LECTURE 8

MT-groups over a complete noetherian ring

18. MT-groups over infinitesimal thickenings

Remark 18.0. Beware that assertion (1) of the proposition below cannot be derived from Th. 17.1 because in that theorem there is a smoothness assumption on the target group, whereas a MT-group is not necessarily smooth: μ_n is not smooth over S is some residual characteristic of S divides n.

Proposition 18.1. *Let S be a scheme and S_0 a closed subscheme having the same underlying topological space. Then:*

1. *The functor $H \longmapsto H_0 = H \times_S S_0$ from the category of* MT-*groups over S to the analogous one over S_0 is* fully faithful.
2. *It induces an* equivalence *between the subcategory of* quasi-isotrivial*, resp.* isotrivial*,* MT-*groups over S and the analogous one over S_0*

Proof. (1) Let H, G be MT-groups over S. Let us prove that the map

$$u : \mathrm{Hom}_{S\text{-gr.}}(H,G) \longrightarrow \mathrm{Hom}_{S_0\text{-gr.}}(H_0,G_0)$$

is bijective. As this question is local over S, we may assume S affine. Then there exists a faithfully flat morphism $S' \to S$, with S' affine, such that the pullbacks H' and G' are diagonalisable. Let H'' and G'' denote the pullbacks over $S'' = S' \times_S S'$. Let $S'_0 = S \times_S S_0$ and define similarly H'_0, G'_0, etc. One has then a commutative diagram with exact rows:

$$\begin{tikzcd}
\mathrm{Hom}_{S\text{-gr.}}(H,G) \arrow[r] \arrow[d, "u"] & \mathrm{Hom}_{S'\text{-gr.}}(H',G') \arrow[r, shift left] \arrow[r, shift right] \arrow[d, "u'"] & \mathrm{Hom}_{S''\text{-gr.}}(H'',G'') \arrow[d, "u''"] \\
\mathrm{Hom}_{S_0\text{-gr.}}(H_0,G_0) \arrow[r] & \mathrm{Hom}_{S'_0\text{-gr.}}(H'_0,G'_0) \arrow[r, shift left] \arrow[r, shift right] & \mathrm{Hom}_{S''_0\text{-gr.}}(H''_0,G''_0),
\end{tikzcd}$$

hence to prove that u is bijective, it suffices to do so for u' and u''. We are therefore reduced to the case where H and G are diagonalisable, say $G = D(M)_S$ and $H = D(N)_S$. Then $G_0 = D(M)_{S_0}$ and $H_0 = D(M)_{S_0}$. By Cor. 6.3 and the proof of Prop. 7.5 in Lecture 3, we obtain a commutative diagram:

$$\begin{tikzcd}
\mathrm{Hom}_{S\text{-gr.}}(D(N)_S, D(M)_S) \arrow[r, "u"] \arrow[d, equal] & \mathrm{Hom}_{S_0\text{-gr.}}(D(N)_{S_0}, D(M)_{S_0}) \arrow[d, equal] \\
\mathrm{Loc}_{\mathrm{grp}}(M \times S, N) \arrow[r, "D(u)"] & \mathrm{Loc}_{\mathrm{grp}}(M \times S_0, N)
\end{tikzcd}$$

where $\mathrm{Loc}_{\mathrm{grp}}(M \times S, N)$ is the set of maps $M \times S \to N$ which are additive in the first variable and locally constant in the second, and where $D(u)$ is the map induced by the inclusion $S_0 \to S$. Since S_0 and S have the same underlying topological space, $D(u)$ is bijective, hence so is u. This proves (1).

(2) Let H_0 be a quasi-isotrivial (resp. isotrivial) MT-group over S_0. We have to prove that there exists a quasi-isotrivial (resp. isotrivial) MT-group H over S such that $H \times_S S_0 \simeq H_0$.

[0]Version of Sep. 1, 2023. This is the original version prepared for the lecture. The shortcuts discussed during the lecture were not legitimate, see Remark 18.0, so we revert to the original text.

By hypothesis, there exists a surjective étale (resp. finite étale) morphism $S'_0 \to S_0$ such that the pullback H'_0 is a diagonalisable group $D(M)_{S'_0}$. Now, recall that the functor $X \longmapsto X_0 = X \times_S S_0$ is an equivalence between the category of schemes étale over S and that of schemes étale over S_0; see [**SGA1**], Exp. I, 8.3 when S is locally noetherian, and EGA IV_4, 18.1.2 in general. Thus, there exists a surjective (resp. finite étale) étale morphism $S' \to S$ such that $S'_0 = S' \times_S S_0$.

Then $H' = D(M)_{S'}$ is such that $H' \times_{S'} S'_0 = H'_0$. Define as usual S'', S''' and note $S'' \times_S S_0 \simeq S'_0 \times_{S_0} S'_0$ and similarly for S'''_0. As $H'_0 = H_0 \times_{S_0} S'_0$, it is endowed by with a descent datum relative to $S'_0 \to S_0$. Applying the result of (1) to the pairs (S'', S''_0) and (S''', S'''_0), one obtains that this descent datum comes from a descent datum on H' relative to $S' \to S$. Since H' is affine over S', this descent datum is effective, by Theorem 8.18 of Lecture 4. Thus there exists a S-group scheme H such that $H \times_S S' = H' = D_{S'}(M)$, and hence H is a quasi-isotrivial (resp. isotrivial) MT-group over S.

