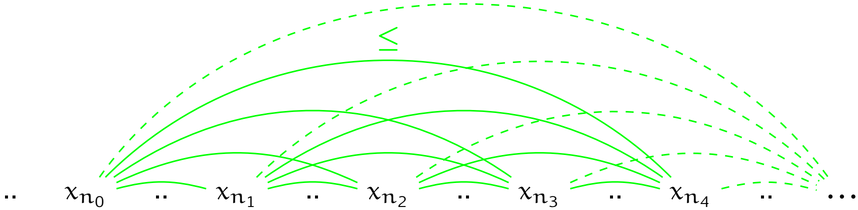


Red ⇒ infinite antichain, contradicts FAC

Proving Def3 $\Rightarrow$ Def2

Infinite Ramsey Theorem:

there is an infinite subset $\{x_i\}_{i \in I}$ that is monochromatic



Must be green $\Rightarrow$ infinite increasing sequence! QED

Spot the WQOs

	linear?	well-founded?	WQO?
$\mathbb{N}, \leqslant$	✓	✓	
$\mathbb{Z}, \_ \mid \_$	×	✓	
$\mathbb{N} \cup \{\omega\}, \leqslant$	✓	✓	
$\mathbb{N}^3, \leqslant_\times$	×	✓	
$\Sigma^*, \leqslant_{\text{pref}}$	×	✓	
$\Sigma^*, \leqslant_{\text{lex}}$	✓	×	
$\Sigma^*, \leqslant_*$	×	✓	

Spot the WQOs

	linear?	well-founded?	WQO?
$\mathbb{N}, \leqslant$	✓	✓	✓
$\mathbb{Z}, \_ \mid \_$	×	✓	
$\mathbb{N} \cup \{\omega\}, \leqslant$	✓	✓	
$\mathbb{N}^3, \leqslant_\times$	×	✓	
$\Sigma^*, \leqslant_{\text{pref}}$	×	✓	
$\Sigma^*, \leqslant_{\text{lex}}$	✓	×	
$\Sigma^*, \leqslant_*$	×	✓	

Spot the WQOs

	linear?	well-founded?	WQO?
$\mathbb{N}, \leqslant$	✓	✓	✓
$\mathbb{Z}, \_ \mid \_$	×	✓	
$\mathbb{N} \cup \{\omega\}, \leqslant$	✓	✓	✓
$\mathbb{N}^3, \leqslant_\times$	×	✓	
$\Sigma^*, \leqslant_{\text{pref}}$	×	✓	
$\Sigma^*, \leqslant_{\text{lex}}$	✓	×	
$\Sigma^*, \leqslant_*$	×	✓	

More generally

Fact. For linear orderings: Well-founded $\Leftrightarrow$ WQO

Cor. Any ordinal is WQO

SPOT THE WQOS

	linear?	well-founded?	WQO?
$\mathbb{N}, \leqslant$	✓	✓	✓
$\mathbb{Z}, \_ \mid \_$	×	✓	×
$\mathbb{N} \cup \{\omega\}, \leqslant$	✓	✓	✓
$\mathbb{N}^3, \leqslant_\times$	×	✓	
$\Sigma^*, \leqslant_{\text{pref}}$	×	✓	
$\Sigma^*, \leqslant_{\text{lex}}$	✓	×	
$\Sigma^*, \leqslant_*$	×	✓	

$(\mathbb{Z}, \_ \mid \_)$: The prime numbers $\{2, 3, 5, 7, 11, \ldots\}$ are an infinite antichain

Spot the WQOs

	linear?	well-founded?	WQO?
$\mathbb{N}, \leqslant$	✓	✓	✓
$\mathbb{Z}, \_ \mid \_$	×	✓	×
$\mathbb{N} \cup \{\omega\}, \leqslant$	✓	✓	✓
$\mathbb{N}^3, \leqslant_\times$	×	✓	✓
$\Sigma^*, \leqslant_{\text{pref}}$	×	✓	
$\Sigma^*, \leqslant_{\text{lex}}$	✓	×	
$\Sigma^*, \leqslant_*$	×	✓	

More generally

(Generalized) Dickson's lemma. If $(X_1, \leqslant_1), \ldots, (X_n, \leqslant_n)$'s are WQOs, then $\prod_{i=1}^{n} X_i, \leqslant_\times$ is WQO

Proof. Easy with Def2. Otherwise, an application of the Infinite Ramsey Theorem

(Usual) Dickson's Lemma. $(\mathbb{N}^k, \leqslant_\times)$ is WQO for any k

Spot the WQOs

	linear?	well-founded?	WQO?
$\mathbb{N}, \leqslant$	✓	✓	✓
$\mathbb{Z}, \_ \mid \_$	×	✓	×
$\mathbb{N} \cup \{\omega\}, \leqslant$	✓	✓	✓
$\mathbb{N}^3, \leqslant_\times$	×	✓	✓
$\Sigma^*, \leqslant_{\text{pref}}$	×	✓	×
$\Sigma^*, \leqslant_{\text{lex}}$	✓	×	×
$\Sigma^*, \leqslant_*$	×	✓	

