Normality: 16/11/23

§1. Condition (S_2)

§2. normal schemes.

§3. reflexive modules

§4. seminormal rings

§5 demi-normal rings.

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§1. (S_2): – Bruns–Herzog
– Matsumura
– Eisenbud

Defn. Let (R, m) a noetherian ring and M a f.g R-module. The depth of M ($\operatorname{depth} M$) is the length of the longest M-regular sequence in m.

$x_1, \dots, x_t \in m$ is $\underline{M\text{-regular}}$ if

(a) x_1 is a nzd on M, and

(b) $\forall\ 2 \le i \le t$, x_i is a nzd on $\dfrac{M}{(x_1, \dots, x_{i-1})M}$.

Example. (1) If (R, m) is a domain and not a field, then $\operatorname{depth} R > 0$.

(2) $R = k[x,y]_{(x,y)}$ (or $k[[x,y]]$)

Then x, y is a maximal R-reg. sequence

$\therefore$ $\operatorname{depth} R = 2$

(3) $R = k[[x,y]]/(x^2, xy) = \dfrac{k[[x,y]]}{(x) \cap (x,y)^2}$.

has depth 0 since $m \in \operatorname{Ass} R$

(In general $\operatorname{depth} M = 0 \iff m \in \operatorname{Ass} M$.

(4) If R is a reduced ring that is not a field, then $\operatorname{depth} R > 0$.

Rmk ① $\operatorname{depth} M \leq \min\{\dim R/p \mid p \in \operatorname{Ass} M\}$
$\leq \dim R$

② If $x \in m$ is a nzd on M, then $\operatorname{depth} M = \operatorname{depth} M/xM + 1$

Defn. Let $k \in \mathbb{N}$. R noeth, M f.g.

Say that M satisfies condition (S_k)

if $\quad \operatorname{depth} M_p \geq \min\{\dim R_p, k\}$.

instead of this, some books might use $\dim M_p$.

But we need this defn...

It doesnot matter much for us, since most of the time, we will have $\operatorname{Supp} M = \operatorname{Spec} R$

(or a coh sh $\mathcal{F}$ on a noeth scheme X st $\operatorname{Supp} \mathcal{F} = X$)

(S_0): satisfied by all modules.

(S_1): $\quad M$ satisfies (S_1)

$\iff \forall p \in \operatorname{Spec} R$ with $\dim R_p > 0$, $\operatorname{depth} M_p > 0$

$\iff \forall p \in \operatorname{Spec} R$, if $\dim R_p > 0$, then $p \notin \operatorname{Ass} M$

$\iff \operatorname{Ass} M \subseteq \operatorname{Min} R$

Example. R reduced $\Rightarrow$ R satisfies (S_1).

(S_2) : more tricky. (S_1)

Propn. R satisfies (S_2) $\iff$ $\boxed{\operatorname{Ass} R = \operatorname{Min} R}$ and

$\forall a \in R$ nzd and $\forall p \in \operatorname{Ass} R/(a)$, $\operatorname{ht} p = 1$.

"principal ideals gen. by nzd are unmixed"

Proof: R satisfies (S_2) iff $\forall p \in \operatorname{Spec} R$, the ff holds:

(1) $\operatorname{ht} p = 1 \Rightarrow p \notin \operatorname{Ass} R$

(2) $\operatorname{ht} p \geq 2 \Rightarrow \operatorname{depth} R_p \geq 2$

"$(\Rightarrow)$": Let $a \in R$ be a nzd, $\& \ p \in \operatorname{Ass} R/(a)$.

BWOC assume that $\operatorname{ht} p \neq 1$.

$\Rightarrow \operatorname{ht} p \geq 2 \Rightarrow \operatorname{depth} R_p \geq 2$

$\Rightarrow \operatorname{depth} R_p / aR_p \geq 1$

On the other hand $p \in \operatorname{Ass} R/(a)$ so

$\operatorname{depth} R_p / aR_p = 0$

$\therefore \operatorname{ht} p = 1$.

($\Leftarrow$) Let $p \in \operatorname{Spec} R$. If $\operatorname{ht} p = 1$, then $p \notin \operatorname{Ass} R$ since $\underline{\operatorname{Ass} R = \operatorname{Min} R}$

Now assume that $\operatorname{ht} p \geq 2$. $\therefore \exists\, a \in p$ that is a nzd on R. $\Rightarrow$ depth $R_p / aR_p \geq 1$

$\Rightarrow$ depth $R_p \geq 2$

$\therefore$ R satisfies (S_2)

<u>Rmks</u>

(1) Let $k \geq 1$. Then R satisfies (S_k) iff $R/(a)$ satisfies (S_{k-1}) $\forall$ nzd $a \in R$.

