Mock modularity of Calabi-Yau threefolds and BPS black holes

review 2505.02572

Introduction to modular and mock modular forms

- Modular group

$SL(2,\mathbb{Z})$ $\quad \rho = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$: $a,b,c,d \in \mathbb{Z}$, $ad-bc=1$

This is a modular group of T^2 : $\tau \to \frac{a\tau+b}{c\tau+d}$

compl. str.

(or $PSL(2,\mathbb{Z})$ where $g \sim -g$)

Two generators $T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ $\quad S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ $\quad S^2=-1$ $\quad (ST)^3=-1$

- Modular forms

$\exists\ \tau = \tau_1 + i\tau_2 \in \mathbb{H} = \{\tau \in \mathbb{C}, \tau_2 > 0\}$

1) $h(\tau)$ - holomorphic on $\mathbb{H}$ and bounded as $\tau_2 \to \infty$

2) $h\left(\frac{a\tau+b}{c\tau+d}\right) = (c\tau+d)^w h(\tau)$

w - modular wight

Consequence: $h(\tau+1) = h(\tau) \Rightarrow h(\tau) = \sum_{n=0}^{\infty} h_n q^n$, $q = e^{2\pi i \tau}$

Ex. Eisenstein series

$$k>1 : \quad E_{2k}(\tau) = \frac{1}{2\zeta(2k)} \sum_{(m,n)\in\mathbb{Z}^2\setminus\{0,0\}} \frac{1}{(m\tau+n)^{2k}}$$
– modular form of weight $2k$

(follows from $m\tau'+n = \frac{m'\tau+n'}{c\tau+d}$ where $\begin{pmatrix} m' \\ n' \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} m \\ n \end{pmatrix}$)

E_4 and E_6 generate the full ring of modular forms!

Generalizations

This definition is too restrictive.

Generalizations:

- relax holomophicity
  allow polynomial growth in q^{-1} at $\tau_2 \to \infty$ → weakly holomorphic m.f.

  $\longrightarrow h(\tau) = \sum_{n=-N}^{\infty} h_n q^n$ terms wih $n<0$ - polar terms

Ex. Inverse discriminant $\Delta^{-1}(\tau) = q^{-1} + 24 + \dots$ m.f. of weight -12

where $\Delta(\tau) = \frac{E_4^3 - E_6^2}{1728}$

- multiplier system

  $$h\left(\frac{a\tau+b}{c\tau+d}\right) = (c\tau+d)^w \underset{\substack{\uparrow \\ \text{should satisfy a cocycle condition} \\ \text{defined by } \mathcal{M}(T) \text{ and } \mathcal{M}(S)}}{M(g)}\, h(\tau) \longrightarrow$$

  non-integer powers in Fourier
  half-integer weigt

Ex. Dedekind eta function

$$\eta(\tau) = \Delta^{1/24}(\tau) = q^{1/24} \prod_{n=1}^{\infty} (1-q^n) = q^{1/24} \sum_{n\in\mathbb{Z}} (-1)\, q^{\frac{1}{2}(3n^2-n)}$$

weigh $\frac{1}{2}$, $\mathcal{M}(T) = e^{\frac{\pi i}{12}}$, $\mathcal{M}(S) = e^{-\pi i/4}$

- vector valued m.f.

  $$h_\mu\left(\frac{a\tau+b}{c\tau+d}\right) = (c\tau+d)^w \sum_\nu \mathcal{M}_{\mu\nu}(\rho)\, h_\nu(\tau)$$

Ex. $\theta_\mu(\tau) = \left(\theta_3(2\tau), \theta_2(2\tau)\right) = \sum_{n\in\mathbb{Z}+\mu/2} q^{n^2}$ $\quad \mu = 0,1$

$w = \frac{1}{2}$, $\mathcal{M}_{\mu\nu}(T) = e^{\frac{\pi i}{2}} \delta_{\mu\nu}$, $\mathcal{M}_{\mu\nu}(S) = \frac{e^{-\frac{\pi i}{4}}}{\sqrt{2}} (-1)^{\mu\nu}$

- non-holomorphic m.f. of weight $(w, \bar{w})$

  $$h\left(\frac{a\tau+b}{c\tau+d}, \frac{a\bar\tau+b}{c\bar\tau+d}\right) = (c\tau+d)^w (c\bar\tau+d)^{\bar w} \sum_\nu \mathcal{M}_{\mu\nu}(g)\, h_\nu(\tau,\bar\tau)$$

Ex. τ_2 ~ m.f. of weight $(-1,-1)$

Mock modular forms

Mock m.f. have a modular anomaly determined by another modular form – shadow

m.f. of weight $2-w$

$$h_\mu\left(\frac{a\tau+b}{c\tau+d}\right) = (c\tau+d)^w \sum_\nu M_{\mu\nu}(g)\left(h_\nu(\tau) - \int_{-d/c}^{-i\infty} \frac{\overline{g_\nu(\bar z)}}{(\tau-z)^w}\, dz\right)$$

$$\Longleftrightarrow\ \hat h_\mu(\tau,\bar\tau) = h_\mu(\tau) - g^*_\mu(\tau,\bar\tau)$$ – modular form (completion) of weight $(w,0)$

$$g^*_\mu(\tau,\bar\tau) = \int_{\bar\tau}^{-i\infty} \frac{\overline{g_\mu(\bar z)}}{(\tau-z)^w}\, dz$$ – Eichler (period) integral

$$g_\mu(\tau) = (2i\tau_2)^w\, \partial_{\bar\tau} \hat h_\mu$$

Ex. 1 Quasimodular Eisenstein series

$$E_2(\tau) = \frac{1}{2\pi i}\partial_\tau \log\Delta(\tau) = 1 - 24\sum_{n=1}^{\infty} \frac{nq^n}{1-q^n}$$ – mock m.f. of $w=2$ with shadow $g = 6i/\pi$

$$\hat E_2(\tau,\bar\tau) = E_2(\tau) - \frac{3}{\pi\tau_2}$$ – modular

Ex. 2 Gen. fun. of Hurwitz class numbers $H(n)$

(# $PSL(2,\mathbb{Z})$ equiv. classes of integral binary quad. f. of discriminant n, divided by # automorphisms)

$$H_0(\tau) = \sum_{n\ge 0} H(4n)\, q^n = -\frac{1}{12} + \frac{1}{2}q + q^2 + \dots$$

$$H_1(\tau) = \sum_{n>0} H(4n-1)\, q^n = q^{3/4}\left(\frac{1}{3} + q + q^2 + \dots\right)$$

} v.v. mock m.f. of $w=\frac{3}{2}$ and shadow $\sim \theta_\mu = (\theta_3(2\tau), \theta_2(2\tau))$

$H(4n+1) = H(4n+2) = 0$ and therefore don't appear

$3H_\mu$ = normalized gen. fun. of $SU(2)$ VW invariants on $\mathbb{P}^2$

- Mixed mock m.f.

$$\hat h = h - \sum_j \underset{w+r_j}{f_j}\, \underset{2+r_j}{g^*_j}$$ i.e. shadow is a sum of products of hol. and anti-hol. modular forms

Important fact: weakly hol. modular and mock modular forms are determined by polar terms (form a finite dimensional vector space)

This is not true for mixed mock m.f.