$(\Sigma^*, \leqslant_{\text{pref}})$ has an infinite antichain

$$\mathtt{bb}, \mathtt{bab}, \mathtt{baab}, \mathtt{baaab}, \ldots$$

$(\Sigma^*, \leqslant_{\text{lex}})$ is not well-founded:

$$\mathtt{b} >_{\text{lex}} \mathtt{ab} >_{\text{lex}} \mathtt{aab} >_{\text{lex}} \mathtt{aaab} >_{\text{lex}} \cdots$$

SPOT THE WQOS

	linear?	well-founded?	WQO?
$\mathbb{N}, \leqslant$	✓	✓	✓
$\mathbb{Z}, \_ \mid \_$	×	✓	×
$\mathbb{N} \cup \{\omega\}, \leqslant$	✓	✓	✓
$\mathbb{N}^3, \leqslant_\times$	×	✓	✓
$\Sigma^*, \leqslant_{\text{pref}}$	×	✓	×
$\Sigma^*, \leqslant_{\text{lex}}$	✓	×	×
$\Sigma^*, \leqslant_*$	×	✓	✓

$(\Sigma^*, \leqslant_*)$ is WQO (Haine's Theorem)

Also by the more general Higman's Lemma (see later)

More equivalent definitions

Def4. (Finite Basis Property). $(X, \leqslant)$ is a WQO $\overset{\text{def}}{\Leftrightarrow}$ every subset $Y \subseteq X$ contains a finite basis B, i.e., such that $\forall y \in Y : \exists b \in B : b \leqslant y$

Def5. (Ascending Chain Condition). $(X, \leqslant)$ is a WQO $\overset{\text{def}}{\Leftrightarrow}$ every strictly increasing sequence $U_0 \subsetneq U_1 \subsetneq U_2 \ldots$ of upward-closed subsets (also: final segments) of X is finite

Def6. $(X, \leqslant)$ is a WQO $\overset{\text{def}}{\Leftrightarrow}$ every linear extension of $\leqslant$ on $X/\equiv$ is a well-ordering

Def7. $(X, \leqslant)$ is a WQO $\overset{\text{def}}{\Leftrightarrow}$ the powerset $\mathcal{P}(X)$ ordered by embedding is well-founded

Def8. *etc.*

Applications in Computer Science

Termination proofs, automated or by hand: WQOs more versatile than well-orderings

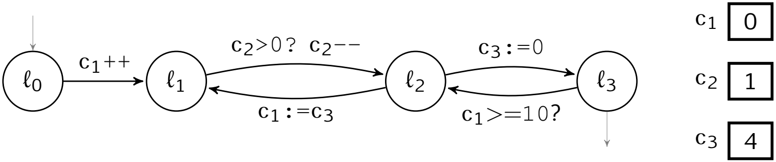
Language theory: any language closed by subwords (or by superwords) is regular

Graphs algorithms: see later

Complexity: WQO-based algorithms have known complexity upper bounds

Program verification: safety properties are decidable for monotonic systems

Monotonic Counter Machines



A run of M: $(\ell_0,0,1,4) \to (\ell_1,1,1,4) \to (\ell_2,1,0,4) \to (\ell_3,1,0,0)$

Ordering states: $(\ell_1,0,0,0) \leqslant (\ell_1,0,1,2)$ but $(\ell_1,0,0,0) \not\leqslant (\ell_2,0,1,2)$.
This is WQO as a product of WQOs: $(Loc,=) \times (\mathbb{N}^3,\leqslant_\times)$

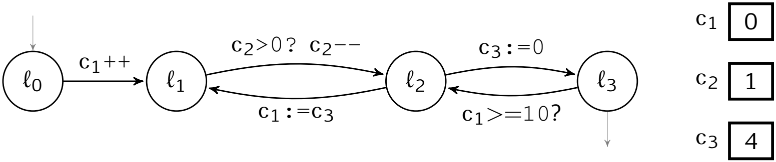
Monotonicity:

if $s_1 \to s_2$ and $s'_1 \geqslant s_1$ then $s'_1 \to \cdots \to s'_2$ for some $s'_2 \geqslant s_2$

Holds because guards are upward-closed and assignments are monotonic functions of the variables

Thm. Safety and termination properties are decidable for monotonic systems over a WQO (Finkel 1987, Abdulla *et al*. 1997, . . .)

