Lecture 2

MMP in Relative Settings

☑ **Definitions:** Let $f : X \to Y$ be an a proper morphism between two varieties.

(i) A line bundle $\mathcal{L}$ on X is called f-ample if $\mathcal{L}|_{X_y}$ is ample $\forall\, y \in Y$.

(ii) $\mathcal{L}$ is called f-nef if $\mathcal{L}|_{X_y}$ is nef $\forall\, y \in Y$. (D is nef: $D \cdot C \geq 0$ $\forall_{C \subseteq X}$)

(iii) $\mathcal{L}$ is called f-big if $\mathcal{L}|_F$ is big for every irreducible components F of very general fibers X_y.

* Recall that a line bundle M on a proper variety X is called big if $\varphi_k : X \dashrightarrow \mathbb{P}H^0(X, M^{\otimes k})$ is birational to it's image for some $k > 0$

(iv) $\mathcal{L}$ is called f-pseudo-effective if $\mathcal{L}|F$ is pseudo-effective for every irreducible components F of very general fibers X_y

* Recall that a Cartier div D or equiv. the bundle $\mathcal{L} \cong \mathcal{O}_X(D)$ on a proper var. X is called pseudo-effective if its numerical class $[D] \in \overline{\mathrm{Eff}}(X)$, where $\overline{\mathrm{Eff}}(X) = \overline{\left\{ \sum a_i D_i : a_i \geq 0 \right\}}$

$\equiv$

Relative N^1 and N_1:

Def: Let $f: X \to Y$ be a proper morphism of varieties.

(i) $N_1(X/Y)_{\mathbb{Q}} := \left\{ \sum a_i C_i \;\middle|\; \begin{array}{l} C_i \subseteq X \text{ curves}, \\ f_*(C_i) = 0 \ \forall i, \\ a_i \in \mathbb{Q} \end{array} \right\} \Big/ \equiv$

$\gamma \equiv \gamma'$
$\iff$
$D \cdot \gamma = D \cdot \gamma'$
$\forall$ $\mathbb{Q}$-Cartier div. D

If $Y = \{pt\}$,
then $N_1(X/Y)_{\mathbb{Q}} = N_1(X)_{\mathbb{Q}}$.

(ii) $N^1(X/Y)_{\mathbb{Q}} = \{\mathbb{Q}\text{-Cartier divisors } D \text{ on } X\} \Big/ \equiv_Y$

$D \equiv_Y D'$
$\iff$
$D \cdot C = D' \cdot C$
$\forall\, C \subseteq X$ s.t. $f_* C = 0$

$\circledast$ $N_1(X/Y)_{\mathbb{R}} = N_1(X/Y)_{\mathbb{Q}} \otimes \mathbb{R}$

$\circledast$ $N^1(X/Y)_{\mathbb{R}} = N^1(X/Y)_{\mathbb{Q}} \otimes \mathbb{R}$

We will simply denote $N_1(X/Y)_{\mathbb{R}}$ and $N^1(X/Y)_{\mathbb{R}}$ by $N_1(X/Y)$ and $N^1(X/Y)$.

$\circledast$ $N^1(X/Y) \times N_1(X/Y) \longrightarrow \mathbb{R}$

$(D, \gamma) \longmapsto D \cdot \gamma$

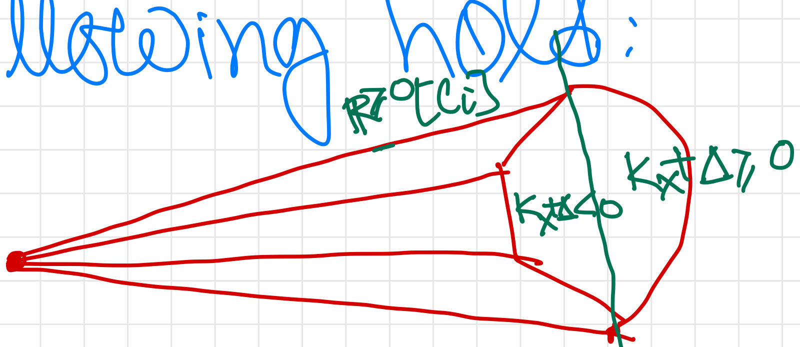
is a perfect pairing

$\overline{NE}(X/Y) = \left\{ \gamma = \sum a_i C_i \;\middle|\; a_i \geq 0,\ f_*(C_i) = 0\ \forall i \right\} \subseteq N_1(X/Y).$

Relative Mori Cone of curves.

