Defining Gromov Witten invariants

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Chennai Mathematical Institute
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5. Gromov-Witten invariants, definition and examples.

If time allows, we will mention other invariants defined in a similar way.

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- Sketch of proof that $\overline{M}_{g,n}(V,d)$ is algebraic and proper.

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Z_1 is flat over S_1 since flatness is preserved by base change.

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This determines $(HilbX, Z_H)$ up to canonical isomorphism.

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Example

The Grassmann variety and the projective space of degree d hypersurfaces are both Hilbert schemes of $\mathbb{P}^N$ (exercise: find P).

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Proof.

Let h be the Hilbert functor of $C \times V$; we can define a natural transformation $m \to h$ by associating to a morphism $f : C \times S \to V$ its graph Γ_f. This map is an open embedding. □

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Prestable pointed curves

Definition

A *prestable n-pointed (or n-marked) genus g curve* is a tuple $(C, x_1, \ldots, x_n)$ such that

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Let $\nu : \widetilde{C} \to C$ be the normalisation morphism. A point $x \in \widetilde{C}$ is *special* if $\nu(x)$ is either singular for C or a marked point x_i.

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Definition

If any of these conditions is satisfied (or, equivalently, all of them are) the prestable curve (C, x_i) is called *stable*.

Families of prestable curves -1

Definition

A *family* of prestable n-pointed, genus g curves over a base scheme S is a tuple $(C, \pi, x_1, \ldots, x_n)$ where

- $\pi : C \to S$ is a flat, projective morphism;
- $x_1, \ldots, x_n : S \to C$ are sections of π;
- for every $s \in S$, $(C_s, x_1(s), \ldots, x_n(s))$ is a prestable n-pointed, genus g curve.

A *family of stable curves* is defined by replacing prestable with stable in the definition above.

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3. Define isomorphisms for families of prestable curves over a base S;
4. Given a family (C, x_i) of prestable n-pointed, genus g curves over a base scheme S and a morphism $f : S' \to S$ of schemes, define a pullback family (C', x_i') over S'.
   Hint: start with $C' := C \times_S S'$.

Moduli stacks of pointed curves

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The stack $\overline{M}_{g,n}$ *is a Deligne-Mumford (DM) algebraic stack and it is proper over* $\mathbb{C}$*.*

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We want to find a good compactification $\overline{M}_{g,n}(V, d)$ of the space of tuples $(C, x_1, \ldots, x_n)$ where $C \subset V$ is a nonsingular connected curve of genus g and degree d, and the $x_i \in C$ are distinct points.

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Good means we want to use it to do enumerative geometry; in particular, we want the compactification to be smooth, or at least pure-dimensional, so we have a fundamental cycle against which to integrate cohomology classes pulled back from V via the maps $ev_i : M \to V$ sending a tuple $(C, x_1, \ldots, x_n)$ to $x_i \in V$.

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We can find such a compactification using the Hilbert scheme; however we have no idea how to choose a homology cycle on it in a natural way, or even how to choose a dimension.

Key idea

The key idea of Gromov Witten theory is to combine the scheme of morphisms, the stack of pointed prestable curves and the stability condition to compactify naturally the space of tuples $(C, x_1, \dots, x_n, f)$ where C is a smooth genus g curve, $x_1, \dots, x_n \in C$ are distinct points, and $f : C \to V$ is a morphism of degree d which may not be an embedding.

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A morphism of degree $d = 0$ is a constant map; so we can define such a compactification to be

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- $f : C \to V$ is a degree d morphism (i.e., $\deg f^*(\mathcal{O}_V(1)) = d$).
- an irriducible component $\widetilde{Z}$ of $\widetilde{C}$ i *contracted by* f, or a *contracted component*, if $f \circ \nu(\widetilde{Z})$ is a point.

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Definition

If any of these conditions is satisfied (or, equivalently, all of them are) the prestable map (C, x_i, f) is called *stable*.

Families of prestable maps -1

Definition

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Definition

If moreover for every $s \in S$ the prestable map $(C_s, x_i(s), f|_{C_s})$ is stable, we say that $(C, \pi, x_1, \ldots, x_n, f)$ is a *family of stable maps*.

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4. Given a family $(C, x_i), f$ of prestable n-pointed, genus g maps over a base scheme S and a morphism $f : S' \to S$ of schemes, define a pullback family (C', x_i', f') over S'.
   Hint: start with pulling back the family of prestable curves.

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5. Show that the pullback of a family of stable maps is also a family of stable maps.

Moduli stacks of pointed maps -1

Definition

The *stack* $\mathfrak{M}_{g,n}(V,d)$ *of prestable, genus* g*,* n *pointed maps of degree* d is the pseudofunctor $(sch) \to (grpd)$ associating to each scheme S the groupoid of families of prestable genus g, n-pointed maps over S with their isomorphisms.

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We denote by $F : \mathfrak{M}_{g,n}(V,d) \to \mathfrak{M}_{g,n}$ the *forgetful morphism*, mapping a family of stable maps to its family of prestable curves and forgetting the map.