- Higher depth mock modular – by iterations (f_j – depth $r-1$)

Jacobi forms

$\varphi(\tau,z) : \mathbb{H}\times\mathbb{C}\to\mathbb{C}$ is a Jacobi form of weight w and index m if it is holomorphic and

1) $\varphi(\tau, z+a\tau+b) = e^{-2\pi i m(a^2\tau+2az)}\varphi(\tau,z)$ $\quad a,b\in\mathbb{Z}$
elliptic property

2) $\varphi\left(\frac{a\tau+b}{c\tau+d}, \frac{z}{c\tau+d}\right) = (c\tau+d)^w e^{\frac{2\pi i m c z^2}{c\tau+d}}\varphi(\tau,z)$
modularity

They have a double Fourier expansion in τ and z

Moreover, they have a "theta expansion"

$$\varphi(\tau,z) = \sum_{\mu=0}^{2m-1} h_\mu(\tau)\,\Theta_\mu^{(m)}(\tau,z)$$

v.v.m.f. of weight $w-\frac{1}{2}$

$$\Theta_\mu^{(m)}(\tau,z) = \sum_{k\in 2m\mathbb{Z}+\mu} q^{\frac{k^2}{4m}}y^k \qquad y=e^{2\pi i z}$$

Obvious generalizations:

- weak holomorphicity
- multiplier system
- vector valuedness
- non-holomorphicity
- mock (of higher depth)

via theta expansion

Less obvious:

- multi-variabl Jacobi: $z\in\mathbb{C}^n$
  $\to$ index = matrix
  $mz^2 \to \sum_{i,j} z^i m_{ij} z^j$
- Jacobi-like
  drops elliptic property
  $\to$ loses theta expansion
  but still gets an infinite set of m.f. via Laurent expansion in z

Ex. Jacobi theta function

$$\Theta_1(\tau,z) = \sum_{k\in\mathbb{Z}+\frac{1}{2}} q^{k^2/2}(-y)^k$$

$w=\frac{1}{2}$, $m=\frac{1}{2}$, $M(T)=e^{\pi i/4}$, $M(S)=e^{-3\pi i/4}$

Indefinite theta series

Λ - d-dimensional lattice with a bilinear form $x\cdot y$

Λ^* - dual lattice $\{x : x\cdot y \in \mathbb{Z} \text{ for } \forall y \in \Lambda\}$

$\mu \in \Lambda^*/\Lambda$ - residue class

$p \in \Lambda$ - characteristic vector : $k^2 + k\cdot p \in 2\mathbb{Z}$ for $\forall k \in \Lambda$

If quadratic form k^2 is positive definite, then

$$\theta_\mu(\tau, \vec{z}) = \sum_{k \in \Lambda + \mu + \frac{1}{2}p} (-1)^{k\cdot p}\, q^{k^2/2}\, e^{2\pi i z\cdot k}$$

- (multi-variable) Jacobi form of $w = \frac{d}{2}$ and $m = \frac{1}{2}$. matrix of bilinear form

What if the quadratic form is indefinite?

- Convergence

If we want a holomorphic theta series, the only possibility is to restrict the sum to the positive definite cone using sign functions.

Ex Lorentzian signature $(d-1, 1)$

insert the kernel $\Phi(k) = \mathrm{sgn}(v_1\cdot k) - \mathrm{sgn}(v_2\cdot k)$

convergence holds for $v_1^2, v_2^2, v_1\cdot v_2 < 0$

- Modularity

The sign functions spoil modularity

$\rightarrow \theta_\mu$ - mixed mock.

Recipe for completion : $\mathrm{sgn}(v\cdot k) \longmapsto \mathrm{Erf}\left(\sqrt{\frac{2\pi\tau_2}{-v^2}}\, v\cdot(k-\beta)\right)$ where we represented $z = \alpha - \tau\beta$

What happens for other signatures?

- Convergence is ensured by (for $\mathrm{sgn} = (d-n, n)$)

$$\Phi(k) = \prod_{i=1}^{n} \left(\mathrm{sgn}(v_{1,i}\cdot k) - \mathrm{sgn}(v_{2,i}\cdot k)\right)$$

with some conditions on $v_{s,i}$

- Recipe for completion

$$\prod_{i=1}^{n} \mathrm{sgn}(v_i\cdot k) \longmapsto \Phi_n^E\left(\{v_i\}; \sqrt{2\tau_2}(k-\beta)\right)$$

(boosted) generalized error functions

- These are mock Jacobi forms of depth n.

Generalized error functions

- $$E_n(\mathcal{M};u) = \int_{\mathbb{R}^n} du' \, e^{-\pi\sum_{k=1}^{n}(u_k-u'_k)^2} \prod_{i=1}^{n} \mathrm{sgn}(\mathcal{M}^T_{ij}u'_j)$$

  $\nearrow$ n×n matrix of parameters $\nwarrow$ n-vector

  - $E_1(u) = \mathrm{Erf}(\sqrt{\pi}\,u)$
  - $E_n(\mathcal{M};u) \xrightarrow[u\to\infty]{} \prod_{i=1}^{n} \mathrm{sgn}(\mathcal{M}^T_{ij}u_j)$

    (for $\mathcal{M}^T_{ij}u_j \neq 0$ - important!)
  - many symmetries ($n=2 \to 1$ parameter
    $3 \to 3$)

- $\exists$ $\{v_i\}$ – set of n vectors in $\mathbb{R}^{d-m,m}$ spanning $\mathbb{R}^{0,n}$, $n\leq m$

  $B=\{e_i\}$ - basis in $\mathbb{R}^{0,n}$

  $$\Phi^E_n(\{v_i\};x) = E_n(\mathcal{M};B\cdot x) \qquad \mathcal{M}_{ij}=e_i\cdot v_j$$

  $\nearrow$ d-vector

  $$= \int_{\mathrm{Span}\{v_i\}} dx' \, e^{+\pi(P(v)\cdot x-x')^2} \prod_{i=1}^{n} \mathrm{sgn}(v_i\cdot x')$$

  - do not depend on choice of B
  - ensure modularity
  - $v_\ell^2=0 \Rightarrow \Phi^E_n(\{v_i\};x) = \mathrm{sgn}(v_\ell\cdot x)\,\Phi^E_{n-1}(\{v_i\}_{i\neq\ell};x)$

As a result, null vectors do not require completion!

But convergence requires:

1) v_ℓ (or its rescaling) belongs to Λ
   → rational coefficients
2) presence of $\beta\neq 0$
   → works only for Jacobi

Calabi - Yau manifolds

- Calabi-Yau manifold – $2n$ real dimensional
  with SU(n) holonomy group

(Parallel transport operators $U_\gamma = \mathcal{P} e^{\int \omega}$ form a group

$\gamma = \gamma_1 \vee \gamma_2 \Rightarrow U_\gamma = U_{\gamma_1} U_{\gamma_2}$

γ - closed curve $\longrightarrow U_\gamma$ - holonomy

group $\longrightarrow \mathrm{Hol}(\mathcal{M}) \subseteq O(2n)$)

This is not very practical definition. One can show:

$\mathrm{CY} \longleftrightarrow$ Kähler and Ricci flat

$\updownarrow$ U(n) holonomy

$\updownarrow$ $c_1(\mathcal{M}) = 0$

$R_{\mu\nu}$ plays the role of the curvature for U(1) in U(n).

Inverse statement – famous theorem of Yau conjectured by Calabi: there is a unique Ricci flat metric for a Kähler manifold with $c_1 = 0$.

- (1,0)-forms belong to $[n]$ - fund. repr. of SU(n)

$$\underbrace{[n] \otimes \cdots \otimes [n]}_{n \text{ times}} = \underbrace{[1]}_{\text{singlet} \sim \varepsilon_{i_1 \ldots i_n}} \oplus \ldots \quad \Rightarrow \exists \underbrace{\text{covariantly constant}}_{\Omega \in H^{n,0}(\mathcal{M})}$$

In fact: $c_1 = 0 \Leftrightarrow \nabla \Omega = 0 \Leftrightarrow \Omega$ is nowhere vanishing

- CY is characterized by 2 compatible structures:

ω	Ω
Kähler str.	complex str.
gives the metric $g(u,v) = \omega(u, \mathcal{J} v)$	determines holomorphic coordinates $\Omega \sim dz^1 \wedge \ldots \wedge dz^n$

Compatibility: $d\omega = d\Omega = 0$

$$\Omega \wedge \omega = 0 \,, \quad \Omega \wedge \bar{\Omega} \sim \underbrace{\omega \wedge \ldots \wedge \omega}_{n}$$

- Examples

$n=1$: T^2 (since curvature must vanish)

$n=2$: K3 – the only 4d CY, realized as $x^4 + y^4 + z^4 + w^4 = 0$ in $\mathbb{CP}^3$

$n=3$: most interesting case.