The Cone and Contraction Theorem

Let $(X, \Delta \geq 0)$ be a klt or lc pair and $f: X \to Y$ be a projective morphism. Then there exist a countable collection of rational curves $\{C_i\}_{i \in I}$ s.t. $f_*(C_i) = 0$, $0 < -(K_X + \Delta) \cdot C_i \leq 2 \dim X$, and the following hold:



(i)

$\overline{NE}(X/Y) = \overline{NE}(X/Y)_{(K_X+\Delta) \geq 0} + \sum_{i \in I} \mathbb{R}^{\geq 0}[C_i]$

$I \subseteq \mathbb{N}$ finite

(ii) If $R = \mathbb{R}^{\geq 0}[C_i]$ is a extremal ray, then $\exists$ a projective morphism $g_R : X \to Z$ over Y with ~~connected~~ fibers s.t. a curve $C \subseteq X$ is contracted by g_R (i.e. $(g_R)_*(C) = 0$) if and only if $[C] \in R = \mathbb{R}^{\geq 0}[C_i]$.

$\Downarrow$ $(g_R)_* \mathcal{O}_X = \mathcal{O}_Z$

$$\begin{tikzcd} C \subseteq X \arrow[r, "g_R"] \arrow[dr, "f"'] & Z \ni g_R(C) = p \arrow[d] \\ & Y \end{tikzcd}$$

$C \equiv \lambda \cdot C_i$ $\lambda > 0$.

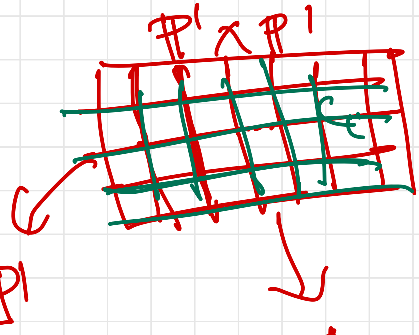
$C_i \not\equiv \lambda C_j$

$\mathbb{R}^{\geq 0}[C_i]$ $i \neq j$.

$X = \mathbb{P}^1 \times \mathbb{P}^1$ $Y = \{pt\}$

Types of contraction:

$$\begin{tikzcd} X \arrow[rr, "g_R"] \arrow[dr, "f"'] & & Z \arrow[dl] \\ & Y & \end{tikzcd}$$



(i) If $\dim Z < \dim X$, then g_R is not birational.
In this case $g_R : X \to Z$ is called a Mori fiber space over Y. $-(K_X + \Delta)$ is g_R-ample

(ii) If $\dim Z = \dim X$, then g_R is birational.
Assume that $\operatorname{Codim}_X \operatorname{Ex}(g_R) = 1$.
In this case g_R is called a divisorial contraction.

(iii) If g_R is birational, and $\operatorname{Codim} \operatorname{Ex}(g_R) \geq 2$, the $g_R : X \to Z$ is called a flipping contraction over Y.

$\cap$

{ Small morphism }

$X = \mathbb{P}^2$

$Y = Bl_{\{p_1, \dots, p_8\}}(X)$

$Z = Bl_{\{p_9\}} Y$.

