Lecture 1

* Singularities of MMP:

Let X be a normal variety $/\mathbb{C}$

$U = X_{sm}$, $\omega_U := \wedge^{\dim X} \Omega^1_U$

Ω^1_U locally free.

Let K_U be a Cartier div corres. to ω_U. $\operatorname{codim}_X(X \setminus U) \geq 2$.

Let K_X be the unique extension of K_U on X.

K_X is called a canonical divisor of X.

(*) The canonical div K_X is NOT unique.

Definition: Let X be a normal variety. X is called $\boxed{\mathbb{Q}\text{-Gorenstein}}$ ~~if~~ if

$\mathrm{d}\, mK_X$ is Cartier for some $m > 0$.

$\mathrm{d}\text{-}7,1$

(*) Let $f : Y \longrightarrow X$ be a birational morphism from a normal variety on Y. Let $E \subseteq Y$ be a prime div and $e \in E$ a general point of E.

$e \in E_{sm} \cap X_{sm}$

Let $(y_1, \dots, y_n)$ be a local coordinate system at $e \in Y$ s.t. $E = \{y_1 = 0\}$. Then, locally near e,

$$f^*(\text{local generator of } \mathcal{O}_X(mK_X) \text{ at } f(e)) = y_1^{c(E,X)} \cdot (\text{unit}) \cdot \underbrace{(dy_1 \wedge \cdots \wedge dy_n)^{\otimes m}}_{mK_Y}$$

$c(E, X) \in \mathbb{Z}$.

The rational number $\in \mathbb{Q}$

$$\boxed{a(E,X) := \frac{1}{m} c(E,X)}$$

is called the Discrepency of E w.r.t. X.

It is independent of m.

$$(*) \quad mK_Y \sim f^*(mK_X) + \sum_{\substack{E \subseteq Y \\ \text{div}}} m \cdot a(E,X) E$$

(finite sum) $E \subseteq X$ f-exceptional

$$\Rightarrow K_Y \underset{\mathbb{Q}}{\sim} f^* K_X + \sum a(E,X) E.$$

Log Equation.

Lemma: Let X be a smooth variety. Then for every prime exceptional div E over X, the discrepancy $a(E, X, \Delta)$ is an integer, and

$$a(E, X, \Delta) \geq 1.$$

Proof: Let $f: Y \to X$ birational, $E \subseteq Y$ prime exceptional div and $e \in E$ a general point. Choose a local coordinate $(y_1, \ldots, y_n)$ near $e \in Y$, and $(x_1, \ldots, x_n)$ near $f(e) \in X$. Then

$$f^*(\underbrace{dx_1 \wedge \cdots \wedge dx_n}_{K_X}) = \mathrm{Jac}\left(\frac{x_1, \ldots, x_n}{y_1, \ldots, y_n}\right) \cdot \underbrace{dy_1 \wedge \cdots \wedge dy_n}_{K_Y}$$

Then $a(E,X)$ = order of vanishing of the Jacobian along E.

Since f is Not an isomorphism near $e \in E$, $\underline{a(E,X) \in \mathbb{Z}^+}$

$a(E,X) > 0$

i.e. $a(E,X) \geq 1$. $a(E,X) \in \mathbb{Z}$

$K_Y \sim f^* K_X + \sum \underbrace{a(E,X)E}_{\in \mathbb{Z}^+}$

$\circledast$ If $\underline{a(E,X)} \notin \mathbb{Z}^+$, then
X is NOT Smooth.

Simple Normal Crossing Div

Let X be a smooth variety and $D = \sum_{i=1}^{m} D_i$ be a reduced Weil div.

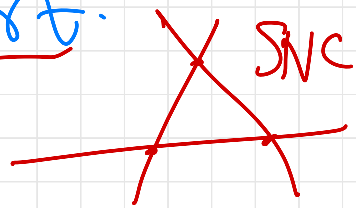
We say D is a simple normal crossing divisor (or SNC div), if every D_i is smooth and all possible intersections of D_i's are transversal intersect.

equivalently, for any $p \in \mathrm{Supp}(D)$, $\exists$ an analytic nbhd $p \in U_p \subseteq X$ with local coordinates $(x_1, \dots, x_n)$ s.t. $D|U_i = \{x_{i_1} \cdots x_{i_k} = 0\}$ for $i_1, \dots, i_k \in \{1, \dots, n\}$

