MT-groups over a complete noetherian ring

19. Projective limits of homomorphisms: the algebrisation theorem

DEFINITION 19.1. Let I be an ideal in a ring A. One says that A is *separated and complete for the I-adic topology* if the natural ring homomorphism $A \to \varprojlim_n A/I^n$ is bijective. In other words, $\bigcap_{n\in\mathbb{N}} I^n = \{0\}$ and every sequence $(a_n)_{n\in\mathbb{N}}$ of elements of A such that $a_{n+1} - a_n \in I^{n+1}$ converges to an element a of A (equivalently, every series $\sum_{n\geq 0} b_n$ with $b_n \in I^n$ converges to an element b of A).

THEOREM 19.2. *Let A be a noetherian ring, with an ideal I such that A is separated and complete for the I-adic topology. Set $S = \operatorname{Spec}(A)$ and $S_n = \operatorname{Spec}(A/I^{n+1})$ for $n \geq 0$.*

Let G be an affine S-group scheme and H an isotrivial MT-group over S. *For each n, we denote by G_n, H_n their pullbacks to S_n.*

(1) *The canonical map $\theta : \operatorname{Hom}_{S\text{-gr.}}(H,G) \longrightarrow \varprojlim_n \operatorname{Hom}_{S_n\text{-gr.}}(H_n,G_n)$ is bijective.*

(2) *Suppose further that* G is flat over S at each point of G_0 and G_0 is an isotrivial MT-group over S_0. *Then the map* $\operatorname{Hom}_{S\text{-gr.}}(H,G) \longrightarrow \operatorname{Hom}_{S_0\text{-gr.}}(H_0,G_0)$ *is bijective.*

PROOF. (1) Suppose first the result proved when H is diagonalisable. In the general case, there exists by hypothesis a surjective *finite* étale morphism $A \to A'$ such that the pull-back H' of H to $S' = \operatorname{Spec} A'$ is diagonalisable. Then so are H'' and H'_n, H''_n, with obvious notation. Moreover, since A' and A'' are finite over A, they are separated and complete for the topology defined by the ideals IA' and IA''. Thus, assuming the result proved in the diagonalisable case, the second and third vertical maps in the commutative diagram below are bijective:

$$\begin{tikzcd}
\operatorname{Hom}_{S\text{-gr}}(H,G) \arrow[r] \arrow[d, "u"] & \operatorname{Hom}_{S'\text{-gr}}(H',G') \arrow[r, shift left] \arrow[r, shift right] \arrow[d, "u'"] & \operatorname{Hom}_{S''\text{-gr}}(H'',G'') \arrow[d, "u''"] \\
\varprojlim_n \operatorname{Hom}_{S_n\text{-gr}}(H_n,G_n) \arrow[r] & \varprojlim_n \operatorname{Hom}_{S'_n\text{-gr}}(H'_n,G'_n) \arrow[r, shift left] \arrow[r, shift right] & \varprojlim_n \operatorname{Hom}_{S''_n\text{-gr}}(H''_n,G''_n).
\end{tikzcd}$$

Further, $A \to A'$ is a morphism of descent (being faithfully flat and quasi-compact) hence the first row is exact, as well as the analogous row for a given n. Since projective limits are left exact, the bottom row is also exact. It follows that the first vertical map is bijective.

Thus, it suffices to prove the theorem when H is diagonalisable, say $H = D(M)_S$. Set $B = A[M]$ and let C be the A-Hopf algebra of the affine group scheme G. Denote by Δ_B and Δ_C their comultiplication maps. Denoting by $(e_m)_{m\in M}$ the canonical basis of $A[M]$, recall that $\Delta_B(e_m) = e_m \otimes e_m$.

For $n \in \mathbb{N}$, set $A_n = A/I^{n+1}$ and let B_n and C_n be obtained by base change. Note that $B_n = A_n[M]$. The morphisms of S-group schemes $H \to G$, resp. $H_n \to G_n$, correspond to the morphisms of A-Hopf algebras $\varphi : C \to B$, resp. $\varphi_n : C_n \to B_n$. Set $\widehat{B} = \varprojlim_n B_n$ and $\widehat{C} = \varprojlim_n C_n$ and let $\tau_B : B \to \widehat{B}$ and $\tau_C : C \to \widehat{C}$ be the canonical maps.