Monotonic Counter Machines



A run of M: $(\ell_0,0,1,4) \to (\ell_1,1,1,4) \to (\ell_2,1,0,4) \to (\ell_3,1,0,0)$

Ordering states: $(\ell_1,0,0,0) \leqslant (\ell_1,0,1,2)$ but $(\ell_1,0,0,0) \not\leqslant (\ell_2,0,1,2)$.
This is WQO as a product of WQOs: $(Loc,=) \times (\mathbb{N}^3,\leqslant_\times)$

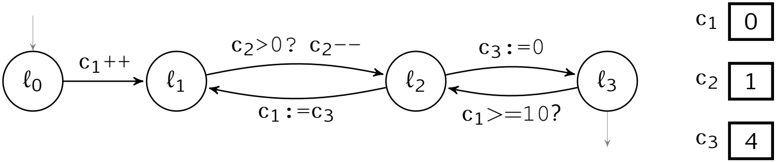
Monotonicity:

if $s_1 \to s_2$ and $s_1' \geqslant s_1$ then $s_1' \to \cdots \to s_2'$ for some $s_2' \geqslant s_2$

Holds because guards are upward-closed and assignments are monotonic functions of the variables

Thm. Safety and termination properties are decidable for monotonic systems over a WQO (Finkel 1987, Abdulla *et al.* 1997, . . .)

Monotonic Counter Machines



A run of M: $(\ell_0,0,1,4) \to (\ell_1,1,1,4) \to (\ell_2,1,0,4) \to (\ell_3,1,0,0)$

Ordering states: $(\ell_1,0,0,0) \leqslant (\ell_1,0,1,2)$ but $(\ell_1,0,0,0) \not\leqslant (\ell_2,0,1,2)$.
This is WQO as a product of WQOs: $(Loc,=) \times (\mathbb{N}^3,\leqslant_\times)$

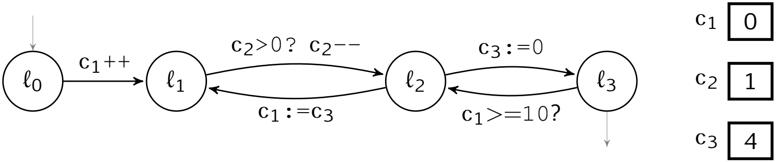
Monotonicity:

if $\quad s_1 \to s_2$ and $s_1' \geqslant s_1 \quad$ then $\quad s_1' \to \cdots \to s_2'$ for some $s_2' \geqslant s_2$

Holds because guards are upward-closed and assignments are monotonic functions of the variables

Thm. Safety and termination properties are decidable for monotonic systems over a WQO (Finkel 1987, Abdulla *et al.* 1997, ...)

Monotonic Counter Machines



A run of M: $(\ell_0,0,1,4) \to (\ell_1,1,1,4) \to (\ell_2,1,0,4) \to (\ell_3,1,0,0)$

Ordering states: $(\ell_1,0,0,0) \leqslant (\ell_1,0,1,2)$ but $(\ell_1,0,0,0) \not\leqslant (\ell_2,0,1,2)$.
This is WQO as a product of WQOs: $(Loc,=) \times (\mathbb{N}^3,\leqslant_\times)$

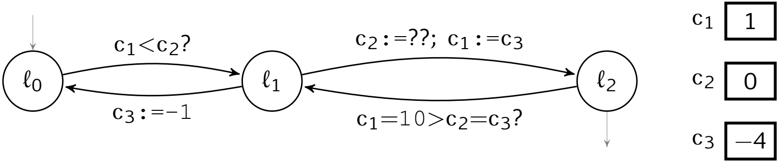
Monotonicity:

if $\quad s_1 \to s_2$ and $s_1' \geqslant s_1 \quad$ then $\quad s_1' \to \cdots \to s_2'$ for some $s_2' \geqslant s_2$

Holds because guards are upward-closed and assignments are monotonic functions of the variables

Thm. Safety and termination properties are decidable for monotonic systems over a WQO (Finkel 1987, Abdulla *et al*. 1997, . . .)

RELATIONAL AUTOMATA



Guards: comparisons between counters and constants

Updates: assignments with counter values, constants, and "??"

One does not use $\leqslant_\times$ to compare states!! Rather

$$(a_1,\dots,a_k) \leqslant_{\text{sparse}} (b_1,\dots,b_k)$$

$$\overset{\text{def}}{\Leftrightarrow} \forall i,j = 1,\dots,k : \big(a_i \leqslant a_j \text{ iff } b_i \leqslant b_j\big) \wedge \big(|a_i - a_j| \leqslant |b_i - b_j|\big)$$

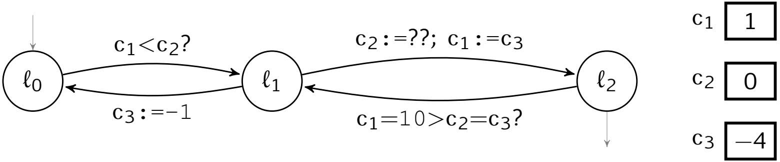
Fact. $(\mathbb{Z}^k, \leqslant_{\text{sparse}})$ is WQO

Monotonicity: using

$$(\ell, a_1,\dots,a_k) \leqslant (\ell', b_1,\dots,b_k) \overset{\text{def}}{\Leftrightarrow}$$

$$\ell = \ell' \wedge (a_1,\dots,a_k,-1,10) \leqslant_{\text{sparse}} (b_1,\dots,b_k,-1,10)$$

Relational automata



Guards: comparisons between counters and constants

Updates: assignments with counter values, constants, and "??"