Many examples: hypersurfaces in projective spaces

Quintic: $\{ x = [x_0 : x_1 : x_2 : x_3 : x_4] \in \mathbb{CP}^4 : P(x) = 0 \}$ where P - degree 5 hom. polynom

Hodge numbers

- On complex manifolds $d = \partial + \bar\partial$, $\partial^2 = \bar\partial^2 = 0$

$$\Lambda^k = \bigoplus_{p+q=k} \Lambda^{(p,q)} \leftarrow \begin{array}{l} p \text{ holomorphic} \\ q \text{ anti-holom-c} \\ \text{indices} \end{array}$$

Dolbeault cohomology

$$H^{p,q}_{\bar\partial} = \frac{\mathrm{Ker}(\bar\partial(\Lambda^{(p,q)}))}{\mathrm{Im}(\bar\partial(\Lambda^{(p,q-1)}))} = \frac{\bar\partial\text{-closed}}{\bar\partial\text{-exact}}$$

$h^{p,q} = \dim H^{p,q}_{\bar\partial}$ – Hodge number

$= \{\text{harmonic forms of } \Delta_{\bar\partial} = \bar\partial\bar\partial^+ + \bar\partial^+\bar\partial\}$

For Kähler manifolds $\Delta_{\bar\partial} = \Delta_{\partial} = \frac{1}{2}\Delta_d$

$\Rightarrow H^{p,q}_{\bar\partial} = H^{p,q}_{\partial} \equiv H^{p,q}$

$H^k = \bigoplus_{p+q=k} H^{p,q}$, $b_k = \sum_{p=0}^{k} h^{p,k-p}$ – Betti numbers

$h^{p,q} = h^{q,p} = h^{n-p,n-q}$

- For CY manifolds

$h^{n,0} = h^{0,n} = 1 \quad \leftarrow \exists!\,\Omega$

$h^{0,0} = h^{n,n} = 1 \quad \leftarrow$ for any complex manifold

$h^{k,0} = h^{0,k} = 0$ for $1 \leq k \leq n-1$

- Hodge diamond for CY with $n=3$

$$\begin{array}{ccccccc} & & & 1 & & & \\ & & 0 & & 0 & & \\ & 0 & & h^{1,1} & & 0 & \\ 1 & & h^{2,1} & & h^{2,1} & & 1 \\ & 0 & & h^{1,1} & & 0 & \\ & & 0 & & 0 & & \\ & & & 1 & & & \end{array}$$

- only 2 unfixed numbers: $h^{2,1}$, $h^{1,1}$
- Euler characteristic $\chi = 2(h^{1,1} - h^{2,1})$
- Symmetry $h^{1,1} \leftrightarrow h^{2,1}$ – mirror symmetry

for any CY there is its mirror image (except rigid CYs with $h^{21} = 0$)

Ex. Quintic

$h^{1,1} = 1$, $h^{2,1} = 101$

Moduli spaces of CY

Let us fix a topology of a CY

What are its possible continuous deformations?

$\longrightarrow$ Moduli space

$$\text{Moduli space} = \begin{matrix}\text{deformations}\\ \text{of Kähler str.}\end{matrix} \times \begin{matrix}\text{deformations}\\ \text{of complex str.}\end{matrix}$$

Kähler moduli space $\mathcal{M}_K$

For a fixed complex structure, $\mathcal{M}_K \subset H^{1,1}(CY;\mathbb{R})$

such that $\displaystyle\int_{\Gamma_{2k}} \omega^k > 0$ for all $2k$-cycles Γ_{2k}, $k=1,\dots,n$

$\rightarrow$ $\mathcal{M}_K$ - Kähler cone

Locally, these conditions are irrelevant, so

$$\underset{\substack{\text{space of infinitesimal}\\ \text{deformations}}}{T(\mathcal{M}_K)} = H^{1,1}(CY;\mathbb{R})$$

More natural to consider complexification

$\mathcal{M}_K \sim H^{1,1}(CY;\mathbb{C})$

Complex structure moduli space $\mathcal{M}_c$

Complex structure is determined by $\Omega = f(z)\, dz^1 \wedge dz^2 \wedge dz^3$

Change of complex str. $\leftrightarrow$ change of $z^i \rightarrow z^i + \varepsilon^i_{\bar{j}} \bar{z}^{\bar{j}}$

$$\Rightarrow \delta\Omega = \varepsilon^i_{\bar{j}} \Big(\underset{(3,0)\text{-form}}{\bar{z}^{\bar{j}}\, \partial_i f\, dz^1 \wedge dz^2 \wedge dz^3} + \underset{(2,1)\text{-form}}{\tfrac{1}{2} f\, \varepsilon_{ik\ell}\, d\bar{z}^{\bar{j}} \wedge dz^k \wedge dz^\ell} \Big)$$

$\Rightarrow$ deformations are classified by (2,1)-forms

$\Rightarrow$ locally $\mathcal{M}_c \sim H^{2,1}(CY;\mathbb{R})$

These results agree with the fact that mirror symmetry exchanges $\mathcal{M}_K$ with $\mathcal{M}_c$

Metric on $\mathcal{M}_K$

On $\mathcal{M}_K$ and $\mathcal{M}_C$ one can introduce natural metrics such that:

- they are exchanged under mirror symmetry
- they become special Kähler manifolds

This means that the Kähler potential $K(z,\bar z)$, where $z^{\dot a}$, $\dot a = 1,\dots,n$ are complex coordinates on this manifold, is captured by a holomorphic function $F(X^I)$ $I=0,1,\dots n$ homogeneous of degree 2:

$$K = -\log\left[i\left(\bar X^I F_I - X^I \bar F_I\right)\right]\Big|_{\substack{X^{\dot a}=z^{\dot a}\\ X^0=1}}$$

F - holomorphic prepotential

Consider $\mathcal{M}_K$ and let ω_i - a basis of $H^2(CY;\mathbb{Z})$

$$\exists \quad B + i\omega = (b^{\dot a} + i t^{\dot a})\,\omega_{\dot a} = z^{\dot a}\omega_{\dot a}$$

B-field in string theory

Kähler form

coordinates on $\mathcal{M}_K$

The most natural metric: $\quad g_{\dot a \dot b} = \dfrac{1}{V}\displaystyle\int_{CY} \omega_{\dot a} \wedge * \omega_{\dot b}$ where $V = \dfrac{1}{6}\displaystyle\int_{CY}\omega^3$

Exercise: show that $g_{i\bar j}$ is special Kähler with $K = -\log(8V)$ and $F(X) = -\dfrac{1}{6X^0}\kappa_{\dot a\dot b\dot c}X^{\dot a}X^{\dot b}X^{\dot c}$

where $\kappa_{\dot a\dot b\dot c} = \displaystyle\int_{CY}\omega_{\dot a}\wedge\omega_{\dot b}\wedge\omega_{\dot c}$ - intersection numbers of CY

[Hint: $\quad *\omega_{\dot a} = -\left(\omega\wedge\omega_{\dot a} - \left(\dfrac{1}{4V}\displaystyle\int_{CY}\omega_{\dot a}\wedge\omega\wedge\omega\right)\omega\wedge\omega\right)$]

