Families of varieties of general type

as part of classification of algebraic varieties

(reduced schemes of finite type / field)

Restrict to char = 0

X alg var

Def: model = a repr of a birational equiv. class.

Two step process

① Find a "nice" model in every birat. class

② Classify these models.

Expectations: ① • should be relatively simple

• should be unique

• should be a way to find this model

② • Find invariants: ...

• After finding enough invariants: the remaining objects form finitely many families.

Searching for "nice"

- the chosen model should be projective

<u>Nagata</u>: $\forall$ variety $\exists$ proper closure

<u>Chow</u>: $\exists$ proj resolution of the proper closure.

<u>Hironaka</u>: $\exists$ smooth, proj birational model. $\leftarrow$

For curves we can stop here: a smooth proj curve is unique in its birat. class.

In higher dim $\rightsquigarrow$ MMP

$X \rightsquigarrow$ contraction or flip $\rightsquigarrow X \sim_{bir} X_1 \longrightarrow X_2$

Fano fibre space (Mori fibration)

If $\kappa(X) \geq 0$ this is trivial

After iterating: up to bir equivalence

$X \underset{bir}{\sim} X_1 \longrightarrow X_2$ $\qquad$ $\underline{\kappa(X_2) \geq 0}$

tower of Fano fibrations

$X_2 \qquad \kappa(X_2) \geq 0$

<u>Iitaka fibration</u>

$$X_2 \sim_{bir} X_3 \longrightarrow X_4$$

- ~~$\dim X_4 = \kappa(X_2)$~~ $\boxed{\dim X_4 = \kappa(X_4)}$
- the gen. fibre has $\kappa = 0$

Again, iterate this $\rightsquigarrow$ tower of fibrations with the gen. fibre having $\kappa = 0$

<u>MMP again</u>

Run the MMP on X_4 (of gen. type)

$\dim = \kappa \rightsquigarrow X_4 \sim_{bir} X_{min} \longrightarrow X_{can}$

X_{min} = gen. type

$\exists$ a canonical model

Kodaira dim: X smooth proj $\rightsquigarrow K_X$ canonical class (divisor)

ω_X – canonical sheaf $\qquad T_X$ tangent bdle

$$\omega_X = (\det T_X)^\vee$$

$$m^{\kappa(X)} \sim^{\dim} H^0(X, \omega_X^{\otimes m}) : X \overset{\phi_m}{\dashrightarrow} \mathbb{P}^N \geq \overline{\phi_m(X)}$$

$$\kappa(X) := \max_m \underline{\dim \overline{\phi_m(X)}} \qquad \kappa(\mathbb{P}^n) < 0 \qquad \dim \emptyset = -1 \text{ or } -\infty$$

$$-1 \leq \kappa(X) \leq \dim X$$

__Def__: X is of general type if $\kappa(X) = \dim X$.

Example $X \subseteq \mathbb{P}^n$ is a hypersurface of deg d

$\kappa(X) < 0 \quad d \leq n$

$\kappa(X) = 0 \quad d = n+1$

$\kappa(X) = \dim X \quad d \geq n+2$

HW: do this

Hint: use adjunction for K_X

$$X_1 \rightsquigarrow X_2 \rightsquigarrow X_4 \xrightarrow{\text{bir}} X_{min} \xrightarrow{\text{bir}} X_{can}$$

X_1: Fano fibrations

X_2: fibrations with gen. fibre $\kappa = 0$

X_4: gen type

X_{can}: canonically polarized variety

ω_X is ample

For surfaces min models are smooth

Even for surfaces can. models behave better.

In higher dimensions: min models are not unique in dim ≥ 3

min models are singular

<u>Canonical models are always unique</u>

$\rightsquigarrow$ For varieties of gen. type "nice" = canonical model

In order to do a classification (K-stability)
we need to understand
- families of Fano varieties (ω_X^{-1} ample)
- families of varieties with $\kappa = 0$ (ex: $\omega_X \simeq \mathcal{O}_X$
e.g. CY, Abelian
- $\boxed{\text{families of canonically polarized varieties}}$

Curves: $\mathcal{M}_g \subseteq \overline{\mathcal{M}}_g \leftarrow$ compactification of $\mathcal{M}_g$

it is actually projective

Goal: do this in higher dim's

main objects: stable curves

a proj curve C is stable if

local: · the sings of C are nodes $\times$ ($\Rightarrow C$ is Gorenst)

global: · ω_C is ample $\leftarrow$ this is what we want in all dimensions.