[0]version of Sept. 2, 2023, after the lecture.

Note first that one has $I^nB = \bigoplus_m I^n e_m$ for each n, hence $\bigcap_{n\in\mathbb{N}} I^nB = \{0\}$. That is, τ_B is injective.[1] Since $B \otimes_A B = A[M \times M]$, the same argument shows that $\tau_{B\otimes_A B}$ is injective too. The injectivity of τ_B immediately gives that the map θ is **injective**. Indeed, for any morphism of Hopf algebras $\varphi : C \to B$, the projective system $\theta(\varphi) = (\varphi_n)_{n\geq 0}$ of morphisms of Hopf algebras induces a morphism of algebras $\widehat{\varphi} : \widehat{C} \to \widehat{B}$ such that the diagram

$$\begin{tikzcd} C \arrow[r, "\varphi"] \arrow[d, "\tau_C"'] & B \arrow[d, "\tau_B"] \\ \widehat{C} \arrow[r, "\widehat{\varphi}"] & \widehat{B} \end{tikzcd}$$

is commutative. Since τ_B is injective, this shows that the map $\theta : \varphi \mapsto (\varphi_n)_{n\geq 0}$ is injective.

Let us prove that θ is surjective. Let $(\varphi_n)_{n\geq 0}$ be a projective system of Hopf algebra morphisms $C_n \to B_n$. It induces a morphism of algebras $\widehat{\varphi} : \widehat{C} \to \widehat{B}$.

What we want is a morphism of Hopf algebras $C \to B$, but a difficulty is that taking the projective limit of the comultiplication maps

$$\Delta_{B_n} : \quad B_n = A_n[M] \to B_n \otimes B_n = A_n[M \times M]$$

gives a map $\widehat{\Delta_B} : \widehat{B} \to \widehat{B \otimes B}$. As noted in footnote (1) the latter algebra is the A-submodule of the product $A^{M\times M}$ consisting of families $(a_{m,m'})$ which tend to zero. Further, the projective system of morphisms $\widehat{B} \otimes \widehat{B} \to B_n \otimes B_n$ gives a morphism of algebras $\eta : \widehat{B} \otimes \widehat{B} \to \widehat{B \otimes B}$ but this morphism is not surjective in general. However, we have the following commutative diagram:

$$\begin{tikzcd}
C_n \arrow[rrrrr, "\varphi_n"] \arrow[ddd, "\Delta_{C_n}"'] & & & & & B_n \arrow[ddd, "\Delta_{B_n}"] \\
& C \arrow[ul] \arrow[r, "\tau_C"] \arrow[d, "\Delta_C"'] \arrow[drrr, "\Psi"] & \widehat{C} \arrow[rr, "\widehat{\varphi}"] & & \widehat{B} \arrow[ur] \arrow[d, "\widehat{\Delta_B}"] & \\
& C \otimes C \arrow[dl] \arrow[r, "\tau_C\otimes\tau_C"] & \widehat{C} \otimes \widehat{C} \arrow[r, "\widehat{\varphi}\otimes\widehat{\varphi}"] & \widehat{B} \otimes \widehat{B} \arrow[r, "\eta"'] & \widehat{B \otimes B} \arrow[dr] & \\
C_n \otimes C_n \arrow[rrrrr, "\varphi_n\otimes\varphi_n"] & & & & & B_n \otimes B_n
\end{tikzcd}$$

Set $\Phi = \widehat{\varphi} \circ \tau_C$ and let Ψ be the composed map indicated in the diagram:

$$C \xrightarrow{\Delta_C} C \otimes C \xrightarrow{\Phi\otimes\Phi} \widehat{B} \otimes \widehat{B} \xrightarrow{\eta} \widehat{B \otimes B}.$$

For each $f \in C$, $\Phi(f)$ is a family (a_m) of $\widehat{B}$ whose image by $\widehat{\Delta_B}$ is a family $(a_{m,m'})$ of $\widehat{B \otimes B}$ which satisfies the hypotheses of Lemma 19.3 below. Hence the support of the families (a_m) and $(a_{m,m'})$ are finite. Therefore $\Phi(C) \subset B$ and $\Psi(C) \subset B \otimes B$ (recall that $\tau_{B\otimes B}$ is injective) and we obtain the commutative diagram below:

$$\begin{tikzcd} C \arrow[r, "\Phi"] \arrow[d, "\Delta_C"'] & B \arrow[d, "\Delta_B"] \\ C \otimes_A C \arrow[r, "\Phi\otimes\Phi"] & B \otimes B. \end{tikzcd}$$

[1] Moreover $\widehat{B}$ identifies with the A-submodule of the product A^M consisting of families $(a_m)_{m\in M}$ which tend to zero in the sense such that for each $n \in \mathbb{N}$, all but a finite number of the a_m belong to I^n.