However, this metric is naive because it does not satisfy mirror symmetry. The complete metric follows from

$$F(X) = -\frac{1}{6X^0}\kappa_{\dot a\dot b\dot c}X^{\dot a}X^{\dot b}X^{\dot c} + \underbrace{\frac{1}{2}A_{IJ}X^I X^J}_{\text{topological term}} + \overset{\text{Euler character}}{\chi_{CY}}\,\frac{\zeta(3)(X^0)^2}{2(2\pi i)^3}$$

perturbative α'-correction

$$-\frac{(X^0)^2}{(2\pi i)^3}\sum_{\beta\in H^2_+(CY;\mathbb{Z})} GV^{(0)}_\beta\, \mathrm{Li}_3\left(e^{2\pi i\beta_{\dot a}X^{\dot a}/X^0}\right)$$

genus 0 GopaKumar-Vafa inv.

contribution of world-sheet instantons

The additional terms - α' quantum corrections in string theory determined by topological invariants of CY

DT invariants

One can define the so-called generalized Donaldson-Thomas invariants $\Omega(\gamma)$

Math	Physics (explanations - next time)
• count semi-stable coherent sheaves	• BPS states in Type IIA/CY (bound states of D6-D4-D2-D0 branes)
• γ - Chern character (ch_0, ch_1, ch_2, ch_3)	• γ - e/m charge (p^0, p^a, q_a, q_0)

$$\gamma(E) = ch(E)\sqrt{Td(TX)} = p^0 + p^a\omega_a - q_a\omega^a + q_0\omega_X \qquad \left(Td(TX) = 1 + \frac{c_1}{2} + \frac{c_1^2 + c_2}{12} \quad \text{but } c_1 = 0 \right)$$

$p^0 = ch_0$ - rank of the sheaf

$p^0 = 1$ - standard DT

$p^0 = -1$ - PT

$p^0 = 0$ - the case we are interested in (the sheaf is supported on a divisor - 4 cycle of CY)

In this case $\Omega(\underset{\nwarrow \text{ specifies the divisor}}{p^a}, q_a, q_0)$

But $\Omega(\gamma)$ is invariant under $ch(E) \to e^{\varepsilon^a\omega_a} ch(E)$, $\varepsilon^a \in \mathbb{Z}$

(math: tensoring with a line bundle

phys: spectral flow symmetry)

$\to$ $q_a = \kappa_{ab}\varepsilon^b + \mu_a + \frac{1}{2}\kappa_{ab}p^b$ where $\kappa_{ab} = \kappa_{abc}p^c$ - quadratic form on the lattice $\Lambda_p = H_4(X, \mathbb{Z})$

$\hat{q}_0 = q_a - \frac{1}{2}\kappa^{ab}q_a q_b$ - invariant $\qquad \mu \in \Lambda^*_p / \Lambda_p$ - residue class

$$\Omega(\gamma) = \Omega_{p,\mu}(\hat{q}_0)$$

Mathematical fact: Bogomolov bound $\hat{q}_0 \leq \frac{1}{24}\chi(\overset{\swarrow \text{divisor}}{\mathcal{D}_p})$

$$\chi(\mathcal{D}_p) = \kappa_{abc}p^a p^b p^c + c_{2,a}p^a$$

$\to$ One can define v.v. generating function

$$h_{p,\mu} = \sum_{\hat{q}_0 \leq \hat{q}_0^{max}} \bar{\Omega}_{p,\mu}(\hat{q}_0)\, e^{-2\pi i \hat{q}_0 \tau} \qquad \text{where } \bar{\Omega}(\gamma) = \sum_{d|\gamma} \frac{1}{d^2}\Omega(\gamma/d)$$

rational invariant

These functions possess nice modular properties!

Elements of string theory

- The main idea is that fundamental objects are 1-dimensional

Usual particles — different excitations
→ possibility of unification

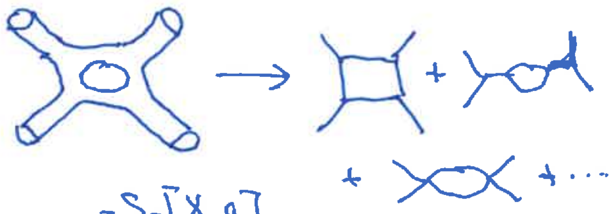
In particular, the spectrum of closed string always has massless spin 2 particle – graviton → quantum gravity

- How to describe?

Propagating strings → 2d surfaces (worldsheets) in spacetime
→ 2d theory on worldsheet (Polyakov action)

Note: • string interactions are determined by the theory of a single string

- a single string amplitude can generate many particle diagrams



$$Z_{str} = \sum_{\text{string config.}} e^{-S_p} = \sum_{\text{2d topol.}} \int dX^\mu \, dg_{ab} \, e^{-S_p[X,g]}$$

$e^{-S_p[X,g]}$ ← generalizes the worldsheet area

dg_{ab} ← metric on worldsheet

dX^μ ← embedding coordinates into target space

→ 2 types of quantum effects:
1) on worldsheet → powers of α' (inverse string tension)
2) in target space → loops counted by topologies of worldsheet
powers of string coupling g_s

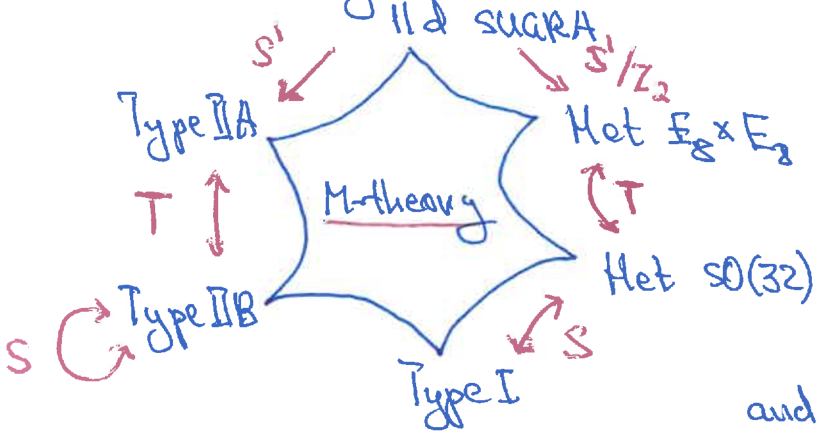
- Polyakov action is invariant under Weyl rescaling $g_{ab} \to \Lambda g_{ab}$
→ 2d confomal field theory

This symmetry must be preserved in quantum theory (no anomaly)
→ β-functions of various couplings must vanish
The coupligs — functions on target space $g_{\mu\nu}(X), B_{\mu\nu}(X), \Phi(X)$.
describe a background
↕
string modes

→ Equations on background fields
(Einstein equations + ...)
⇓
Effective field theory

Different string theories and dualities

- In bosonic string theory, the conformal anomaly cancellation requires $d=26$
  → String theory predicts the dimension of spacetime!
  But there are other bad features:
  - tachyon $(m_T^2 < 0)$
  - no fermions
  - no exceptional gauge groups which often appear in grand unified theories
- All problems are solved by adding supersymmetry
  (It is introduced on worldsheet, but also arises in target space.)
  Various possibilities how to realize, but also new possible anomalies due to fermions
  → All anomalies cancel in 5 theories all living in $d=10$ and reducing at low energies to SUGRA in 10d.