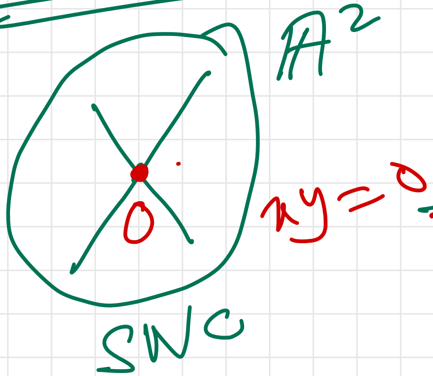
$\boxed{\circledast\ (X, D) \text{ log Smooth pair.}}$

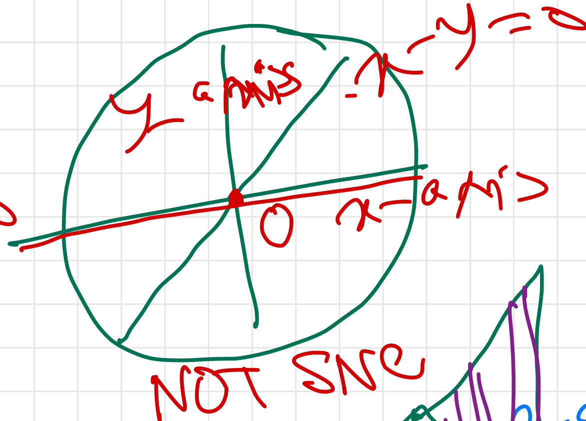
A $\mathbb{Q}$-div $D = \sum a_i D_i$, $a_i \in \mathbb{Q}$ is called a SNC div if the corresponding reduced div $\sum D_i$ is SNC.

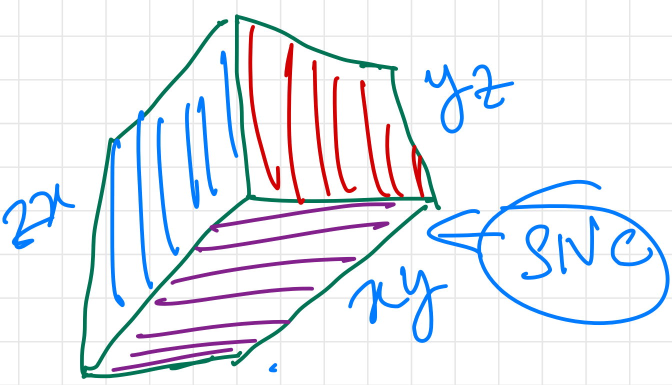
In this case we also say D has SNC support.

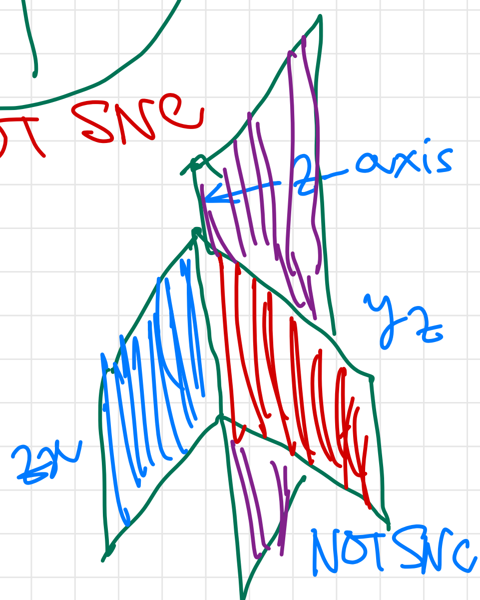


Examples:









Resolution of Singularities

Definition: Let X be an algebraic variety $/\mathbb{C}$.
A resolution of singularities of X is a projective birational morphism $f: Y \to X$ from smooth variety Y s.t. $Ex(f) = \bigcup_{i=1}^{k} E_i$ is a SNC div. *

* Res. of Sing. exists due to Hironaka.

MMP Singularities

Def: Let X be a normal $\mathbb{Q}$-Gorenstein variety and $f: Y \to X$ a resolution of singularities of X. and $E_1, \dots, E_k$ are the all exceptional divisors of f.

$$K_Y = f^*K_X + \sum_{i=1}^{k} a_i E_i$$

where $a_i = a(E_i, X)$ $\forall i$

We will say X has

(i) Terminal singularities if $a_i > 0$ $\forall i$ $\in \mathbb{Q}$

(ii) Canonical " if $a_i \geq 0$ $\forall i$

$\mathbb{C}^2/G$, $G \subseteq SL_2(\mathbb{C})$ finite $\leftarrow \mathbb{Q}_{\geq 0}$

(iii) Log Terminal or Kawamata Log Terminal (KLT is short) singularities if $a_i > -1$ $\forall i$

$\mathbb{C}^2 / G$, $G \leq GL_2(\mathbb{C})$

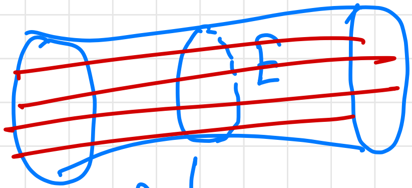
(iv) Log Canonical of LC fruit sing. if $a_i \geq -1$ $\forall i$.