One does not use $\leqslant_\times$ to compare states!! Rather

$$(a_1,\ldots,a_k) \leqslant_{\text{sparse}} (b_1,\ldots,b_k)$$

$$\overset{\text{def}}{\Leftrightarrow} \forall i,j=1,\ldots,k : \big(a_i \leqslant a_j \text{ iff } b_i \leqslant b_j\big) \wedge \big(|a_i - a_j| \leqslant |b_i - b_j|\big)$$

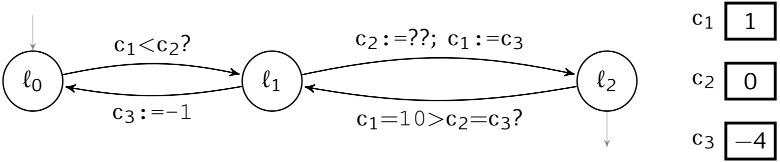
Fact. $(\mathbb{Z}^k, \leqslant_{\text{sparse}})$ is WQO

Monotonicity: using

$$(\ell, a_1,\ldots,a_k) \leqslant (\ell', b_1,\ldots,b_k) \overset{\text{def}}{\Leftrightarrow}$$

$$\ell=\ell' \wedge (a_1,\ldots,a_k,-1,10) \leqslant_{\text{sparse}} (b_1,\ldots,b_k,-1,10)$$

Relational automata



Guards: comparisons between counters and constants

Updates: assignments with counter values, constants, and "??"

One does not use $\leqslant_\times$ to compare states!! Rather

$$(a_1,\dots,a_k) \leqslant_{\text{sparse}} (b_1,\dots,b_k)$$
$$\overset{\text{def}}{\Leftrightarrow} \forall i,j = 1,\dots,k : \big(a_i \leqslant a_j \text{ iff } b_i \leqslant b_j\big) \wedge \big(|a_i - a_j| \leqslant |b_i - b_j|\big)$$

Fact. $(\mathbb{Z}^k, \leqslant_{\text{sparse}})$ is WQO

Monotonicity: using

$$(\ell, a_1,\dots,a_k) \leqslant (\ell', b_1,\dots,b_k) \overset{\text{def}}{\Leftrightarrow}$$
$$\ell = \ell' \wedge (a_1,\dots,a_k,-1,10) \leqslant_{\text{sparse}} (b_1,\dots,b_k,-1,10)$$

Building more WQOs

SEQUENCES AND HIGMAN'S LEMMA

Def. The sequence extension of a QO $(X, \leqslant)$ is the QO $(X^*, \leqslant_*)$ —also: $X^{<\omega}$— of finite sequences over X ordered by embedding:

$$u = x_1 \cdots x_n \leqslant_* y_1 \cdots y_m = v \overset{\text{def}}{\Leftrightarrow} \begin{array}{l} x_1 \leqslant y_{l_1} \wedge \cdots \wedge x_n \leqslant y_{l_n} \\ \text{for some } 1 \leqslant l_1 < l_2 < \cdots < l_n \leqslant m \end{array}$$

$$\overset{\text{def}}{\Leftrightarrow} u \leqslant_\times v' \text{ for a length-}n\text{ subsequence } v' \text{ of } v$$

Higman's Lemma (1952). X WQO implies X^* WQO

With $(\Sigma^*, \leqslant_*)$ and Haines' Theorem, we were considering the sequence extension of $(\Sigma, =)$ which is finite, hence necessarily WQO

Higman's Lemma applies to the sequence extension of more complex WQOs, e.g., $\mathbb{N}^2$:

Does $\begin{vmatrix}0\\1\end{vmatrix}\begin{matrix}2\\0\end{matrix}\begin{vmatrix}0\\2\end{vmatrix} \leqslant_* \begin{vmatrix}2\\0\end{vmatrix}\begin{matrix}0\\2\end{matrix}\begin{vmatrix}0\\2\end{vmatrix}\begin{matrix}2\\2\end{matrix}\begin{vmatrix}2\\0\end{vmatrix}\begin{matrix}0\\1\end{matrix}$?

Sequences and Higman's Lemma

Def. The sequence extension of a QO $(X, \leqslant)$ is the QO $(X^*, \leqslant_*)$ —also: $X^{<\omega}$— of finite sequences over X ordered by embedding:

$$u = x_1 \cdots x_n \leqslant_* y_1 \cdots y_m = v \overset{\text{def}}{\Leftrightarrow} \begin{array}{l} x_1 \leqslant y_{l_1} \wedge \cdots \wedge x_n \leqslant y_{l_n} \\ \text{for some } 1 \leqslant l_1 < l_2 < \cdots < l_n \leqslant m \end{array}$$

$$\overset{\text{def}}{\Leftrightarrow} u \leqslant_\times v' \text{ for a length-}n\text{ subsequence } v' \text{ of } v$$