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We also denote by F the restriction of the forgetful morphism to $\overline{M}_{g,n}(V, d)$.

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Let S be a scheme and $S \to \mathfrak{M}_{g,n}$ a morphism, i.e., a family of prestable genus g, n-pointed curves. We need to prove that the fiber product $S \times_{\mathfrak{M}_{g,n}} \mathfrak{M}_{g,n}(V,d)$ is a scheme, quasiprojective over S.

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It follows easily from the definition of fiber product for stacks that it is isomorphic to $Mor_S(C, V \times B)^d$. □

Algebraicity of $\overline{M}_{g,n}(V,d)$ -1

Theorem

1. *The stack $\overline{M}_{g,n}(V,d)$ is a Deligne-Mumford (DM) algebraic stack.*
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$\overline{M}_{g,n}(V,d)$ is an algebraic stack because it is open in $\mathfrak{M}_{g,n}(V,d)$ which is algebraic; in particular, the forgetful morphism is quasi projective. □

Algebraicity of $\overline{M}_{g,n}(V, d)$-2

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By assumption the automorphism group is finite; since we are in characteristic zero, it is smooth, hence its tangent space is zero-dimensional.

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This argument fails in positive characteristic, and indeed in that case the stack is not DM in general. □

Properness of $\overline{M}_{g,n}(V, d)$ -1

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Let $\overline{B}$ be any smooth affine curve, $b_0 \in \overline{B}$ any point, and $B = \overline{B} \setminus b_0$.
We need to show that any family of stable maps (C, π, x_i, f) over B can be uniquely extended to $\overline{B}$, after possibly a finite base change.

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We first use a base change to extend (C, π, x_i) to $(\overline{C}, \bar{\pi}, \bar{x}_i)$ over $\overline{B}$.
This can be done in analogy with the proof of properness for $\overline{M}_{g,n}$.

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This can be done in analogy with the proof of properness for $\overline{M}_{g,n}$.

Assume for simplicity that $\overline{C}$ is a smooth surface. Then $f : C \to V$ induces a rational map $\overline{C} \to V$; after a finite number of blow-ups $\varepsilon : \overline{C}' \to \overline{C}$, we can assume that the map $f' := f \circ \varepsilon$is regular.

Properness of $\overline{M}_{g,n}(V,d)$ -2

The fibres of $\bar{\pi} \circ \varepsilon : \overline{C}' \to \overline{B}$ are nodal curves but may be non reduced. This can be fixed by a finite base change and normalisation.

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We have thus proven that, up to base change, we can extend (C, π, x_i, f) to a family of prestable maps over $\overline{B}$. We now need to show that among such extensions there is a unique stable one.

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First we prove existence. Let $(\overline{C}, \bar{\pi}, \bar{x}_i, \bar{f})$ be a prestable extension to B. If it isn't stable, there is a rational curve Z in C_{b_0}, contracted by $\bar{f}$, whose normalisation $\widetilde{Z}$ contains at most two special points.

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The same argument applies when there are two special points, one a node and one marked.

Properness of $\overline{M}_{g,n}(V, d)$ -3

Proof.

$\square$

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To conclude the proof, we have to consider components Z as above whose normalisation contains exactly two special points, both nodes.

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Again by connectedness, Z must also be smooth, and one can prove that $N_{Z/C} = \mathcal{O}_Z(-2)$. Hence Z can be contracted as above, but this time to an A_1 singularity.

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Repeating the process, we get in the situation we had before, except now $\overline{C}'$ has rational double points. This proves existence.

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Properness of $\overline{M}_{g,n}(V, d)$ -4

Proof.

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Properness of $\overline{M}_{g,n}(V, d)$ -4

Proof.

To prove uniqueness, any other extension must be birational to the one we started with; if they are both smooth, a birational map factors uniquely as a sequence of blow-ups and blow-downs.
One can prove by induction on the total number of bloe-ups and blow-downs that the birational map must be an isomorphism.

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One can prove by induction on the total number of bloe-ups and blow-downs that the birational map must be an isomorphism.
For the general case, we cannot assume that $\overline{C}$ is smooth but its singularities are very limited, either nodes

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One can extend the previous argument to this case, working with the minimal resolution of singularities of $\overline{C}$, which is easy to construct explicitly. □

Summary of the lecture

For any projective smooth variety (indeed, any projective scheme) V we have defined a proper DM algebraic stack of (families of) stable maps $\overline{M}_{g,n}(V,d)$; by definition it carries a universal genus g, n-pointed stable map (C, π, x_i, f) of degree d.

The forgetful morphism $F : \overline{M}_{g,n}(V,d) \to \mathfrak{M}_{g,n}$ is quasiprojective.

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1. A similar argument with minimal models of surfaces proves the properness of $\overline{M}_{g,n}$.
2. Replacing minimal models with semistable reduction gives properness of $\overline{M}_{g,n}$ and $\overline{M}_{g,n}(V, d)$ in any characteristic.
3. However while $\overline{M}_{g,n}$ is DM in any characteristic (and indeed over $\mathbb{Z}$), $\overline{M}_{g,n}(V, d)$ is not.