Examples

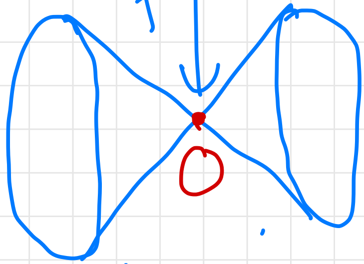
(1) Let X be the affine cone over the d-uple embedding of $\mathbb{P}^1$ in $\mathbb{P}^{d+1}$

$$\mathbb{P}^1 \hookrightarrow \mathbb{P}^{d+1}$$

$$[x:y] \longmapsto [x^d : x^{d-1}y : \dots : y^d]$$

$Bl_0(X) = Y =$  $\longrightarrow \mathbb{P}^1$

$\Big\downarrow f$

$X =$ 

$\underline{E \cong \mathbb{P}^1}$

$f(E) = 0$

$\mathcal{H}_d.$

$$E^2 = -d$$

Adjunction formula

$$K_Y = f^*K_X + aE$$

$$(K_Y + E)\cdot E = f^*K_X \cdot E + (a+1)E^2$$

$$\Rightarrow \underbrace{2g(E) - 2}_{g(E)=0 \text{ as } E \cong \mathbb{P}^1} = 0 + (a+1)(-d)$$

$$\Rightarrow \underline{-2} = -d(a+1)$$

$$\Rightarrow a = -1 + \frac{2}{d} > -1$$

$\therefore$ X has $\underline{\text{KLT}}$ sing.

$d > 0.$

$$\{\text{Terminal}\} \subseteq \{\text{Canonical}\} \subseteq \{\text{LC}\} \subseteq \{\text{KLT}\} \subseteq \{\text{LC}\}.$$

If $d=2$, then X is a quadratic cone and it has Canonical sing, as $a=-1+\frac{2}{2}=0$.

⊛ $\boxed{\text{Question}}$: What is X if $d=1$?

$X=\mathbb{A}^2$

$\boxed{\text{A Surface has terminal sing} \Leftrightarrow \text{it is smooth.}}$

$\mathbb{P}^1 \hookrightarrow \mathbb{P}^2$

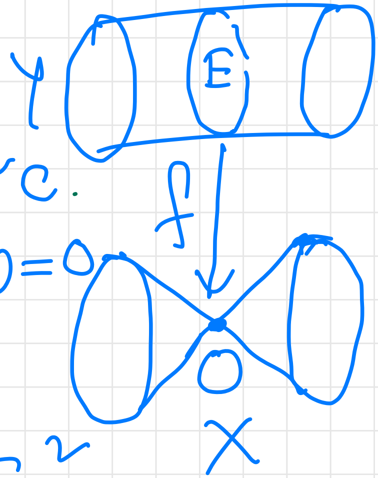
$[x:y] \mapsto [x:y:0]$

$a=-1+\frac{2}{1}=1>0$.

Terminal Sing.

$X=\mathbb{A}^2$.

② Let X be an cone over an elliptic curve C.



$E \cong C$.

$f(E) = 0$

$$K_Y = f^*K_X + aE$$

$$(K_Y + E)\cdot E = (a+1)E^2$$

$$\Rightarrow 2\underbrace{g(E)}_{=1} - 2 = (a+1)E^2 \qquad \boxed{g(E) = 1}$$

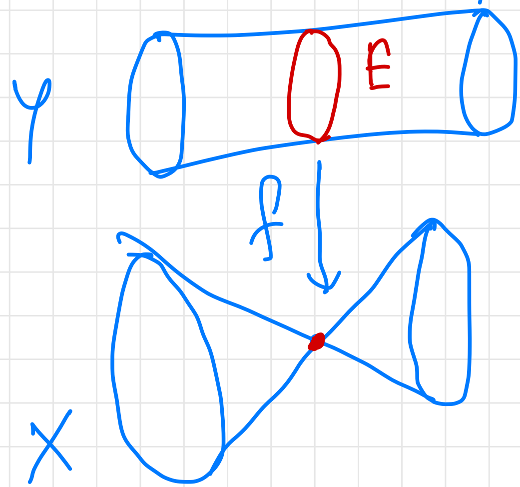
$$\Rightarrow 0 = (a+1)\underbrace{E^2}_{<0}$$

$$\Rightarrow a = -1$$

i.e. $K_Y = f^*K_X - E$ is LC

but NOT KLT.

