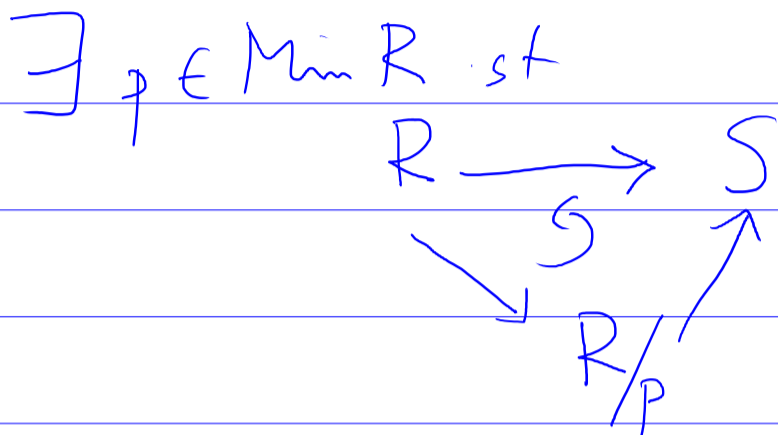
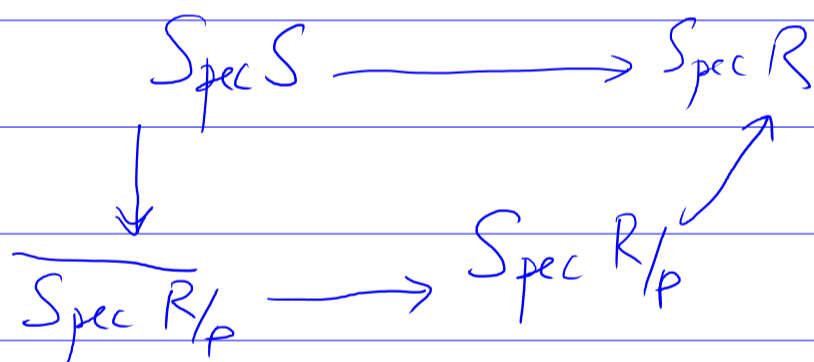


# Non-normal Varieties II. 18/11/23

Recall:  $R$  reduced,  $S$  normal domain  
 $R \rightarrow S$  integral



If  $\text{Im}(\text{Spec } S \rightarrow \text{Spec } R)$  contains a generic point of  $\text{Spec } R$ , then we have



Conductor ideal. Let  $R$  be a domain,  
 $K = \text{frac Fied}$ ,  $\bar{R}$  int closure of  $R$  in  $K$ .

$e_R := \{a \in K \mid a \bar{R} \subseteq R\}$  is called the conductor of  $R$   
 $\uparrow$   
 mult. in  $K$ .

$C_R$  is an  $\bar{R}$ -ideal.

Suppose  $a \in C_R$ . Then  $a \cdot 1 \in R$  so  $C_R \subseteq R$

$C_R$  is the largest abelian subgp of  $R$   
that is both an  $R$ -ideal and an  $\bar{R}$ -ideal.

Example:  $R = k[t^3, t^5] \subseteq \bar{R} = k[t]$

$\uparrow$   
UFD, so int closed.

$\therefore \text{Frac}(R) = \text{Frac}(\bar{R})$ , so  
 $\bar{R}$  is the normalization of  $R$ .

$$t^8, t^9, t^{10}, \dots \in R$$

$$t^7 \notin R.$$

$$\delta = \min \{c \in \mathbb{N} \mid t^i \in R \forall i \geq c\}.$$

$$t^{\delta} \bar{R} = k\langle t^{\delta}, t^{\delta+1}, \dots \rangle \subseteq R$$

$$C_R = (t^{\delta}, t^{\delta+1}, \dots) R$$

Defn. A noeth local ring  $(R, \mathfrak{m})$  is said to be Gorenstein

if (1) it is CM (ie  $\text{depth } R = \dim R$ )

(2) if  $\underline{x} = x_1, \dots, x_d$  ( $d = \dim R$ ) is a sop,

then  $O_{\mathfrak{m}} := \frac{R}{(\underline{x})}$  is 1-dim'l vectorspace

over  $R/\mathfrak{m}$ .

Example (1)  $k[x] / (x^m)$   $m \geq 1$

(2)  $\frac{k[x, y]}{(x^2, y^3)}$ , in general  $\frac{k[x_1, \dots, x_n]}{(f_1, \dots, f_r)}$

$f_1, \dots, f_r$  is a regular seq.

(3) Non-example.  $R = \frac{k[x, y]}{(x, y)^2}$  is not

Gorenstein since  $\dim_R k\langle x, y \rangle = 2$ .

Fact: Let  $(R, \mathfrak{m})$  be a Gorenstein ring. Then  $\text{Hom}_R(-, R)$  is a duality functor for  $R$ -modules.

Now suppose that  $(R, \mathfrak{m})$  is a Gorenstein domain. Assume that  $\bar{R}$  is a fg.  $R$ -module.

(True in geometric situations.)