This proves that Φ is a morphism of Hopf algebras $C \to B$, which reduces modulo I^{n+1} to the given φ_n. This completes the proof of assertion (1).

(2) By assertion (1), the map $\mathrm{Hom}_{S\text{-gr.}}(H, G) \longrightarrow \varprojlim_n \mathrm{Hom}_{S_n\text{-gr.}}(H_n, G_n)$ is bijective. Further, by Th. 18.3, each G_n is an isotrivial MT-group over S_n and hence, by assertion (1) of Prop. 18.1, each map $\mathrm{Hom}_{S_n\text{-gr.}}(H_n, G_n) \to \mathrm{Hom}_{S_0\text{-gr.}}(H_0, G_0)$ is bijective. This proves assertion (2). □

LEMMA 19.3. *Let A be a noetherian ring, M a set, $(a_{m,m'})$ a family of elements of A indexed by $M \times M$ such that:*

(1) *$a_{m,m'} = 0$ if $m \neq m'$ (i.e. the support of the family is contained in the diagonal of $M \times M$).*

(2) *There exist a finite number of elements $b^i, c^i \in A^M$ such that $a_{m,m'} = \sum_i b^i_m c^i_{m'}$ for every m, m'. (This means that the element $(a_{m,m'})$ of $A^{M\times M}$ is the image of the element $\sum_i b^i \otimes c^i$ under the canonical morphism $A^M \otimes_A A^M \to A^{M\times M}$).*

Then the support of the family $(a_{m,m'})$ is finite.

PROOF. By (1), the family $(a_{m,m'})$ is determined by the $a_m = a_{m,m}$. Define a homomorphism $u : A^{(M)} \to A^M$ as follows: for every $x = (x_m)_{m\in M} \in A^{(M)}$,

$$u(x)_{m'} = \sum_m a_{m',m}\, x_m.$$

Denote by (e_m) the canonical basis of $A^{(M)}$. By (1), one has simply $u(e_m) = a_m e_m$. On the other hand, by (2) one has

$$u(e_m)_{m'} = \sum_i c^i_m b^i_{m'}$$

hence the $u(e_m) = a_m e_m$ are contained in the finitely generated A-module $\sum_i Ab^i$. Since A is noetherian, they generate a finitely generated A-module. Since the e_m are linearly independent, it follows that $a_m = 0$ for all but a finite number of m. □

COROLLARY 19.4. *Let A, I, S, S_0 and H be as in* Th. 19.2, *suppose that G is a* smooth affine *S-group scheme and let $u_0 : H_0 \to G_0$ be a morphism of S_0-group schemes. Then:*

(1) *There exists a morphism of S-group schemes $u : H \to G$ that lifts u_0.*

(2) *If v is another such lifting, there exists $g \in \mathrm{Ker}\big(G(S) \to G(S_0)\big)$ such that $v = \mathrm{int}(g)\circ u$.*

PROOF. (1) Using Theorem 17.1, one can lift u_0 to a projective system of morphisms (u_n). Then assertion (1) of theorem 19.2 gives a morphism $u : H \to G$ lifting u_0.

(2) Let u, v be two liftings of u_0. By Theorem 17.1, one obtains a projective system of elements $g_n \in \mathrm{Ker}\big(G(S_n) \to G(S_0)\big)$ such that $v_n = \mathrm{int}(g_n) \circ u_n$ for all n. That is, we have a projective system of morphisms of algebras $g_n : C \to A/I^{n+1}$. Since A is separated and complete the sequence $g_n(c)$ converges, for each $c \in C$, to an element $g(c)$. This gives an element $g \in \mathrm{Ker}\big(G(S) \to G(S_0)\big)$ such that $v_n = (\mathrm{int}(g) \circ u)_n$ for all n. Since the morphism θ of Theorem 19.2 is injective, it follows that $v = \mathrm{int}(g) \circ u$. □

20. The density theorem

Recall that we have always assumed that MT-groups be of finite type, i.e. that the corresponding abelian group M be finitely generated. As this is important in the next theorem, we write this hypothesis explicitly.