$$\begin{array}{c} Z \\ g \downarrow \\ Y \\ f \downarrow \\ X \end{array}$$

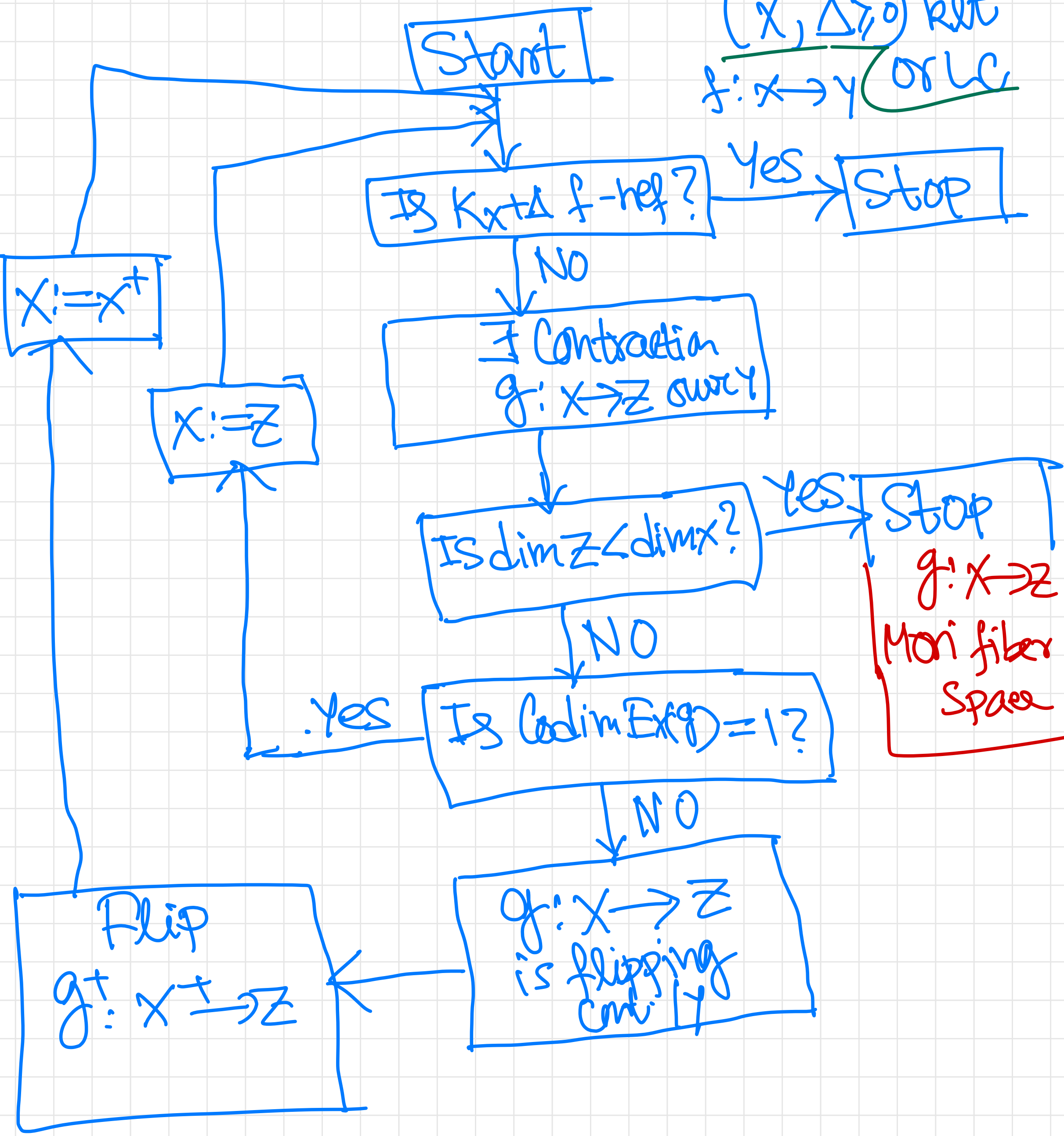
$\overline{NE}(Y)$ has finitely many external rays.

$\overline{NE}(Z)$ has infinitely many K_Z-negative extremal rays.

$g: Z \to Y$

Algorithm of MMP

X is $\mathbb{Q}$-factorial
$(X, \Delta \geq 0)$ klt or LC
$f: X \to Y$



(*) If $g: X \to Z$ is a $(K_X+\Delta)$-flipping contraction, then $g_*(K_X+\Delta)$ $= K_Z + \Delta_Z$ is NOT $\mathbb{Q}$-Cartier. So we cannot replace X by Z in this case. We need some-thing else.

Definition: Let $g: X \to Z$ be a $K_X+\Delta$-flipping contraction /Y.
Then $g^+: X^+ \to Z$ is called a flip of g over Y if the following hold:

$$\begin{tikzcd} X \arrow[rr, dashed, "\Phi"] \arrow[dr, "g"'] & & X^+ \arrow[dl, "g^+"] \\ & Z \arrow[d, "f"] & \\ & Y & \end{tikzcd}$$

$-(K_X+\Delta)$ is g-ample

$K_{X^+}+\Delta^+$ is g^+-ample.

$\Delta^+ := \Phi_* \Delta$.

(i) X^+ is $\mathbb{Q}$-factorial

(ii) (X^+, Δ^+) is klt (res. lc).

(iii) $K_{X^+}+\Delta^+$ is g^+-ample.

(iv) $\rho(X^+/Y) = 1$.