Let  $k = \text{Frac}(R) = \text{Frac}(\bar{R})$

$e_R = (R :_k \bar{R}) = \text{Hom}_R(\bar{R}, R)$  || not assuming G

$= \omega_{\bar{R}}$  "canonical module" of  $\bar{R}$

If  $\bar{R}$  is CM, then  $\omega_{\bar{R}}$  defined as above satisfies the ff:

$\text{Hom}_{\bar{R}}(-, \omega_{\bar{R}})$  is a duality functor for  $\bar{R}$ .

If  $\dim R \leq 2$ , then  $\bar{R}$  is CM since  $\bar{R}$  satisfies  $(S_2)$ .

### §3. Reflexive modules/sheaves.

Let  $R$  be a noeth. ring  $M$  f.g.  $R$ -module

def.  $M$  is reflexive if  $M \rightarrow M^{**}$   
is an isomorphism, where  
 $(-)^* = \text{Hom}_R(-, R)$ .

$$\begin{array}{ccc} M & \longrightarrow & M^{**} \\ m & \longmapsto & [\varphi \longmapsto \varphi(m)] \end{array}$$

Fact (Bruns-Herzog § 1.4).

$M$  is reflexive iff

- (1)  $M_p$  is a reflexive  $R_p$  module  $\forall p$  with  $\text{depth } R_p \leq 1$
- (2)  $\text{depth } M_p \geq 2 \quad \forall p \in \text{Spec } R$  with  $\text{depth } R_p \geq 2$

Thm. Let  $R$  be a normal domain. Then  $M$  is reflexive  
iff  $M$  satisfies  $(S_2)$

Sketch "If": (1) Let  $p \in \text{Spec } R$  be s.t.  $\text{depth } R_p \leq 1$ .

Since  $R$  satisfies  $(S_2)$ ,  $\text{ht } p \leq 1$ .

$\Rightarrow R_p$  is a RLR (either a fld or a DVR)

and  $M_p$  is a torsion free  $R_p$ -module

$\Rightarrow M_p$  is a f.g.  $R_p$ -module  $\Rightarrow M_p$  is reflexive  $R_p$ -module

(2) Let  $p \in \text{Spec } R$  be s.t.  $\text{depth } R_p \geq 2$ .

$\Rightarrow \text{ht } p \geq 2$  (dim  $R_p \geq 2$ )

$\Rightarrow$  Since  $M$  satisfies  $(S_2)$ ,  $\text{depth } M_p \geq 2$

## Applications

①  $R$  normal domain,  $M$  reflexive  $R$ .

Let  $U$  be the nonsingular locus of  $\text{Spec } R$

Then if we know  $\tilde{M}|_U$ , we can recover  $\tilde{M}$   
( $\cong M$ )

Let  $I \subseteq R$  be an ideal that set theoretically

defines  $X \setminus U$ . Then

$$0 \rightarrow H_I^0(M) \rightarrow M \rightarrow H^0(U, \tilde{M}) \rightarrow H_I^1(M) \rightarrow 0$$

$\text{ht } I \geq 2$        $\downarrow$        $\leftarrow \{m \in M \mid I^n m = 0 \text{ for } n \gg 0\}$        $\downarrow$

$$H_I^2(M) = H_I^1(M) = 0$$

Since  $M$  satisfies  $(S_2)$

② Let  $X$  be a normal variety over  $k$  (fld).  $\dim X = n$

Let  $U = \text{nonsingular locus of } X$ .

Using Kähler differentials, we can define  $\omega_U := \tilde{\Lambda}^n \Omega_{U/k}$

$\omega_U$  is a dualizing sheaf for  $U$ .

We can extend  $\omega_U$  to a reflexive sheaf  $\omega_X$

$$\text{st } \omega_X|_U = \omega_U$$

If  $X$  is  $(M)$ ,  $\omega_X$  is a dualizing sheaf.

## §4 Seminormal & deminormal varieties.

Defn: A noeth integral domain  $R$  is seminormal if  $\forall a \in K$  st  $a^2, a^3 \in R, a \in R$ .

eg  $R = k[t^2, t^3]$  is not seminormal  
 $t \in \text{Frac}(R), t^2, t^3 \in R, \text{ but } t \notin R$

Swanson-Huneke  
book

Example:  $R = k[x^2, y^2, xy, x^2y, xy^2]$   
 is seminormal but not normal.

$$R \subseteq S := k[x, y] \subseteq \text{Frac}(R)$$

integrally  
closed

look for  $x^i y^j$  st  $(x^i y^j)^2, (x^i y^j)^3 \in R$ .

Defn:  $X$  is deminormal if  $\exists$  a dense open

set  $U$  st (a)  $\text{codim}(X \setminus U) \geq 2$

(b)  $\forall$  closed  $X$  singular pts  $x \in U, \widehat{\mathcal{O}}_{x, X} = \frac{k[x_1, x_2]}{(x_1, x_2)}$

(c)  $X \in \mathbb{S}_2$

Simple normal crossing divisor

[ Normal varieties are deminormal.

$U =$  nonsing locus

$$x \in U, \widehat{\mathcal{O}}_{x, X} \cong k[x_1, \dots, x_n]$$