Remark 20.1. Let G be a commutative affine group scheme over S. For each $n \geq 1$, let ${}_nG$ be the kernel of the n-th power map r_n. As we have a cartesian diagram

$$\begin{tikzcd} G \arrow[r, "r_n"] & G \\ {}_nG \arrow[u] \arrow[r] & S \arrow[u, "\varepsilon"'] \end{tikzcd}$$

and the unit section $\varepsilon : S \to G$ is a closed immersion (G being affine hence separated over S), one obtains that ${}_nG$ is a closed sugbgroup of G, hence is defined by a sheaf of ideals that we will denote by $\mathcal{I}_n$.

Theorem 20.2. *Let G be a* MT-*group* of finite type *over S. For each $n \geq 1$, let ${}_nG$ be the kernel of the n-th power map and $\mathcal{I}_n$ the corresponding sheaf of ideals.*

(1) *Let Z be a* closed *subscheme of G containing all ${}_nG$ in the schematic sense, i.e. if $\mathcal{J}$ is the sheaf of ideals of $\mathcal{O}_G$ defining Z, the assumption is that $\mathcal{J} \subset \mathcal{I}_n$ for all n. Then $Z = G$, i.e. $\mathcal{J} = 0$.*

(2) *Let H be a* subgroup *scheme of G such that each ${}_nG$ is a closed subscheme of H. Then $H = G$.*

Proof. [2] (1) Taking a covering of S by affine subsets, we may suppose that $S = \operatorname{Spec} A$ is affine. Denote then by I_n the ideal of $\mathcal{O}(G)$ corresponding to ${}_nG$. Let $A \to A'$ be a faithfully flat morphism such that the pullback G' of G to $S' = \operatorname{Spec}(A')$ is isomorphic to $D(M)_{S'}$ for some finitely generated abelian group M. Since the formation of kernels commutes with base change, we have ${}_nG' = ({}_nG) \times_S S'$ and hence, with obvious notation, $I'_n = I_n \otimes_A A'$.

We have to prove that any $f \in \mathcal{O}(G)$ belonging to all I_n is zero. Since the map $\mathcal{O}(G) \to \mathcal{O}(G) \otimes_A A' = \mathcal{O}(G')$ is injective, it suffices to prove the corresponding result over S'. Thus, replacing S by S' we may assume that $G = D(M)_S$, i.e. $\mathcal{O}(G) = A[M]$.

One has $M \simeq \mathbb{Z}^r \times Q$ for some finite abelian group Q of order q. Denote by B the Laurent polynomial ring $A[T_1^{\pm 1}, \dots, T_r^{\pm 1}]$, then $A[M] \simeq B[Q]$. Let $f = \sum_{x \in Q} b_x\, x \in A[M]$ and suppose that f is zero in each quotient $\mathcal{O}({}_nG) = A[M/nM]$. Let m be the supremum of the absolute values of the exponents of the T_i's in the various b_x. Let n be a multiple of q which is $> 2m$. Then

$$A[M/nM] \simeq \big(A[T_1, \dots, T_r]/(T_1^n - 1, \dots, T_r^n - 1)\big)[Q]$$

and hence in $A[M/nM]$ the elements $T_1^{d_1} \cdots T_r^{d_r} b_x$, with $-m \leq d_i \leq m$ and $x \in Q$ are linearly independent over A. It follows that every coefficient of f is zero, hence $f = 0$. This proves (1).

(2) When S is the spectrum of a field k, one knows that every subgroup scheme is *closed*, hence (2) follows from (1) in that case. In the general case, let H be a subgroup scheme of G containing all the ${}_nG$. Being a subscheme means that H is a closed subscheme of an open subscheme U of G.