Higman's Lemma (1952). X WQO implies X^* WQO

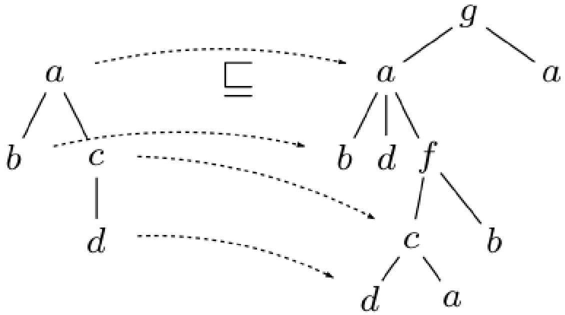
With $(\Sigma^*, \leqslant_*)$ and Haines' Theorem, we were considering the sequence extension of $(\Sigma, =)$ which is finite, hence necessarily WQO

Higman's Lemma applies to the sequence extension of more complex WQOs, e.g., $\mathbb{N}^2$:

$$\text{Does} \quad \left|\begin{smallmatrix}\mathbf{0}\\\mathbf{1}\end{smallmatrix}\right. \left|\begin{smallmatrix}\mathbf{2}\\\mathbf{0}\end{smallmatrix}\right. \left|\begin{smallmatrix}\mathbf{0}\\\mathbf{2}\end{smallmatrix}\right. \leqslant_* \left|\begin{smallmatrix}\mathbf{2}\\\mathbf{0}\end{smallmatrix}\right. \left|\begin{smallmatrix}\mathbf{0}\\\mathbf{2}\end{smallmatrix}\right. \left|\begin{smallmatrix}\mathbf{0}\\\mathbf{2}\end{smallmatrix}\right. \left|\begin{smallmatrix}\mathbf{2}\\\mathbf{2}\end{smallmatrix}\right. \left|\begin{smallmatrix}\mathbf{2}\\\mathbf{0}\end{smallmatrix}\right. \left|\begin{smallmatrix}\mathbf{0}\\\mathbf{1}\end{smallmatrix}\right. \quad ?$$

TREES

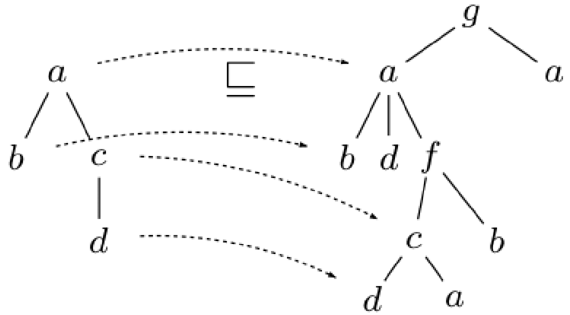
$(\mathrm{T}[\mathrm{X}], \sqsubseteq)$ has all finite rooted trees with labels from a WQO, ordered with embedding:



Kruskal's Tree Theorem (1960). X WQO implies T[X] WQO

Trees

$(\mathrm{T}[\mathrm{X}], \sqsubseteq)$ has all finite rooted trees with labels from a WQO, ordered with embedding:



Kruskal's Tree Theorem (1960). X WQO implies $\mathrm{T}[\mathrm{X}]$ WQO

From WQOs to BQOs

WELL-QUASI ORDERING INFINITARY CONSTRUCTIONS?

All the above constructions $\prod_{i=1}^{n} X_i$, or X^*, or $T[X]$, or .. have a restriction of a finitary kind

Very early, Rado showed that X WQO does not imply that X^ω —infinite sequences over X ordered by embedding,— or even $\mathcal{P}(X)$, is WQO

This was the starting point of better-quasi-orderings (BQO)

Well-quasi ordering infinitary constructions?

All the above constructions $\prod_{i=1}^{n} X_i$, or X^*, or $T[X]$, or .. have a restriction of a finitary kind

Very early, Rado showed that X WQO does not imply that X^ω —infinite sequences over X ordered by embedding,— or even $\mathcal{P}(X)$, is WQO

This was the starting point of better-quasi-orderings (BQO)

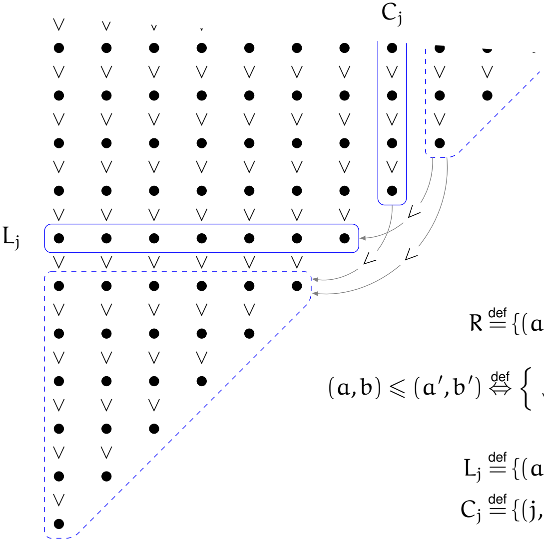
WELL-QUASI ORDERING INFINITARY CONSTRUCTIONS?