(2) (R, m) is Cohen Macaulay (ie depth M = dim M) iff it satisfies (S_k) $\forall k \geq 1$.
((R, m) CM $\Rightarrow$ (R_p, pR_p) CM $\forall p \in \operatorname{Spec} R$).

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§2 Normal rings / schemes

Say that R is <u>normal</u> if $\forall p \in \operatorname{Spec} R$, R_p is a domain that is integrally closed in its fraction fld.

S domain K - fraction fld.

$\bar{S} := \{\alpha \in K \mid \alpha \text{ int}/S\}$ is a ring with
$S \subseteq \bar{S} \subseteq K$

Say that S is integrally closed (in K)

if $\bar{S} = S$

Propn Every (noetherian) normal ring is a finite product of normal domains

Proof. Let R be a normal.
If R is a domain ✓
Otherwise Write $0 = J_1 \cap \cdots \cap J_r$
an irredundant primary decomposition of 0
Say that $\exists\ p \in \operatorname{Spec} R$ st $p \supseteq J_i$ & $p \supseteq J_j$
for some $i \neq j$. Then R_p is not even a domain. $\Rightarrow$ $\{J_i\}$ is pairwise comax'l (ie $J_i + J_j = R$)

$$\Rightarrow \quad R \longrightarrow \prod_{i=1}^{r} \underbrace{R/J_i}_{R_i} \quad \text{is an iso (of rings)}$$

$$R \simeq \prod R_i \quad \text{normal} \overset{(*)}{\Rightarrow} R_i \text{ is normal } \forall i$$
$$\Rightarrow R_i \text{ is a normal domain } \forall i$$

Rmk. With notation as in the proof above,

$$\operatorname{Spec} R = \coprod_{\text{disjoint}} \operatorname{Spec} R_i$$

Cor. Every noeth normal scheme is a disjoint union of noeth normal integral schemes.

explanation for (*)

Let $p \in \operatorname{Spec} R$. Then $\exists ! i$ and $\exists ! p_i \in \operatorname{Spec} R_i$ st under this

$$\operatorname{Spec} R = \coprod \operatorname{Spec} R_i$$
$$p \longmapsto p_i.$$

$$R_p = (R_i)_{p_i}$$

Thm. Let R be a noeth domain. Then R is normal iff it satisfies the ff:

(1) (R_1): R_p is a RLR $\forall p \in \operatorname{Spec} R$ with $\operatorname{ht} p \leq 1$

(2) (S_2).

Rmk. (R_1): R_p RLR $\Rightarrow \forall \overset{\text{prime ideals}}{q} \subseteq p$, R_q is a RL

$\therefore$ for a domain R, R satisfies (R_1) iff R_p is a DVR $\forall p$ of $\operatorname{ht} 1$

Cor. Let X be a noetherian normal scheme. Then the singular locus of X $\left(:= \{x \in X \mid \mathcal{O}_{X,x} \text{ is not a RLR}\}\right)$ has codim ≥ 2 in X.

Examples: (1) isolated hypersurface singularities.

$$R = \cancel{\frac{k[[x_1, \ldots, x_n]]}{(f)}} \quad n \geq 3$$

st singular locus of R is isolated

ie R_p is RLR $\forall p \neq$ max'l ideal.

$k[[x_1,\dots,x_n]]$ is Cohen Macaulay
n-dim'l

$\Rightarrow$ R satisfies (S_{n-1}) $((S_k)\ \forall k)$

Sing R is 0-dim'l. (want this to be of codim ≥ 2, so dim $R \geq 2$ ie $n \geq 3$)

(2) isolated complete intersection singularities of dim ≥ 2
(c equations gen. a ht c ideal)