Then, on each fiber one has $H_s = G_s$. Thus H has the same underlying space as G, hence $U = G$ and H is a closed subscheme of G, and we conclude by (1) that $H = G$. □

Remark 20.3. (1) Note that the assumptions and conclusions in the previous theorem are *schematic* and not purely topological. Let us illustrate this in the case where $S = \operatorname{Spec}(k)$ for an algebraically closed field of characteristic $p > 0$.

a) Let $G = \mu_{p,S}$; then $\mathcal{O}(G) = k[T]/(T^p - 1) \simeq k[T]/(T-1)^p$. Here the reduced scheme $G_{\text{red}} = S$ has the same topological space as G, but is not equal to G.

[2] This neat proof is due to Joseph Oesterlé, see [**Oes14**], §8. In [**SGA3**$_2$], IX, Th. 4.7, Grothendieck proves the stronger result that the family of subschemes $({}_nG)_{n \geq 1}$ is *schematically dense* in G.

b) On the other hand, let $G = \mathbb{G}_{m,S} \times_S \mu_{p,S} = D(M)_S$, where $M = \mathbb{Z} \times \mathbb{Z}/p\mathbb{Z}$. The above proof shows that it suffices to consider the ${}_{n_i}G$ for a sequence of integers $(n_i)_{i\in\mathbb{N}^*}$ divisible by the order of the torsion group of M such that $\lim_{i\to\infty} n_i = +\infty$. Here we can take $n_i = p^i$; then the subgroup schemes ${}_{p^i}G \simeq \mu_{p^i,S} \times_S \mu_{p,S}$ have the same topological space as S, but any subgroup scheme of G containing them must equal G.

(2) One could be tempted to call "schematic density" the notion considered in the theorem. In fact, this terminology is used for a stronger property: one says that a family $(Y_i)_{i\in I}$ of subschemes of a scheme X is *schematically dense* if for every open subset U of X, every closed subscheme Z of U which contains all $Z_i \cap U$ must equal U. In [**SGA3**$_2$], Th. 4.5, Grothendieck proves a stronger version of Th. 20.2: the family $({}_nG)_{n\in\mathbb{N}^*}$ is schematically dense in G.

Remark 20.4. The theorem does not hold for the (non-finitely generated) abelian group $\mathbb{Q}$. Indeed, setting $G = D(\mathbb{Q})_S$, one has ${}_nG = \{1\}$ for all $n \geq 1$.

21. Free actions of MT-groups on schemes affine over S

This section was meant to be given in Lecture 6, after the results on faithfully flat descent of Section 9 and their applications in Section 10. In order to go quickly into deformation theory, we postponed it till now and perhaps, due to lack of time, this material will not be covered in an actual lecture. In a later reorganisation of this notes, this section will probably be moved to an earlier place.

Theorem 21.1. *Let H be a* MT*-group scheme over S acting freely, say on the right, on a scheme X affine over S. Then there exists a scheme Y affine over S, together with a faithfully flat, H-invariant, morphism $p : X \to Y$, which represents the quotient X/H.*

In particular, p makes X into a H_Y-torsor over Y, where $H_Y = H \times_S Y$.

Proof. See [**SGA3**$_2$], VIII Th. 5.1 together with IX, Prop. 2.3, or [**Oes14**], §10. □

Corollary 21.2. *Let $u : H \to G$ be a* monomorphism *of S-group schemes, where is H is a* MT*-group and G is affine over S. Then:*

(1) *u is a closed immersion.*
(2) *There exists a scheme Y affine over S, together with a faithfully flat morphism $p : G \to Y$, which represents the quotient G/H.*
(3) *Further, if H is a normal subgroup of G then Y has a structure of S-group scheme such that p is a morphism of group schemes.*

Proof. Assertions (1,2) are in Exp. IX, Cor 2.5, whereas assertion (3) follows from Exp. IV, Prop. 5.2.3. □

Notes for this Lecture

Lemma 19.3 is Lemma 7.2 of Exp. IX.

Assertion (1) of Th. 19.2 is Th. 7.1 of Exp. IX, while assertion (2) is Lemma 3.1 of Exp. X.

The proof of Th. 20.2 is that given by Oesterlé in [**Oes14**], §8. In [**SGA3**$_2$], IX, Th. 4.7, a stronger result is proved (with a much longer proof).

The references for the results of Section 21 are given in the text.