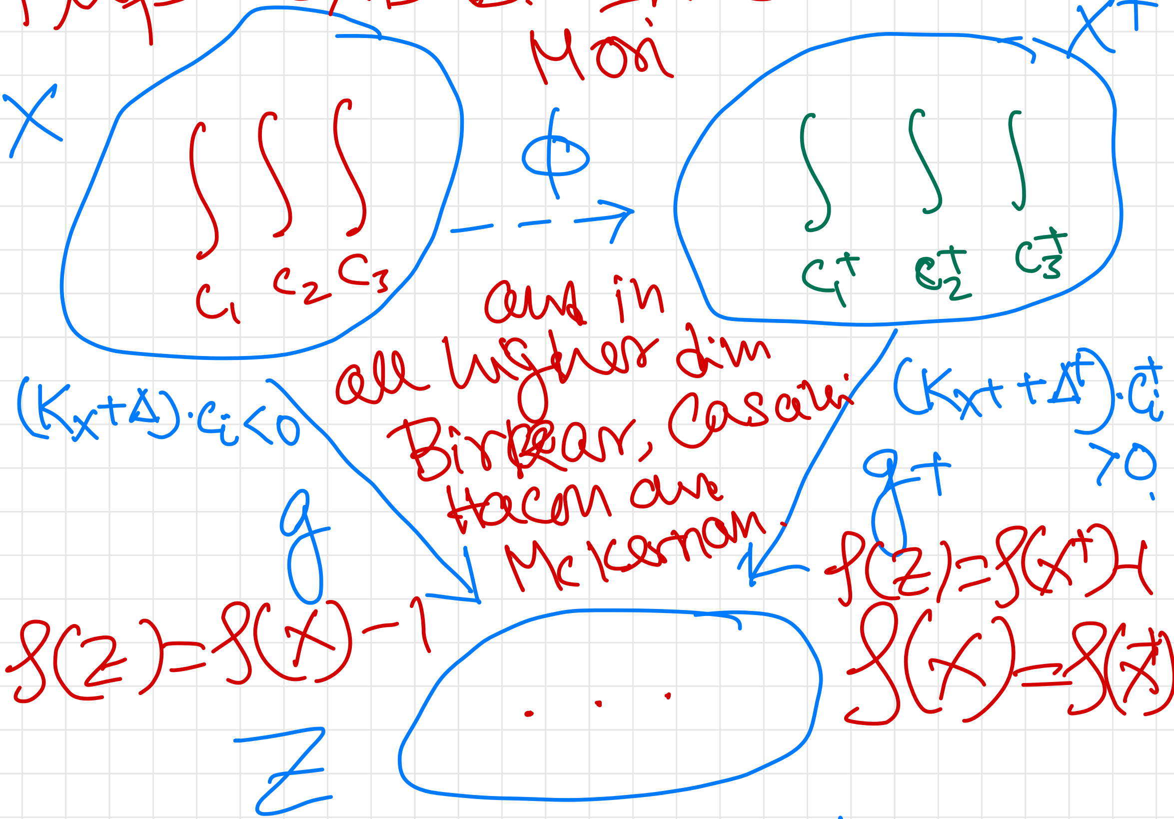
(relative Picard No.)

(v) $\operatorname{Codim}_{X^+} \operatorname{Ex}(g^+) \geq 2$.

i.e. $\phi := (g^+)^{-1} \circ g : X \dashrightarrow X^+$

is an isomorphism in Codim 1.

Flips exists. In dim 3 due to Mori and in all higher dim Birkar, Cascini, Hacon and McKernan.



$\rho(Z)=\rho(X)-1$

$\rho(Z)=\rho(X^+)-1$

$\rho(X)=\rho(X^+)$

✱ Divisional contraction reduces Picard number by 1. But flip keeps the Picard number same.

$$X^+ := \mathrm{Proj}_{\mathcal{O}_Z} \bigoplus_{m\geq 0} g_*\mathcal{O}_X(m(K_X+\Delta)).$$

Termination Conjecture

Every sequence of flips terminate, i.e. the MMP algorithm always terminates.

* Termination of the MMP algorith is highly sensitive to the choice of order of extremal rays.

(*) (i) It is known in dim 3 due to Mori. Various partial cases are also known in dim 4.

(ii) If $(X, \Delta \geq 0)$ is a klt pair any $K_X + \Delta$ is f-big, then any "Directed sequence of flips w.r.t. an ample/y div" always terminates.

This is due to Birkar, Cascini, Hacon and McKernan.

(This holds in all dimensions)

(This is however NOT known if $K_X + \Delta$ is LC).