All the above constructions $\prod_{i=1}^{n} X_i$, or X^*, or $T[X]$, or .. have a restriction of a finitary kind

Very early, Rado showed that X WQO does not imply that X^ω —infinite sequences over X ordered by embedding,— or even $\mathcal{P}(X)$, is WQO

This was the starting point of better-quasi-orderings (BQO)

RADO'S STRUCTURE [1954]



$$R \overset{\text{def}}{=} \{(a,b) \in \mathbb{N}^2 \mid a < b\}$$

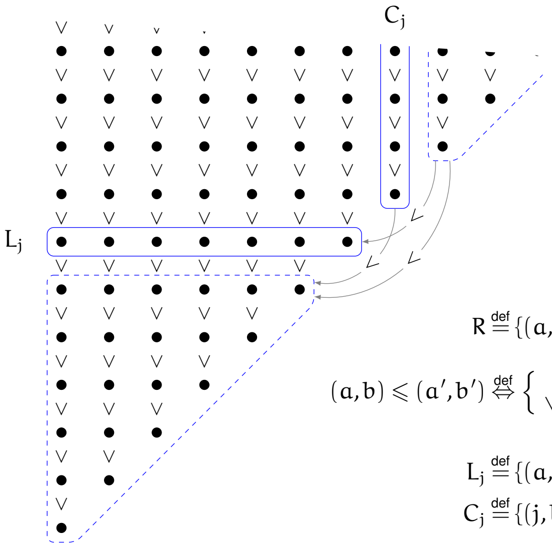
$$(a,b) \leqslant (a',b') \overset{\text{def}}{\Leftrightarrow} \begin{cases} & a = a' \wedge b \leqslant b' \\ \vee & b < a' \end{cases}$$

$$L_j \overset{\text{def}}{=} \{(a,j) \in R \mid a < j\}$$

$$C_j \overset{\text{def}}{=} \{(j,b) \in R \mid j < b\}$$

Fact. $(R,\leqslant)$ is a WQO but $(R^\omega,\leqslant_\omega)$ or $(\mathcal{P}(R),\sqsubseteq)$ are not

RADO'S STRUCTURE [1954]



$$R \stackrel{\text{def}}{=} \{(a,b) \in \mathbb{N}^2 \mid a < b\}$$

$$(a,b) \leqslant (a',b') \stackrel{\text{def}}{\Leftrightarrow} \begin{cases} & a = a' \wedge b \leqslant b' \\ \vee & b < a' \end{cases}$$

$$L_j \stackrel{\text{def}}{=} \{(a,j) \in R \mid a < j\}$$

$$C_j \stackrel{\text{def}}{=} \{(j,b) \in R \mid j < b\}$$

Fact. $(R, \leqslant)$ is a WQO but $(R^\omega, \leqslant_\omega)$ or $(\mathcal{P}(R), \sqsubseteq)$ are not

TOWARDS BQOS

Question: On what condition is X^ω WQO?

Answer [Rado 1954]: If and only if X does not contains (an isomorphic copy of) Rado's structure

Question: Now assume X does not contain R. Then X^ω is WQO. But what about $(X^\omega)^\omega$?

Answer: (X^ω) is WQO but may still contain R :-(

One can characterize the WQOs X such that X^ω does not contain R, again using forbidden substructures

NB. Recall that WF $\Leftrightarrow$ "does not contain $(\omega, \geqslant)$"
and that FAC $\Leftrightarrow$ "does not contain $(\omega, =)$"

Caveat. This may go on forever (it does)

TOWARDS BQOS

Question: On what condition is X^ω WQO?

Answer [Rado 1954]: If and only if X does not contains (an isomorphic copy of) Rado's structure

Question: Now assume X does not contain R. Then X^ω is WQO. But what about $(X^\omega)^\omega$?

Answer: (X^ω) is WQO but may still contain R :-(

One can characterize the WQOs X such that X^ω does not contain R, again using forbidden substructures

NB. Recall that WF $\Leftrightarrow$ "does not contain $(\omega, \geqslant)$"
and that FAC $\Leftrightarrow$ "does not contain $(\omega, =)$"

Caveat. This may go on forever (it does)

TOWARDS BQOS

Question: On what condition is X^ω WQO?

Answer [Rado 1954]: If and only if X does not contains (an isomorphic copy of) Rado's structure

Question: Now assume X does not contain R. Then X^ω is WQO. But what about $(X^\omega)^\omega$?

Answer: (X^ω) is WQO but may still contain R :-(

One can characterize the WQOs X such that X^ω does not contain R, again using forbidden substructures

NB. Recall that WF ⇔ "does not contain $(\omega, \geqslant)$"
and that FAC ⇔ "does not contain $(\omega, =)$"

Caveat. This may go on forever (it does)

A SMALL GLIMPSE AT BQOs

The general question is when is X^{α} WQO for a given countable α?
Or more generally: when is $X^{<\omega_1}$ WQO?