③ Let X be an affine cone over a curve of genus $g \geq 2$.



$E^2 = -d < 0$

$$K_Y = f^* K_X + aE$$

$$(K_Y + E) \cdot E = (a+1) E^2 \qquad g \geq 2$$

$$\Rightarrow (-d)(a+1) = 2g - 2 \geq 2 > 0$$

$$\Rightarrow a + 1 < 0$$

$$a < -1$$

$\Rightarrow$ ∴ the singularities of X

is worse than log canonical.

Pairs: Let X be a normal variety and $\Delta = \sum a_i D_i$, $a_i \in \mathbb{Q}$

D_i's are prime divisors.

If $K_X + \Delta$ is $\mathbb{Q}$-Cartier, i.e. $\exists\ m \in \mathbb{Z}^+$ s.t. $m(K_X + \Delta)$ is a Cartier divisor, then (X, Δ) is called a $\boxed{\text{Pair}}$

Log resolution:

Let X be a variety and $Z \subseteq X$ be a closed subset. A projective birational morphism $f: \tilde{X} \longrightarrow X$ from a smooth variety $\tilde{X}$ is called a Log Resolution of (X, Z) if the following hold:

(i) $Ex(f) \cup f^{-1}Z$ is a SNC div.

(ii) $X \setminus (Ex(f) \cup f^{-1}Z) \cong X \setminus (X_{sing} \cup Z)$

Definitions:

Let (X, Δ) be a pair and $f: Y \longrightarrow X$ a log resolution of (X, Δ).

$\Delta = \sum a_i D_i$

$E_i = f^{-1}_* D_i$

$a(E_i, X, \Delta) = -a_i$

write

$K_Y + f^{-1}_* \Delta = f^*(K_X + \Delta) + \sum a(E, X, \Delta) E$

$- f^{-1}_* \Delta \qquad\qquad - f^{-1}_* \Delta$

$$K_Y = f^*(K_X + \Delta) + \sum a(E, X, \Delta) E, \quad \ldots\ldots \ (*)$$

$*$ Note that E's are either f-exceptional or components of the strict transform of Δ, i.e. components of $f^{-1}_* \Delta$.

We say that (X, Δ) have

(i) KLT sing. if $a(E, X, \Delta) > -1$
$\forall E$ in (*)

(ii) LC sing. if $a(E, X, \Delta) \geq -1$
$\forall E$ in (*)

(iii) Terminal sing if $\mathrm{Supp}(f_*^{-1}\Delta)$ is smooth, and $a(E, X, \Delta) > 0$ for all f-exceptional div. E in (*)

(iv) Canonical sing. if $\mathrm{Supp}(f_*^{-1}\Delta)$ is smooth and $a(E, X, \Delta) \geq 0$ $\forall$ f-exep div E in (*).

$\{\text{Terminal}\} \subseteq \{\text{Canonical}\} \not\subseteq \{\text{KLT}\}$

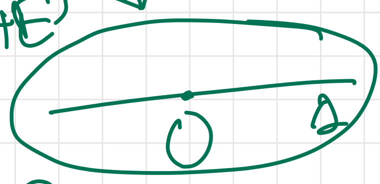
$\subseteq \{\text{LC}\}.$

⊛ If (X, Δ) is canonical and Δ has a component with co-effi. 1, then (X, Δ) may not be klt:

Example: $X = \mathbb{A}^2$, $\Delta = \{y = 0\}$

$\tilde{X} =$ 

$\downarrow f$

$X =$ 

$K_{\tilde{X}} = f^* K_X + E$

$K_{\tilde{X}} = f^*(K_X + \Delta) - f^*\Delta + E$

$= f^*(K_X + \Delta) - \tilde{\Delta} + 0 \cdot E.$

~~Silly examples:~~

① $\left(\mathbb{A}^2,\ a_1\{y=0\}+a_2\{x=0\}\right)$

is KLT $\iff a_1<1,\ a_2<1$

is LC $\iff a_1\le 1,$

$a_2\le 1$

$i:\mathbb{A}^2\to\mathbb{A}^2$ is a log res. in this case.

