Non-normal Varieties · II · 18/11/23

<u>Recall:</u> R reduced $\quad S$ normal domain

$R \to S$ surjmap

$\exists\, p \in \mathrm{Min}\, R$ st

$$\begin{tikzcd} R \arrow[rr] \arrow[dr] & \arrow[d, phantom, "\circlearrowleft"] & S \\ & R/p \arrow[ur] & \end{tikzcd}$$

If $\mathrm{Im}(\mathrm{Spec}\, S \to \mathrm{Spec}\, R)$ contains a generic point of $\mathrm{Spec}\, R$, then we have

$$\begin{tikzcd} \mathrm{Spec}\, S \arrow[rr] \arrow[d] & & \mathrm{Spec}\, R \\ \overline{\mathrm{Spec}\, R/p} \arrow[r] & \mathrm{Spec}\, R/p \arrow[ur] & \end{tikzcd}$$

<u>Conductor ideal.</u> Let R be a domain, $K = $ frac Field, $\bar{R}$ int closure of R in K.

$\mathfrak{c}_R := \{ a \in K \mid a\bar{R} \subseteq R \}$ is called the <u>conductor</u> of R

mult. in K.

c_R is an $\bar{R}$-ideal

Suppose $a \in c_R$. Then $a \cdot \underline{1} \in R$ so $c_R \subseteq R$

c_R is the largest abelian subgp of R that is both an R-ideal and an $\bar{R}$-ideal.

<u>Example</u>: $R = k[t^3, t^5] \subseteq \bar{R} = k[t]$

UFD, so int closed.

$\therefore$ $\mathrm{Frac}(R) = \mathrm{Frac}(\bar{R})$, so $\bar{R}$ is the normalization of R.

$t^8, t^9, t^{10}, \ldots \in R$

$t^7 \notin R$.

$8 = \min\{c \in \mathbb{N} \mid t^i \in R \ \forall i \geq c\}$.

$t^8 \cdot \bar{R} = k\langle t^8, t^9, \ldots \rangle \subseteq R$

$c_R = (t^8, t^9, \ldots)R$ $\blacksquare$

Defn. A noeth local ring (R, m) is said to be <u>Gorenstein</u> if

(1) it is CM (ie depth $R = \dim R$)

(2) if $\underline{x} = x_1, \ldots, x_d$ ($d = \dim R$) is a sop, then $0 :_{R/(\underline{x})} m$ is 1-dim'l vectorspace over R/m.

<u>Example</u>. (1) $k[[x]]/(x^n)$ $n \geq 1$

(2) $\dfrac{k[[x,y]]}{(x^2, y^3)}$, in general $\dfrac{k[[x_1,\dots,x_d]]}{(f_1,\dots,f_d)}$

$f_1,\dots,f_d$ is a regular seq.

(3) Non-example. $R = \dfrac{k[x,y]}{(x,y)^2}$ is not

Gorenstein since $0 :_R m = k\langle x, y\rangle$.

<u>Fact</u>: Let (R, m) be a Gorenstein ring. Then $\mathrm{Hom}_R(-, R)$ is a dualizing functor for R-modules.

Now suppose that (R, m) is a Gorenstein domain.
Assume that $\bar{R}$ is a f.g. R-module.
(True in geometric situations.)

Let $K = \mathrm{Frac}(R) = \mathrm{Frac}(\bar{R})$

$\mathfrak{c}_R = (R :_K \bar{R}) \cong \mathrm{Hom}_R(\bar{R}, R)$ || not assuming Gor

$= \omega_{\bar{R}}$ "canonical module" of $\bar{R}$

If $\bar{R}$ is CM, then $\omega_{\bar{R}}$ defined as above satisfies the ff:

$\mathrm{Hom}_{\bar{R}}(-, \omega_{\bar{R}})$ is a dualizing functor for $\bar{R}$.

If $\dim R \leq 2$, then $\bar{R}$ is CM since $\bar{R}$ satisfies (S_2).