(*) Remarks:

(i) Let $X \dashrightarrow X'$ be the end result of a $K_X+\Delta$-MMP/Y, then $K_{X'}+\Delta'$ is nef over Y (res. Mori fiber space) if and only if $K_X+\Delta$ is f-pseudo-effective (res. NOT f-pseudo-eff).

Definition:

Let (X, Δ) be a pair.

It is called a Divisorially Log Terminal or DLT pair if $\exists$ a log resolution $f: Y \to X$ of (X, Δ) s.t.

$$K_Y = f^*(K_X + \Delta) + \sum a(E, X, \Delta) E$$

and $a(E, X, \Delta) > -1$ $\forall$ f-exceptional divisors E.

(*) A dlt pair (X, Δ) is klt $\iff$ all the co-efficients of $\Delta < 1$.

$$\{klt\} \subseteq \{dlt\} \subseteq \{lc\}.$$

Examples:

(i) If (X, Δ) is a log smooth pair, i.e. X is smooth and Δ has SNC support, and the co-efficients of Δ are in the interval $(-\infty, 1]$, then (X, Δ) is dlt.

(ii) Let $f: X \to Y$ be a proj. mor. and $(X, \Delta \geq 0)$ a log smooth pair, $0 \leq \Delta \leq 1$ and $X \overset{(\text{bir.})}{\dashrightarrow} X'$ is a $K_X + \Delta$-MMP$/Y$. Then (X', Δ') has dlt sing.

Log Terminal and Log Canonical Model

Let (X, Δ) be a lc pair and $f: X \to Y$ a proj. morphism. Let $\phi: X \dashrightarrow X'$ be a birational contraction over Y and $\Delta' := \phi_* \Delta$

Then

(i) (X', Δ') is called a Log Canonical Model of (X, Δ) if (X', Δ') is lc and $K_{X'} + \Delta'$ is ample over Y

$$\begin{tikzcd} X \arrow[r, dashed, "\phi"] \arrow[dr, "f"'] & X' \arrow[d, "f'"] \\ & Y \end{tikzcd}$$

(ii) (X', Δ') is called a a **Log Terminal Model** of (X, Δ) if (X', Δ') has DLT singularities, X' is $\mathbb{Q}$-factorial, $K_{X'} + \Delta'$ is nef over Y and

$$a(E, X, \Delta) < a(E, X', \Delta')$$

for all ϕ-exceptional divisors $E \subseteq X$.

* Remark: While log terminal and minimal models are not unique, (Log) Canonical models are unique if it exists.

Theorem: Let $(X, \Delta \geq 0)$ be a log smooth pair, the co-eff. of Δ are in $[0,1]$ and $f: X \to Y$ a proper morphism. Assume that $K_X + \Delta$ is f-big. Then relative log canonical sheaf of rings

$$R(X/Y, K_X+\Delta) := \bigoplus_{m \geq 0} f_* \mathcal{O}_X(m(K_X+\Delta))$$

is a finitely generated $\mathcal{O}_Y$-alg.

Then the following hold:

(i) $X^{can} = \operatorname{Proj}_Y R(X/Y, K_X+\Delta)$ is normal, $f^{can}: X^{can} \longrightarrow Y$ is a proj. morphism.

(ii) There is a birational map $\phi: X \dashrightarrow X'$ and (X^{can}, Δ^{can}) has lc sing., where $\Delta^{can} = \phi_* \Delta$.

(iii) $K_{X^{can}} + \Delta^{can}$ is f^{can}-ample.

i.e. (X^{can}, Δ^{can}) is the unique log canonical model of (X, Δ) over Y.

$\circledast$ $f^{can}: (X^{can}, \Delta^{can}) \longrightarrow Y$ is called the $\boxed{\text{relative}}$

log canonical model of (X, Δ) over Y.

$\circledast$ **Important Remark:**

If $\Delta = 0$ or more generally the co-eff of Δ are contained in the interval $[0,1)$, then $R(X/Y, K_X+\Delta)$ (X,Δ) klt. is known to be a finitely generated $\mathcal{O}_Y$-algebra, due to (Birkar, Cascini, Hacon and McKernan) and independently by Cascini and Lazić.

In particular the relative (log) canonical model $f^{can}: (X^{can}, \Delta^{can}) \to Y$ of $f: (X, \Delta) \to Y$ always exists.

If $(X, \Delta \geq 0)$ is klt, then (X^{can}, Δ^{can}) always exists.