They are all related by dualities

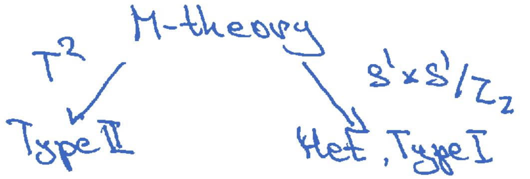
T-duality : $R \to 1/R$

S-duality : $g_s \to 1/g_s$

and can be seen as different vacua of a single mysterious M-theory

M-theory has an additional vacuum where it reduces to 11d SUGRA and appears as a theory of membranes

This explains S-duality



$R_2 \gg R_1 \Rightarrow$ IIA, $g_s \sim R_1$

$R_1, R_2 \to 0 \Rightarrow$ IIB, $g_s \sim \frac{R_1}{R_2}$

$R_2 \gg R_1 \Rightarrow$ Het $E_8 \times E_8$

$R_1 R_2 \to 0 \Rightarrow$ Het SO(32)
Type I

This shows that S-duality is realized by $SL(2,\mathbb{Z})$ modular group of the torus

Dualities require the existense of other extended object - branes

M-theory: M2, M5

Type IIA: Dp-even, NS5

Type IIB: Dp-odd, NS5

(Open strings end on D-branes)

Compactifications

The standard method to relate a 10d (or 11d) theory to 4 dimensions is to compactify it.

$$\mathcal{M}_{10} = \mathcal{M}_4 \times \mathcal{M}_6 \leftarrow \text{compact manifold of "small" size not seen at low energies}$$

$\downarrow$

Effective theory on $\mathcal{M}_4$ strongly depends on $\mathcal{M}_6$

Let us find when it is supersymmetric $\to$ $\exists$ supercharge Q_α

$$\Rightarrow \quad 0 \underset{\text{susy}}{=} \langle 0 | \{ Q_\alpha , \underset{\substack{\uparrow \\ \text{any fermionic field}}}{\chi_\beta} \} | 0 \rangle = \langle 0 | \delta_\alpha \chi_\beta | 0 \rangle \underset{\text{class}}{\sim} \delta_\alpha \chi_\beta \quad - \text{SUSY transf.}$$

In SUGRA:
- dilatino $\delta\lambda \sim (\Gamma^M \partial_M \varphi)\varepsilon \longrightarrow \varphi = \text{const}$
- gravitino $\delta\psi_M \sim \nabla_M \varepsilon \longrightarrow \exists$ covariantly constant spinor

$$\varepsilon = \xi_+ \otimes \eta_+ + (\xi_+)^* \otimes (\eta_+)^*$$

(this corresponds to decomposition
$SO(1,9) \to SO(1,3) \times SO(6)$
$16 \to (2,4) \oplus (\bar{2},\bar{4})$)

$$\nabla_M \varepsilon = 0 \Rightarrow \partial_\mu \xi_+ = 0 \quad \text{and} \quad \nabla_m \eta_+ = 0$$

$\downarrow$ $N=1$ SUSY in 4d

$\downarrow$ cov. const. spinor on $\mathcal{M}_6$

$\Downarrow$

$$\underset{\text{parallel transport}}{U_\gamma}\, \eta = \eta \quad \text{for all loops } \gamma$$

In general $U_\gamma \in SO(6) \simeq SU(4)$, $\eta \in [4]$ - fund. repr. of $SU(4)$

$$\exists\, \eta = \begin{pmatrix} 0 \\ 0 \\ 0 \\ \eta_0 \end{pmatrix} \Rightarrow U_\gamma \in SU(3) \subset SU(4)$$

(corresponds to $[4] \to [3] \oplus \underline{[1]}$)

$$\Rightarrow \text{Hol}(\mathcal{M}_6) = SU(3) \longrightarrow \underline{\mathcal{M}_6 \text{ - Calabi-Yau}}$$

In particular:

$$\omega_{mn} = -2i\, \eta_+^\dagger \Gamma_{mn} \eta_+$$

$$\Omega_{mnk} = -2i\, (\eta_+^*)^\dagger \Gamma_{mnk} \eta_+$$

In Type II string theory, we have two supercharges

One cov. const. spinor on $\mathcal{M}_6$ then gives two SUSY in 4d

IIA:

$$\varepsilon^1 = \xi_+^1 \otimes \eta_+ + (\xi_+^1)^* \otimes \eta_+^*$$

$$\varepsilon^2 = \xi_+^2 \otimes \eta_+^* + (\xi_+^2)^* \otimes \eta_+$$

IIB:

$$\varepsilon^i = \xi_+^i \otimes \eta_+ + (\xi_+^i)^* \otimes \eta_+^*$$

$\longrightarrow$ $N=2$ SUGRA in 4d

4d $\mathcal{N}=2$ SUGRA

It can always be written as a coupling of
- grav. mult. $(g_{\mu\nu}, A^0_\mu)$
- n_V vector mult. (A^a_μ, z^a)
- n_H hypermult. (q^u)

$$S_{bos} = \int_{\mathcal{M}_4} \Big[\tfrac{1}{2} R \star 1 - \tfrac{1}{2} \mathrm{Im}\, \mathcal{N}_{IJ}(z) F^I \wedge \star F^J - \tfrac{1}{2} \mathrm{Re}\, \mathcal{N}_{IJ}(z) F^I \wedge F^J + G_{a\bar b}\, dz^a \wedge \star d\bar z^b + h_{uv}(q)\, dq^u \wedge \star dq^v \Big]$$

$\mathcal{N}=2$ SUSY $\Rightarrow$
- $G_{a\bar b}(z)$ - special Kähler metric determined by a prepotential $F(X)$
- $$\mathcal{N}_{IJ} = \bar F_{IJ} - i \frac{(NX)_I (NX)_J}{(XNX)} \qquad N_{IJ} = -2\, \mathrm{Im}\, F_{IJ}$$
- $h_{uv}(q)$ - quaternion-Kähler metric

→ The effective theory is determined by the moduli space which is a product

$$\begin{tikzcd} & \mathcal{M}_V \times \mathcal{M}_H & \\ \text{sp.K.} \arrow[ur] & & \text{QK.} \arrow[ul] \end{tikzcd}$$

Type IIA		Type IIB	
$n_V = h^{1,1}$	$n_H = h^{2,1} + 1$	$n_V = h^{2,1}$	$n_H = h^{11} + 1$
$\mathcal{M}_V = \mathcal{M}_K(CY)$	$\mathcal{M}_H$ - ext. of $\mathcal{M}_C$	$\mathcal{M}_V = \mathcal{M}_C(CY)$	$\mathcal{M}_H$ - ext. of $\mathcal{M}_K$

mirror symmetry

$$\mathcal{M}^A_V(CY) = \mathcal{M}^B_V(\widetilde{CY}) \qquad \mathcal{M}^A_H(CY) = \mathcal{M}^B_H(\widetilde{CY})$$

$$\mathrm{IIA}/_{CY} \simeq \mathrm{IIB}/\widetilde{CY}$$

In the classical approximation, the metrics can be computed by Kaluza-Klein reduction of 10d SUGRA

(In particular: $F(X) \simeq -\frac{1}{6}$ [illegible] $X^a X^b X^c / X^0$)

But there are also quantum corrections:
- α'-corrections affect $\mathcal{M}^A_V = \mathcal{M}_K(CY)$ and $\mathcal{M}^B_H$
- g_s- corrections affect $\mathcal{M}_H$
  - perturbative: only 1-loop
  - non-perturbative: branes wrapping non-trivial cycles of CY

We know how to compute all D-brane instantons, but not NS5-brane instantons

Modularity of DT invariants

Type IIB is self-dual under S-duality

$\Rightarrow$ $\mathcal{M}_H$ carries an isometric action of $SL(2,\mathbb{Z})$

$\downarrow$

holds classically

imposes restrictions on quantum corr.

generic D-instanton of charge $\gamma = (p^0, p^a, q_a, q_0)$

– bound state of D5-D3-D1-D(-1) branes wrapping 6-4-2-0 cycles

$$\sim \Omega(\gamma)\, e^{-\mathrm{Vol}(\Gamma_\gamma)/g_s}$$

$\uparrow$ DT invariant

S-duality: D5 $\leftrightarrow$ NS5 $\Rightarrow$ no restrictions

D3 $\leftrightarrow$ D3 $\Rightarrow$ restrictions on rank 0 DT

These restrictions can be derived explicitly!