§3. Reflexive modules/sheaves.

Let R be a noeth. ring M fg R-module

def: M is reflexive if $M \to M^{**}$ is an isomorphism, where $(-)^* = \mathrm{Hom}_R(-, R)$.

$$\begin{array}{ccc} M & \longrightarrow & M^{**} \\ m & \longmapsto & [\varphi \longmapsto \varphi(m)] \end{array}$$

Fact (Bruns-Herzog § 1.4).

M is reflexive iff

(1) M_p is a reflexive R_p module $\forall p$ with depth $R_p \leq 1$

(2) depth $M_p \geq 2$ $\forall p \in \mathrm{Spec}\, R$ with depth $R_p \geq 2$

Thm. Let R be a normal domain. Then M is reflexive iff M satisfies (S_2)

Sketch "If": (1) Let $p \in \mathrm{Spec}\, R$ be s.t. depth $R_p \leq 1$.

Since R satisfies (S_2), ht $p \leq 1$.

$\Rightarrow$ R_p is a RLR (either a fld or a DVR)

and M_p is a torsion free R_p-module

$\Rightarrow$ M_p is a free R_p-module. $\Rightarrow$ M_p is reflexive R_p-mod.

(2) Let $p \in \mathrm{Spec}\, R$ be st depth $R_p \geq 2$.

$\Rightarrow$ ht $p \geq 2$ (dim $R_p \geq 2$)

$\Rightarrow$ Since M satisfies (S_2), depth $M_p \geq 2$

Application :-

① R normal domain, M reflexive R.

Let U be the nonsing locus of $\operatorname{Spec} R$

Then if we know $\tilde{M}|_U$, we can recover $\tilde{M}$ (also M)

Let $I \subseteq R$ be an ideal that set theoretically define $X \setminus U$. Then

$$0 \to H^0_I(M) \to M \to H^0(U, \tilde{M}) \to H^1_I(M) \to \cdots$$

$H^0_I(M) = 0$, $H^1_I(M) = 0$; $H^0_I(M) = \{m \in M \mid I^n m = 0 \text{ for some } n\}$

ht $I \geq 2$

$$H^0_I(M), = H^1_I(M) = 0$$

Since M satisfies (S_2)

② Let X be a normal variety over k (field). $\dim X = n$

Let U = nonsing locus of X.

Using Kähler diffs, we can define $\omega_U := \bigwedge^n \Omega_{U/k}$

ω_U is a dualizing sheaf for U.

we can extend ω_U to a reflexive sheaf ω_X

st $\omega_X|_U = \omega_U$

If X is CM, ω_X is a dualizing sheaf.

§4 Seminormal & deminormal varieties.

Defn: A noeth integral domain R is **seminormal** if $\forall a \in K$, s.t $a^2, a^3 \in R$, $a \in R$.

eg $R = k[t^2, t^3]$ is not seminormal

Swanson–Huneke book

$t \in \mathrm{Frac}(R)$, $t^2, t^3 \in R$, but $t \notin R$

Example: $R = k[x^2, y^2, xy, x^2y, xy^2]$

is seminormal but not normal.

$R \subseteq S := k[x,y] \subseteq \mathrm{Frac}(R)$

integrally closed

look for x^iy^j st $(x^iy^j)^2, (x^iy^j)^3 \in R$.

Defn: X is **deminormal** if $\exists$ a dense open set U st (a) $\mathrm{codim}(X \setminus U) \geq 2$

(b) $\forall$ closed $x \in U$, $\widehat{\mathcal{O}}_{X,x} = \dfrac{k[[x_1, \ldots x_n]]}{(x_1 x_2)}$

X singular pts

(c) X is S_2

Simple normal crossing divisor

[Normal varieties are deminormal.

U = nonsing locus

$x \in U$, $\widehat{\mathcal{O}}_{X,x} \cong k[[x_1, \ldots, x_n]]$]