(3) Cusp singularity $y^2 = x^3$

Non example

$$\frac{k[x,y]_{(x,y)}}{(y^2-x^3)} \quad \left(\text{or } \frac{k[[x,y]]}{(y^2-x^3)}\right)$$

$$\cong k[[t^2,t^3]]$$

$$k[x,y] \longrightarrow k[t]$$
$$x \longmapsto t^2$$
$$y \longmapsto t^3$$

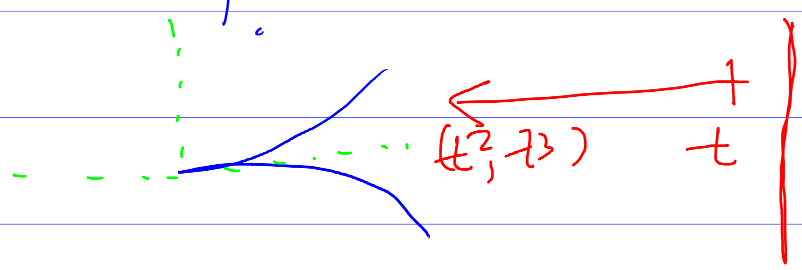
$$R = k[t^2,t^3] \subseteq k[t] \subseteq k(t) = \text{Frac}(R)$$

UFD $\Rightarrow$ normal

$\therefore$ the normalization of R is $k[t]$

t is integral over R. but not in R.

$\operatorname{Spec} R \longleftarrow \operatorname{Spec} k[t]$.



normalization map

(4) nodal cubic. $y^2 = x^2(x+1)$

non example.

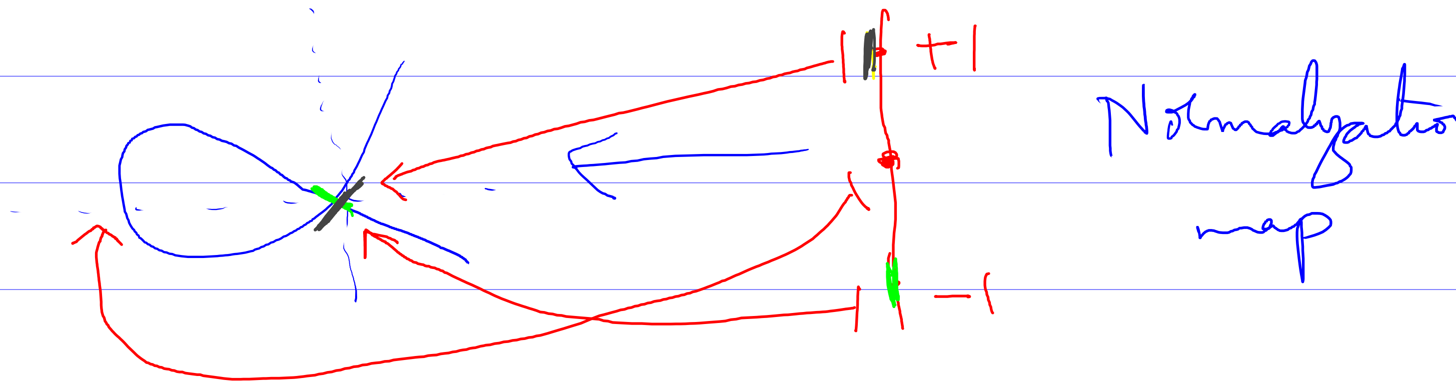
$$k[x,y] \longrightarrow k[t]$$
$$x \longmapsto t^2 - 1$$
$$y \longmapsto t^3 - t$$

$$\left(\frac{y}{x}\right)^2 = x+1$$
$$t^2 = x+1$$

$$R = \frac{k[x,y]}{(y^2 - x^2(x+1))}$$

$\frac{y}{x} = t \in \operatorname{Frac}(R)$ & integral over R.
but $t \notin R$.

$$R \subseteq k[t] \subseteq k(t) = \operatorname{Frac}(R)$$



Every 1-dim'l normal domain R is regular

i.e. R_p is a DVR $\forall p \in \operatorname{Spec} R, p \neq 0$

In general, if R is a finite type domain over a field, $\bar{R}$ is a finite R-alg

$\operatorname{Spec} R \longleftarrow \operatorname{Spec} \bar{R}$ is a finite morphism

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If R is a reduced ring, then we have this map

$$R \hookrightarrow \prod_{p \in \operatorname{Min} R} R/p$$

The normalization of R is $\prod_{p \in \mathrm{Min}\, R} \overline{R/p}$.

[Y normal scheme, X reduced scheme

$$\begin{tikzcd} Y \arrow[rr, "\text{dominant map}"'] \arrow[dr] & & X \\ & \bigsqcup \overline{X_i} \arrow[r] & \bigsqcup X_i \arrow[u] \end{tikzcd}$$

$\{X_i\}$ irred components of X.]