Stable extension

B smooth curve ⇝ base

$B^\circ \subseteq B$ dense open subset $\quad \exists\, B' \xrightarrow{\sigma} B$ finite surj

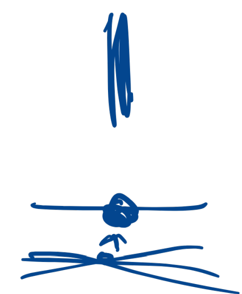
$$\begin{tikzcd}
 & X^\circ \times B' \arrow[r, hook] \arrow[dl] \arrow[d] & X^{ss}_B \arrow[r, "\tau"] \arrow[d] & X^{stab}_B \arrow[dl] \\
X^\circ \arrow[d, "\pi^\circ"'] & B^\circ \times_B B' \arrow[r, hook] \arrow[dl] & B' & \\
B^\circ & & &
\end{tikzcd}$$

flat family of reduced nodal curves

flat fam of stable curves

rel can. model

flat family of smooth proj curves of genus ≥ 2



Kollár – Shepherd-Barron (1988)

$$\begin{tikzcd} X^0 \arrow[d, "\pi^0"'] \\ B^0 \end{tikzcd} \xrightarrow[\text{extend}]{\text{step 1}} \begin{tikzcd} X \arrow[d, "\pi"'] \\ B \end{tikzcd} \xrightarrow{\text{step 2}} \begin{tikzcd} X \arrow[d, "\pi"'] & X' \arrow[l] \arrow[d] \\ B & B' \arrow[l] \end{tikzcd}$$

has reduced fibres

finite

smooth proj fam. of can-ly pol'd varieties

X smooth
X/B proj
will have
big fibres

step 3
Run the rel MMP on X'/B'

$$\begin{tikzcd} X^{ss}_{B'} \arrow[r, "\tau"] \arrow[d] & X^{stab}_{B'} \arrow[dl, bend left] \\ B' & \end{tikzcd}$$

rel can. model

Canonical sings.

flat fam of stable vari's

stable vari's should have slc sings.

KSB stability

X is KSB stable
if (local) X has slc sings
(global) K_X is ample
(K_X is not a Cartier div
but $\mathbb{Q}$-Cartier)

Stable families

curves : stable family = flat fam of stable curves

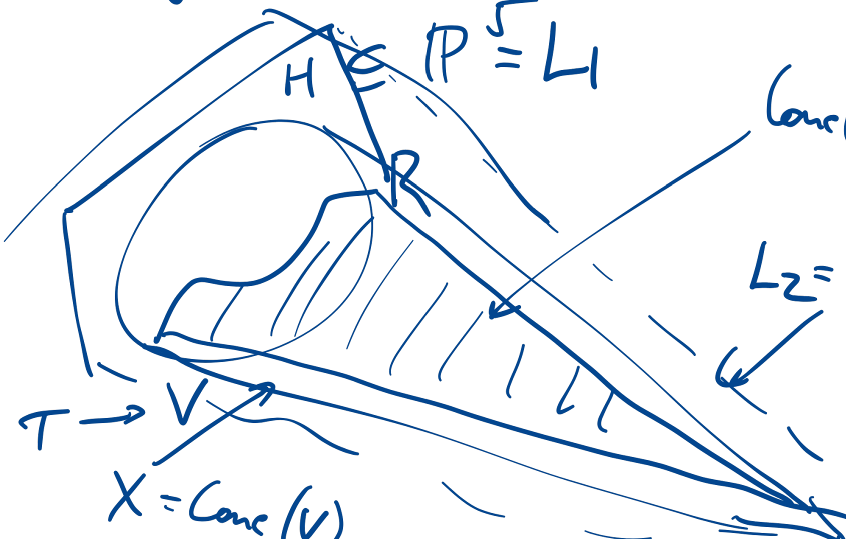
higher dim :

$\mathbb{P}^2 \xrightarrow[\sim]{} V \subseteq \mathbb{P}^5$ Veronese embedding

$\deg V$ $\quad H \subseteq \mathbb{P}^5$ gen hyp. plane

$V \cap H = R \subseteq H \simeq \mathbb{P}^4$

$\simeq$ rtl normal quartic curve [image of $[t^4 : t^3u : t^2u^2 : tu^3 : u^4]$]



Cone(R)

$X := \mathrm{Cone}(V) \subseteq \mathbb{P}^6$

$L_2 = \mathrm{Cone}(H) \subseteq \mathbb{P}^6$ hyperplane

$V = X \cap L_1$

$\mathrm{Cone}(R) = X \cap L_2$

$\rightsquigarrow$ V can be deformed into Cone(R)

$$\begin{tikzcd} \mathbb{P}^1 \times \mathbb{P}^1 \arrow[r, hook, "{(2,1)}"'] \arrow[dr, "\sim"'] & \mathbb{P}^5 \\ & T \arrow[u, phantom, "\cup"] \end{tikzcd}$$

$\mathcal{O}(2) \quad \mathcal{O}(1)$

$\deg T = 4$

$T \cap H = R' \subseteq H$

$H \subseteq \mathbb{P}^5$ gen. h.p.

T can be deformed into $\underline{\underline{\text{Cone}(R)}}$ $\rightsquigarrow$ V

$\wr\wr$ $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ $\wr\wr$

$\mathbb{P}^1 \times \mathbb{P}^1 \quad K_T^2 = -8 \qquad\qquad K_V^2 = -9 \quad \mathbb{P}^2$

Young person's guide to moduli theory