Nash-Williams (1965) defined a BQO as any QO X that does not lead to “bad X-patterns” (with a complex combinatorial definition of patterns)

These BQOs are between well-orderings and WQOs

He proved X BQO implies $X^{<\omega_1}$ BQO

This is the general answer since $X^{<\omega_1}$ WQO implies X BQO (Pouzet 1972)

A SMALL GLIMPSE AT BQOS

The general question is when is X^{α} WQO for a given countable α?
Or more generally: when is $X^{<\omega_1}$ WQO?

Nash-Williams (1965) defined a BQO as any QO X that does not lead to “bad X-patterns” (with a complex combinatorial definition of patterns)

These BQOs are between well-orderings and WQOs

He proved X BQO implies $X^{<\omega_1}$ BQO

This is the general answer since $X^{<\omega_1}$ WQO implies X BQO (Pouzet 1972)

A SMALL GLIMPSE AT BQOs

The general question is when is X^α WQO for a given countable α?
Or more generally: when is $X^{<\omega_1}$ WQO?

Nash-Williams (1965) defined a BQO as any QO X that does not lead to "bad X-patterns" (with a complex combinatorial definition of patterns)

These BQOs are between well-orderings and WQOs

He proved X BQO implies $X^{<\omega_1}$ BQO

This is the general answer since $X^{<\omega_1}$ WQO implies X BQO (Pouzet 1972)

A hint of Graph Minor Theory

GRAPHS TOO CAN BE ORDERED

There are many ways of embedding a smaller graph into a larger graph

Four definitions for $G \preccurlyeq H$ (from stronger to weaker):

induced subgraph: delete some vertices of H (and their edges)

subgraph: delete some vertices and edges of H

topological minor: a subdivision of G is a subgraph of H

minor: take a subgraph of H and contract some edges (fusing adjacent vertices)

Kuratowksi's Theorem (1930):

A graph G is planar iff it does not contain K_5 or $K_{3,3}$ as a minor

Graphs too can be ordered

There are many ways of embedding a smaller graph into a larger graph

Four definitions for $G \preccurlyeq H$ (from stronger to weaker):

induced subgraph: delete some vertices of H (and their edges)

subgraph: delete some vertices and edges of H

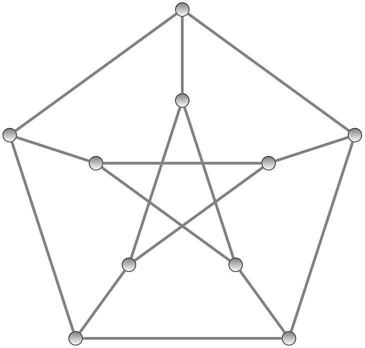
topological minor: a subdivision of G is a subgraph of H

minor: take a subgraph of H and contract some edges (fusing adjacent vertices)

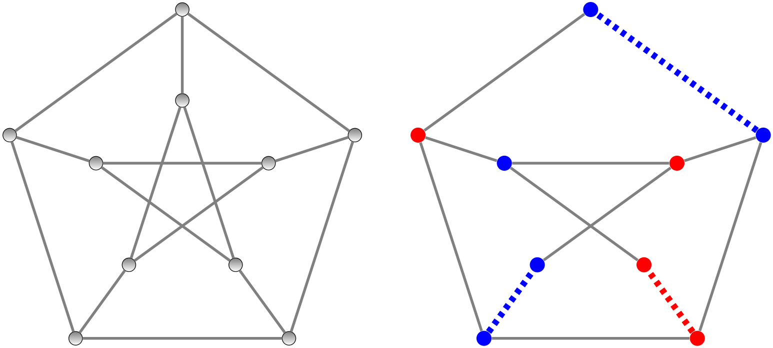
Kuratowksi's Theorem (1930):

A graph G is planar iff it does not contain K_5 or $K_{3,3}$ as a minor

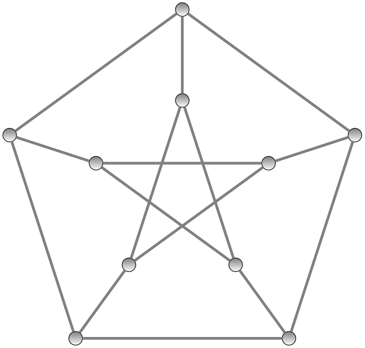
Example: Petersen's graph

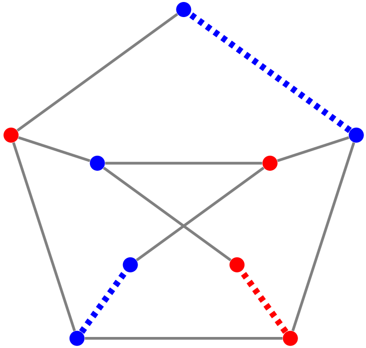


Example: Petersen's graph



Example: Petersen's graph





Contains $K_{3,3}$ as a minor. Hence is **not planar!**

Excluded Minors

Any property of graphs that can be characterized by (finitely many) excluded minors is easy to test algorithmically: e.g., planarity

NB. This must be a minor-closed property but there are many examples: G is a tree (a forest) iff it does not contain K_3 (as a minor), it is series-parallel iff it does not contain K_4, etc.