Result: for reducible divisor $\mathcal{D}_p$ of degree r $\left(\mathcal{D}_p = \sum_{i=1}^{r} \mathcal{D}_{p_i}\right)$ the generating function of rank 0 DT invariants $h_{p,\mu}(\tau)$ is a v.v. mock modular form of depth $r-1$ and weight $-\frac{b_2}{2}-1$

$\rightarrow$ $r=1$ (irreducible divisor) – modular

$r=2$ – mixed mock modular

Consider

$$\begin{tikzcd} \frac{\text{IIB}}{CY\times S^1_R} \arrow[r, leftrightarrow, "\text{T-duality}"] & \frac{\text{IIA}}{CY\times S^1_{1/R}} \\ \hat{\mathcal{M}}_V^B \times \mathcal{M}_H^B \arrow[r, bend left, leftrightarrow] \arrow[r, bend right, leftrightarrow] & \hat{\mathcal{M}}_V^A \times \mathcal{M}_H^A \end{tikzcd}$$

$\hat{\mathcal{M}}_V$ - ext of $\mathcal{M}_V$ which is QK

D-instantons $\leftrightarrow$ instantons from BPS states going around S^1

$\parallel$

bound states of D6-D4-D2-D0 branes wrapping 6-4-2-0 cycles

$\parallel$

BPS black holes

DT invariants count states of BPS black holes

Modular anomaly

- It is best characterized by a formula for the modula completion

$$\hat{h}_{p,\mu}(\tau,\bar\tau) = h_{p,\mu}(\tau) + \sum_{n=2}^{n} \sum_{\sum_{i=1}^{n} p_i = p} \sum_{\{\mu_i\}} R^{\{p_i\}}_{\mu,\{\mu_i\}}(\tau,\bar\tau) \prod_{i=1}^{n} h_{p_i,\mu_i}(\tau)$$

sum over bound states

indefinite theta series on $\oplus_{i=1}^{n} \Lambda_{p_i}/\Lambda_p$ with a kernel given by a sum over trees, weight by sign functions and derivatives of gen.err.fu (details later)

- The original derivation assumed:
  - compact CY
  - ample divisor $\mathcal{D}_p$

$\hookrightarrow$ 1) $\mathrm{sign}(\Lambda_p) = (1, b_2-1) \Rightarrow$ indefinite theta series

2) $p^3 = \kappa_{abc} p^a p^b p^c > 0$

All these assumptions can be relaxed!

- $\mathcal{D}_p$ can be effective

Then it may happen that $\Big[$ $\mathrm{rank}(\Lambda_p) < b_2 \Rightarrow w = -\frac{1}{2}\mathrm{rank}(\Lambda_p) -$

$p^3 = 0 \Rightarrow h_{p,\mu}$ - quasi-modular (anomaly captured by E_2)

- limit of non-compact CY

For example, take CY to be an elliptic fibration over a smooth Fano surface with $b_2^+(S) = 1$, $b_1(S) = 0$.

The limit of large fiber $\longrightarrow X = \mathrm{Tot}(K_S)$

rank 0 DT $\longrightarrow$ VW invariants on S for $G = U(N)$

(The only divisor remaining compact $= [S]$ (base of fibration)

$\hookrightarrow$ $p^a = N \underset{\overset{\shortparallel}{(1,-c_1^a(S))}}{p_0^a} \longrightarrow$ all charges collinear

N - degree of reducibility)

$\Downarrow$

anomaly equation for VW invariants.

Refinement

For non-compact CYs one can introduce refined DT invariants giving access to Betti numbers of moduli spaces of sheaves

$$\Omega(\gamma,y) = \sum_{p=0}^{2d} (-y)^{p-d}\, b_p(\mathcal{M}_\gamma)$$

rational version
$$\bar\Omega(\gamma,y) = \sum_{m|\gamma} \frac{y-1/y}{m(y^m-1/y^m)}\, \Omega(\gamma/m, y^m)$$

compact case - ?

generating function for rank 0
$$h^{\rm ref}_{p,\mu}(\tau,z) = \sum_{\hat q_0 \le \hat q_0^{\max}} \frac{\bar\Omega_{p,\mu}(\hat q_0,y)\, e^{-2\pi i \hat q_0 \tau}}{y-1/y} \qquad y=e^{2\pi i z}$$

← generates a pole in the unrefined limit

Claim: $h^{\rm ref}_{p,\mu}$ - higher depth mock Jacobi form with completion

$$\hat h^{\rm ref}_{p,\mu}(\tau,\bar\tau,z) = h^{\rm ref}_{p,\mu}(\tau,z) + \sum_{n=2}^{n} \sum_{\sum_{i=1}^n p_i = p} \sum_{\{\mu_i\}} R^{\{p_i\},\rm ref}_{\mu,\{\mu_i\}}(\tau,\bar\tau,z) \prod_{i=1}^n h^{\rm ref}_{p_i,\mu_i}(\tau,z)$$

of weight $w = -\frac{1}{2}\operatorname{rank}(\Lambda_p) = -\frac{b_2}{2}$

and index $m = -\chi(\mathcal{O}_{\mathcal{D}_p}) - \lambda_a p^a$

↑ arithmetic genus of $\mathcal{D}_p$ $= \frac{p^3}{6} + \frac{c_{2,a}p^a}{12} \in \mathbb{Z}$

Here

$$R^{\{p_i\},\rm ref}_{\mu,\{\mu_i\}} = \sum_{\substack{q_i \in \Lambda_{p_i}+\mu_i+\frac{1}{2}p_i \\ \sum q_i = \mu + p/2}} \operatorname{Sym}\left\{ R^{\rm ref}_n(\{\hat\gamma_i\};\underline{\tau_2,\beta})\, (-y)^{\sum_{i<j}\gamma_{ij}} \right\} e^{\pi i \tau Q_n(\{\hat\gamma_i\})}$$

$$\downarrow$$

$$\frac{1}{2^{n-1}} \sum_{T\in\mathbb{T}^S_n} (-1)^{n_T-1}\, \mathcal{E}^{(+)\rm ref}_{v_0} \prod_{v\in V_T\setminus\{v_0\}} \mathcal{E}^{(0)\rm ref}_v$$

↑ Schröder trees (rooted, planar with $k_v \ge 2$)

↑ root

↑ set of vertices (w/o leaves)

$\hat\gamma_i = (p_i,q_i)$

$\gamma_{ij} = \langle\hat\gamma_i,\hat\gamma_j\rangle = q_{i,a}p^a_j - q_{j,a}p^a_i$

$Q_n(\{\hat\gamma_i\}) = \kappa^{ab} q_a q_b - \sum_{i=1}^n \kappa_i^{ab} q_{i,a} q_{i,b}$