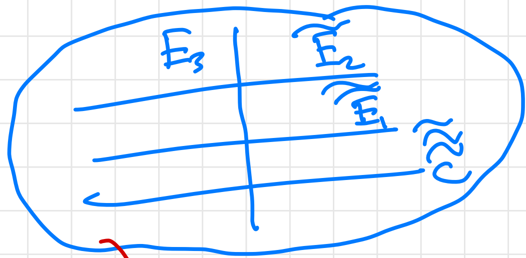
Example:

$X = \mathbb{A}^2,$

$\Delta = C = \{y^2 - x^3 = 0\}$

(X, tC)

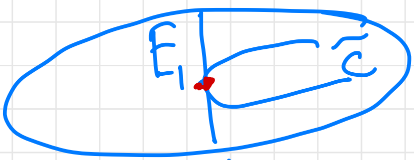
Y_3

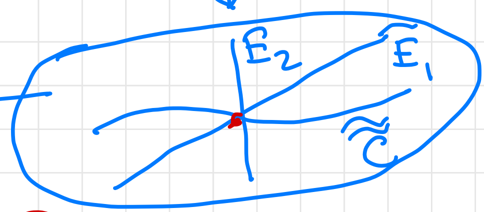


$\downarrow f_3$

$f_3^*(f_2^* E_1 = \tilde{E}_1 + E_2)$

$Y_1 \xleftarrow{f_2} Y_2$

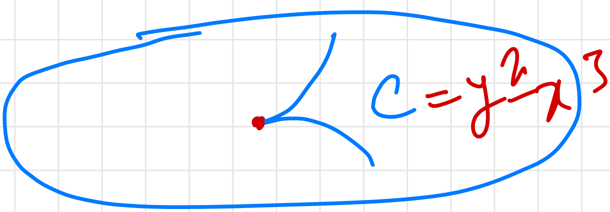




$f_1 \downarrow$

$(f_2 \circ f_3)^* E_1 = \tilde{\tilde{E}}_1 + \tilde{E}_2 + 2E_3$

X



X sm. surface

$p \in X,$

$\tilde{X} = Bl_p(X)$

$\pi \downarrow \quad , \; E = Ex(\pi)$

X

$K_{\tilde{X}} = \pi^* K_X + E.$

$$
\begin{aligned}
K_{Y_3} &= f_3^* K_{Y_2} + E_3 \\
&= f_3^*(f_2^* K_{Y_1} + E_2) + E_3 \\
&= (f_2 \circ f_3)^* K_{Y_1} + \widetilde{E}_2 + 2E_3 \\
&= (f_2 \circ f_3)^* (f_1^* K_X + E_1) + \widetilde{E}_2 + 2E_3 \\
&= (f_1 \circ f_2 \circ f_3)^* K_X + \widetilde{E}_1 + 2\widetilde{E}_2 \\
&\qquad + 4E_3
\end{aligned}
$$

$$
\begin{aligned}
(f_1 \circ f_2 \circ f_3)^* C &= (f_2 \circ f_3)^* (\widetilde{C} + 2E_1) \\
&= f_3^* (\widetilde{\widetilde{C}} + 2\widetilde{E}_1 + 3E_2) \\
&= \widetilde{\widetilde{\widetilde{C}}} + 2\widetilde{\widetilde{E}}_1 + 3\widetilde{E}_2 + 6E_3
\end{aligned}
$$

Let $f = f_1 \circ f_2 \circ f_3 : Y_3 \to X$

$$K y_3 + t\tilde{\tilde{c}} = f^*(Kx + tC) + \tilde{\tilde{E}}_1 + 2\tilde{E}_2 + 5E_3 - t(\tilde{\tilde{E}}_1 + 3\tilde{E}_2 + 6E_3)$$

$$= f^*(Kx + tC) + \underline{(1-2t)}\tilde{\tilde{E}}_1 + \underline{(2-3t)}\tilde{E}_2 + \underline{(4-6t)}E_3 .$$

$$4 - 6t \geq -1$$

$$\Rightarrow 6t \leq 5$$

$$\Rightarrow t \leq \frac{5}{6}$$

$t = \frac{5}{6}$

(x^3+y^3-1)

" Δ

$\left(\mathbb{A}^2, \frac{5}{6}\{y^2-x^3\}\right)$

is LC but NOT

KLT.

$\left(\mathbb{A}^2, \left(\frac{5}{6}+\varepsilon\right)\{y^2-x^3\}\right)$

NOT LC $\forall\, \varepsilon > 0$

$\left(\mathbb{A}^2, t\{x^3+y^3-1\}\right)$ LC

for $\forall\, t \leq 1$

(*) MMP sing are NOT invariant under the numerical equivalence of Δ.

(B) (X, Δ) KLT or LC and X is $\mathbb{Q}$-Gorenstein, then X is KLT or LC respectively.

(X,Δ) LC, X $\mathbb{Q}$-Gor

$\implies X$ is KLT along $\operatorname{Supp}(\Delta)$.