Robertson-Seymour Theorem (1983–2004) Finite graphs are WQOs under the minor ordering
Cor. Any minor-closed property is characterized by excluded minors

Applications. Find the excluded minors for your minor-closed property of interest (e.g., *embeddable on a given surface*, or *embeddable in* $\mathbb{R}^3$ *with no links*, or *no knots*, etc.) and you have a polynomial-time decision algorithm for it

More generally, many hard problems become simpler when restricted to graphs that exclude a minor

Other WQOs on graphs: Graphs are not WQOs under subgraph or even topological minors (Ding 1996) but many subclasses of graphs

Excluded Minors

Any property of graphs that can be characterized by (finitely many) excluded minors is easy to test algorithmically: e.g., planarity

Robertson-Seymour Theorem (1983–2004) Finite graphs are WQOs under the minor ordering
Cor. Any minor-closed property is characterized by excluded minors

Applications. Find the excluded minors for your minor-closed property of interest (e.g., *embeddable on a given surface*, or *embeddable in* $\mathbb{R}^3$ *with no links*, or *no knots*, etc.) and you have a polynomial-time decision algorithm for it

More generally, many hard problems become simpler when restricted to graphs that exclude a minor

Other WQOs on graphs: Graphs are not WQOs under subgraph or even topological minors (Ding 1996) but many subclasses of graphs are. Example: cographs are WQOs under induced subgraph ordering (Damaschke 1990)

Excluded Minors

Any property of graphs that can be characterized by (finitely many) excluded minors is easy to test algorithmically: e.g., planarity

Robertson-Seymour Theorem (1983–2004) Finite graphs are WQOs under the minor ordering
Cor. Any minor-closed property is characterized by excluded minors

Applications. Find the excluded minors for your minor-closed property of interest (e.g., *embeddable on a given surface*, or *embeddable in* $\mathbb{R}^3$ *with no links*, or *no knots*, etc.) and you have a polynomial-time decision algorithm for it

More generally, many hard problems become simpler when restricted to graphs that exclude a minor

Other WQOs on graphs: Graphs are not WQOs under subgraph or even topological minors (Ding 1996) but many subclasses of graphs are. Example: cographs are WQOs under induced subgraph ordering (Damaschke 1990)

Excluded Minors

Any property of graphs that can be characterized by (finitely many) excluded minors is easy to test algorithmically: e.g., planarity

Robertson-Seymour Theorem (1983–2004) Finite graphs are WQOs under the minor ordering
Cor. Any minor-closed property is characterized by excluded minors

Applications. Find the excluded minors for your minor-closed property of interest (e.g., *embeddable on a given surface*, or *embeddable in* $\mathbb{R}^3$ *with no links*, or *no knots*, etc.) and you have a polynomial-time decision algorithm for it

More generally, many hard problems become simpler when restricted to graphs that exclude a minor

Other WQOs on graphs: Graphs are not WQOs under subgraph or even topological minors (Ding 1996) but many subclasses of graphs are. Example: cographs are WQOs under induced subgraph ordering (Damaschke 1990)

Concluding remarks

WQOs are fun !

Every computer scientist is likely to use the basics at some point

See gentle tutorial notes *Algorithmic aspects of WQO Theory* by sylvain schmitz & myself for complexity of WQO-based algorithms

SELECTED BIBLIOGRAPHY

D. Bienstock and M. A. Langston.
Algorithmic implications of the graph minor theorem.
In M. O. Ball, T. L. Magnanti, C. L. Monma, and G.L. Nemhauser, editors, *Network Models*, volume 7 of *Handbooks in Operations Research and Management Science*, chapter 8, pages 481–502. Elsevier Science, 1995.

F. D'Alessandro and S. Varricchio.
Well quasi-orders in formal language theory.
In *Proc. 12th Int. Conf. Developments in Language Theory (DLT 2008), Tokyo, Japan, Sep. 2008*, volume 5257 of *Lecture Notes in Computer Science*, pages 84–95. Springer, 2008.

J. B. Kruskal.
The theory of well-quasi-ordering: A frequently discovered concept.
Journal of Combinatorial Theory, Series A, 13(3):297–305, 1972.

L. Lovász.
Graph minor theory.
Bull. Amer. Math. Soc., 43(1):75–86, 2006.

A. Marcone.
Foundations of BQO theory.
Trans. Amer. Math. Soc., 345(2):641–660, 1994.

E. C. Milner.
Basic WQO- and BQO-theory.
In I. Rival, editor, *Graphs and Order. The Role of Graphs in the Theory of Ordered Sets and Its Applications*, pages 487–502. D. Reidel Publishing, 1985.

S. Schmitz and Ph. Schnoebelen.
The power of well-structured systems.
In *Proc. 24th Int. Conf. Concurrency Theory (CONCUR 2013), Buenos Aires, Argentina, Aug. 2013*, volume 8052 of *Lecture Notes in Computer Science*, pages 5–24. Springer, 2013.