$z = \alpha - \tau\beta$, $\alpha,\beta \in \mathbb{R}$

$$\mathcal{E}^{(\rm ref)}_n(\{\hat\gamma_i\};\tau_2,\beta) = \Phi^E_{n-1}\left(\{v_e\}; \sqrt{2\tau_2}(\mathbf{q}+\beta\Theta)\right)$$

$$\shortparallel$$

$$\mathcal{E}^{(0)\rm ref}_n + \mathcal{E}^{(+)\rm ref}_n \qquad \mathcal{E}_v \equiv \mathcal{E}_{k_v}(\{\hat\gamma_{v'}\})$$

$$\shortparallel$$

$$\lim_{\tau_2\to\infty} \mathcal{E}^{\rm ref}_n(\{\hat\gamma_i\};\tau_2,0) = \prod_{e=1}^{n-1}\operatorname{sgn}(\Gamma_e)$$

provided $\sin^m(0) = \begin{cases} 0 & m \text{ odd} \\ \frac{1}{m+1} & m \text{ even}\end{cases}$

$$\Gamma_e = \sum_{i=1}^{e}\sum_{j=e+1}^{n} \gamma_{ij} \qquad v_e = \sum_{i=1}^{e}\sum_{j=1}^{n} v_{ij}, \quad \Theta = \sum_{i<j} v_{ij}$$

$\mathbf{q}, v_{ij} \in \Lambda = \bigoplus_{i=1}^n \Lambda_{p_i}/\Lambda_p$ such that $v_{ij}\cdot\mathbf{q} = \gamma_{ij}$

Unrefined limit $R = \lim_{y\to1}\left[\frac{R^{\rm ref}}{(y-1/y)^{n-1}}\right]$ ← apply l'Hopital's rule

↳ new trees and derivatives

Example $r=2$

Consider the case where (relevant for VW):

- $p^2 = 2p_0^2$
- Λ_{p_0} – unimodular of signature $(1, b_2-1)$
- $\exists\, v_0 \in \Lambda_{p_0} : v_0^2 = 0$ – null vector

Due to unimodularity, $h_1 \equiv h_{p_0}$ is scalar valued

$\longrightarrow$ one can define $\quad g_{2,\mu} = h_{2p_0,\mu} / h_1^2$

$\longrightarrow$ anomaly equation

$$\hat{g}_{2,\mu} = g_{2,\mu} + \frac{1}{2} \sum_{k\in\Lambda+\mu} \left(\mathrm{Erf}\left(2\sqrt{\frac{\pi\tau_2}{p_0^2}}\,(p_0\cdot k + \beta p_0^2)\right) - \mathrm{sgn}(p_0\cdot k)\right) q^{-k^2}\, y^{2p_0\cdot k}$$

To get a modular theta series, we need a kernel given by the difference

$$\underbrace{\mathrm{Erf}\left(2\sqrt{\frac{\pi\tau_2}{v^2}}\,(v\cdot(k+\beta p_0))\right)}_{v=p_0} - \underbrace{\mathrm{Erf}\left(2\sqrt{\frac{\pi\tau_2}{v_0^2}}\,(v_0\cdot(k+\beta p_0))\right)}_{\substack{\text{impossible to get} \\ \text{due to non-holom.} \\ \text{except if } v_0^2=0 \\ \downarrow \\ \mathrm{sgn}(v_0\cdot(k+\beta p_0))}}$$

$\Rightarrow$ Obvious solution

$$g_{2,\mu} = \frac{1}{2}\sum_{k\in\Lambda+\mu} \left(\mathrm{sgn}(p_0\cdot k) - \mathrm{sgn}(v_0\cdot(k+\beta p_0))\right) q^{-k^2}\, y^{2c\cdot k}$$
$$+ \ \varphi_{2,\mu} \longleftarrow \text{holomorphic modular ambiguity}$$

(Jacobi form)

The ambiguity is severily restricted by the condition of having well-defined unrefined limit $y\to 1$:

$g_{2,\mu}$ must have a zero at $y=1$, but the theta series has a pole

For VW, this is enough to fix the ambiguity.

General strategy

1.) Find the general solution at depth r, which is parametrized by holomorphic modular ambiguities for all smaller depths $r' \leq r$

2) Fix the ambiguities

- For compact CYs:
  - the lattice for irreducible divisors is not in general unimodular
  - does not have null vectors
  - no well-defined refined invariants

The first problem is solved by introducing "anomalous coefficients" (instead of normalized functions)

$$h_{p,\mu} = \sum_{n=1}^{n} \sum_{\sum_{i=1}^{n} p_i = p} \sum_{\{\mu_i\}} \underset{\nearrow}{g^{\{p_i\}}_{\mu,\{\mu_i\}}} \prod_{i=1}^{n} h^{(0)}_{p_i,\mu_i} \nwarrow$$

holomorphic modular ambiguities

depth n-1
mock modular forms
satisfy their own anomaly equation

A solution for $g^{\{p_i\}}_{\mu,\{\mu_i\}}$ has been found in the 1-modulus case $b_2(CY)=1$ through a formal refinement and lattice extension (to get null vectors)

Thus, for $b_2=1$, the problem reduces to the problem of finding $h^{(0)}_{p,\mu}$ – weakly holomorphic modular forms.

But any such form is determined by its polar terms

↳ It remains to compute a finite number of terms

Wall-crossing

A systematic approach to compute polar terms is based on wall-crossing.

In physics, $\Omega(\gamma)$ count BPS bound states of charge γ.
But bound states can decay for some parameters
$\hookrightarrow$ Stability is characterized by central charge Z_γ
$\uparrow$ the same in math. def.

- $\mathcal{N}=2$ SUSY algebra

$\{Q^A_\alpha, Q^{+B}_{\dot\beta}\} = 2\delta^{AB} P_\mu \sigma^\mu_{\alpha\dot\beta}$ $\qquad A,B=1,2 \quad \varepsilon^{12} = -\varepsilon_{12}=1, \ \sigma^0_{\alpha\dot\beta}=\delta_{\alpha\dot\beta}, \ \sigma^{0\dot\beta}_\alpha = \varepsilon^{\alpha\dot\beta}$

$\{Q^A_\alpha, Q^B_\beta\} = 2\varepsilon^{AB}\varepsilon_{\alpha\beta} Z_\gamma$ $\qquad \exists\, P^\mu = (M,0,0,0)$

define $Q^{(\varphi)}_\alpha = \frac{1}{\sqrt{2}}(Q^1_\alpha + e^{-i\varphi}\sigma^{0\dot\beta}_\alpha Q^{+2}_\beta) \Rightarrow \{Q^{(\varphi)}_\alpha, Q^{(\varphi)+}_\beta\} = 2\delta_{\alpha\beta}(M - \mathrm{Re}(e^{-i\varphi}Z_\gamma))$

$$\Rightarrow \quad M \geq \mathrm{Re}(e^{-i\varphi}Z_\gamma) \underset{\varphi=\arg Z_\gamma}{\Rightarrow} M \geq |Z_\gamma|$$

BPS state has $M=|Z_\gamma|$, annihilates $Q^{(\arg Z_\gamma)}_2$ and preserves $\frac{1}{2}$ SUSY.
BPS multiplet has 4 states (short) instead of 16 (long)
This is why they are stable under deformations.
But when $|Z_\gamma| = |Z_{\gamma_1}| + |Z_{\gamma_2}| \Leftrightarrow \arg Z_{\gamma_1} = \arg Z_{\gamma_2}$ for $\gamma=\gamma_1+\gamma_2$
bound states can decay/form.

- Central charge $Z: \Gamma \times \mathcal{M}_K \longrightarrow \mathbb{C}$ - linear in γ
$\uparrow$ charge lattice

$Z_\gamma = z^I q_I - F_I p^I$ $\qquad$ where $F_I(z) = \partial_I F(z)$

- $\mathcal{M}_K$ is divided into chambers by walls of marginal stability
$\arg Z_{\gamma_1}(z) = \arg Z_{\gamma_2}(z)$
Across these walls DT invariants can jump.
The jumps satisfy universal wall-crossing formulas
The simplest case of 2 states (primitive w.c.f.)

$$\Delta\Omega(\gamma) = (-1)^{\langle\gamma_1,\gamma_2\rangle}\langle\gamma_1,\gamma_2\rangle\,\Omega(\gamma_1)\,\Omega(\gamma_2)$$

does not exist - exists

- Modularity holds in the "large volume attractor" chamber
$z^a_\infty = -\kappa^{ab}q_b + i\lambda p^a$ $\qquad$ in math: Gieseker stability
$\lambda \to \infty$

Polar terms from wall-crossing

- Idea: find a chamber with a simple BPS spectrum, compute DT invariants there, and perform wall-crossing.

  Works for non-compact CYs.

  For compact CYs, there is no such chamber on the physical slice!

- In math, one can take F_Λ to be independent parameters

  $\hookrightarrow$ large space of (Bridgeland) stability conditions

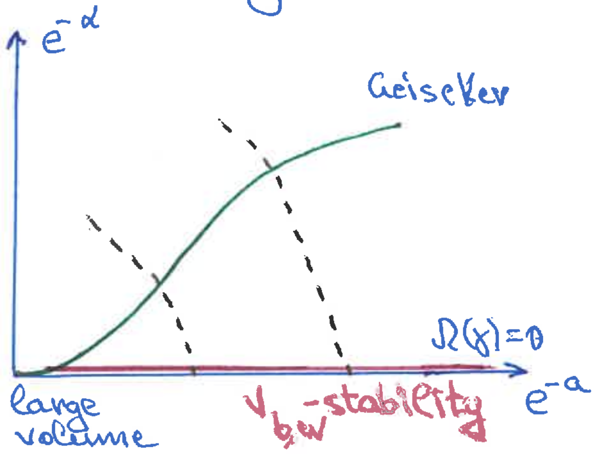
There is another slice ($\nu_{b,w}$ stability) where some $\Omega(\gamma)$ vanish

It is defined by

$$Z_{b,a}(\gamma) = -a\,ch^b_2 + \frac{a^2}{2} ch^b_0 + i a\, ch^b_1$$

where $ch^b_k(E) = \int e^{-bw} \omega^{3-k} ch(E)$

$w = \frac{1}{2}(a^2+b^2)$



For $\nu_{b,w}$, stable objects must satisfy BMT (Bayer, Macrì, Stellary) inequality $ch^b_3 \leq \frac{a^2}{6} ch^b_1$

If it is spoiled (small a region), then $\Omega(\gamma)=0$

$\hookrightarrow$ $\Omega(\gamma; z^*_\gamma)$ via wall-crossing.

Using this logic one can derive a formula expressing any rank 0 DT invariant in terms of PT (rank -1) and rank 0 DT of smaller charges.

(In practice, so far only for $p=1$ and 2)

$$\mathrm{PT}(\overset{\nearrow}{Q}, \overset{\nwarrow}{m})$$

degree (D2-charge)   D0-charge

Drawback: larger D0-charge of rank 0 DT → larger degree Q on the r.h.s.

Next: How to find PT invariants?

PT from GV

- For this we need topological string theory

  This is a topological theory defined on a CY.

  There are two versions (A/B model), quantizing Kähler/complex structure of CY, computing certain couplings in the effective theory of Type IIA/B on CY, and related by mirror symmetry.

  Full partition function $\Psi_{top}(z,\lambda) = \exp\left[\sum_{g=0}^{\infty} \lambda^{2g-2} \mathcal{F}^{(g)}(z\bar{z})\right]$ is defined by perturbative genus expansion.

  $\mathcal{F}^{(0)} = F$ - holomorphic prepotential on $\mathcal{M}_K / \mathcal{M}_C$

  For $g>0$, $\mathcal{F}^{(g)}$ satisfy a holomorphic anomaly equation

  It can be solved by a "direct integration" method, g by g, but the solution depends on a holomorphic ambiguity

  At fixed g, it is parametrized by a finite # of parameters

  We know a # of conditions that allow to fix them, but this # grows slower with g than # of parameters to fix

  $\rightarrow \exists\, g_{max}$ for which this method works.

- In the holomorphic limit $\mathcal{F}^{(g)}(z,\bar{z}) \xrightarrow{\bar{z}\to -i\infty} F^{(g)}(z) = \sum_{Q=1}^{\infty} GW_Q^{(g)} e^{2\pi i Q z}$

  $GW_Q^{(g)} \in \mathbb{Q}$ and in practice it is more convenient to work in term of integer valued GV invariants

  $$\log \Psi_{top}^{hol}(z,\lambda) = \sum_{g=0}^{\infty} \sum_{k=1}^{\infty} \sum_{Q=1}^{\infty} \frac{GV_Q^{(g)}}{k} \left(2\sin\frac{k\lambda}{2}\right)^{2g-2} e^{2\pi i k Q z}$$

  multi covering effects

- Relation to PT (MNOP formula)

  $$\Psi_{top}(z,\lambda) = M(-e^{i\lambda})^{\frac{1}{2}\chi} Z_{PT}(e^{2\pi i z/\lambda}, e^{i\lambda})$$

  where $Z_{PT}(y,q) = \sum_{Q,m} PT(Q,m)\, y^Q q^m$

  $M(q) = \prod_{k>0} (1-q^k)^{-k}$ - Mac-Mahon function

  Also: $Z_{DT}(y,q) = M(-q)^{\chi} Z_{PT}(y,q)$

Results

- Summary of the procedure

$$\begin{tikzcd}
\boxed{\text{GV}} \arrow[loop left, "\text{direct integration}"] \arrow[r, "\text{MNOP}"'] & \boxed{\text{PT}} \arrow[r, "\text{wall-crossing}"'] & \boxed{\text{rank 0 DT}} \arrow[loop right, "\text{modularity}"]
\end{tikzcd}$$

Problem: large D0-charge of rank 0 DT → larger Q of PT → larger g of GV

but $\exists\, g_{max}$!

→ restriction on applicability of this approach

- We analyzed one-parameter smooth complete intersection CYs in weigted projective spaces (13 manifolds)

  $[\, X_{d_1 \dots d_k}(w_1^{m_1}, \dots w_p^{m_p})$ – CY in $\mathbb{P}^{k+3}(w_1, \dots, w_p)$ of multi-degree $(d_1, \dots, d_k)$ where m_i is the number of repetitions of $w_i\,]$

- At $r=1$, we have found all polar terms for 11 CYs

  $\longrightarrow h_{1,\mu}(\tau)$

  Also checked many non-polar terms ← striking check of modularity

- At $r=2$, we have found $h_{2,\mu}(\tau)$ for 2 CYs:

  $X_8(1^4,4)$ and $X_{10}(1^3,2,5)$

  This is the first example of mock modular forms for CYs without any additional structure.

  (the mock part can be expressed in terms of the generating function of Hurwitz class numbers)

- Recent generalization by Mc Govern to quotients like $X_5/\mathbb{Z}_5$, $X_{3,3}/\mathbb{Z}_3$, etc.

Implications for GV (and GW) invariants

The knowledg of the generating functions $h_{r,\mu}(\tau)$ allows to get infinitely many rank 0 DT invariants.
Inverting the procedure allows to compute new GV invariants for genera beyond $g_{\max}$!

↳ Overcomes limitations of the direct integration

Knowledge of $h_{r,\mu} \longrightarrow g^{(r)}_{\rm mod}$ new limitation

Ex. X_5 : $g_{\max}=53 \longrightarrow g^{(1)}_{\rm mod}=69$ $(g_{\rm avail}=64)$

X_{10} : $g_{\max}=50 \longrightarrow g^{(1)}_{\rm mod}=70$, $g^{(2)}_{\rm mod}=95$ $(g_{\rm avail}=71)$

The new bounds are still too restrictive to allow the computation of polar terms for higher r's by this method.

Alternatives?