Further, the pullbacks of $H \times_S S_0$ and H_0 by $S'_0 \to S_0$ are isomorphic. Since $S'_0 \to S_0$ is surjective and étale, it is a morphism of descent (see e.g. Lemme 8.22), hence the previous isomorphism comes from an isomorphism $H \times_S S_0 \simeq H_0$. This completes the proof of (2). $\square$

Remark 18.2. Suppose for simplicity that $S = \operatorname{Spec} \Lambda$ is affine. Then a closed subscheme $S_0 = \operatorname{Spec}(\Lambda/I)$ has the same underlying space topological space if and only if I is a *nilideal*, i.e. for every $a \in I$ there exists an integer n such that $a^n = 0$. If Λ is not noetherian, I need not be nilpotent: for example, let k be a field, A the polynomial ring over k in infinitely many variables $(X_i)_{i \in \mathbb{N}^*}$ and Λ the quotient of A by the relations $X_i^{i+1} = 0$. Then the maximal ideal of Λ is a nilideal which is is not nilpotent.

For simplicity, assume now that $S = \operatorname{Spec} \Lambda$ and $S_0 = \operatorname{Spec}(\Lambda/I)$. Under the additional assumption that I be nilpotent, one obtains the following stronger result.

Theorem 18.3. *Suppose that $S = \operatorname{Spec} \Lambda$ and $S_0 = \operatorname{Spec}(\Lambda/I)$, with I* nilpotent. *Let H be a flat S-group scheme such that H_0 is a quasi-isotrivial, resp. isotrivial,* MT-*group over S_0. Then H is a quasi-isotrivial, resp. isotrivial,* MT-*group over S.*

Proof. Suppose that H_0 is a quasi-isotrivial (resp. isotrivial) MT-group over S_0. Proceeding as in the previous proof, we obtain a surjective étale (resp. finite étale) morphism $S' \to S$ such that $H'_0 \simeq D(M)_{S'_0}$. We want to prove that $H \simeq D(M)_{S'}$. So, replacing S by S', we are reduced to the case where $H_0 = D(M)_{S_0}$.

Set $G = D(M)_S$. Then we have an isomorphism $u_0 : H_0 \xrightarrow{\sim} G_0$. Let us show[1] that u_0 lifts uniquely to a morphism of S-group schemes $u : H \to G$. By Cor. 6.3 one has

$$\operatorname{Hom}_{S\text{-gr.}}(H, G) = \operatorname{Hom}_{S\text{-gr.}}(M_S, \underline{\operatorname{Hom}}_{S\text{-gr.}}(H, \mathbb{G}_{m,S})) = \operatorname{Hom}_{\mathrm{grp}}(M, \operatorname{Hom}_{S\text{-gr.}}(H, \mathbb{G}_{m,S})) \tag{18.1}$$

the second equality coming from $\operatorname{Hom}_{S\text{-gr.}}(M_S, Y) = \operatorname{Hom}_{\mathrm{grp}}(M, Y(S))$ for any S-group scheme Y. Then, we have a commutative diagram:

$$\begin{tikzcd}
\operatorname{Hom}_{S\text{-gr.}}(H, G) \arrow[r, equal] \arrow[d] & \operatorname{Hom}_{\mathrm{grp}}(M, \operatorname{Hom}_{S\text{-gr.}}(H, \mathbb{G}_{m,S})) \arrow[d] \\
\operatorname{Hom}_{S_0\text{-gr.}}(H_0, G_0) \arrow[r, equal] & \operatorname{Hom}_{\mathrm{grp}}(M, \operatorname{Hom}_{S_0\text{-gr.}}(H_0, \mathbb{G}_{m,S_0}))
\end{tikzcd}$$

where the vertical maps are induced by the base change $S_0 \to S$. Since H is flat over S and H_0 of multiplicative type and $G = \mathbb{G}_{m,S}$ is smooth and *commutative* (so that the inner automorphisms are trivial), Theorem 17.1 ensures that the map

$$\operatorname{Hom}_{S\text{-gr.}}(H, \mathbb{G}_{m,S}) \to \operatorname{Hom}_{S_0\text{-gr.}}(H_0, \mathbb{G}_{m,S_0})$$

is *bijective.* Therefore $u_0 : H_0 \to G_0$ lifs to a unique morphism of S-group schemes $u : H \to G$.

[1]Again, we cannot invoke directly Th. 17.1 because H is not necessarily smooth. This is why the duality functor D is used, in order to be in a situation where the target group is $\mathbb{G}_m$, which is smooth (and commutative).

Moreover, u is an isomorphism. Indeed, since u_0 is an isomorphism, it suffices to see that for each $h \in H$, the ring homomorphism $\phi : \mathcal{O}_{G,u(h)} \to \mathcal{O}_{H,h}$ is bijective. Let C and K denote its cokernel and kernel. By assumption, $\phi_I = \phi \otimes (\Lambda/I)$ is bijective. It follows that C satisfies $C = IC$, hence $C = 0$ since I is nilpotent. Then, since $\mathcal{O}_{H,h}$ is flat over Λ, the kernel of ϕ_I is K/IK. It follows, as above, that $K = IK$ and hence $K = 0$. This completes the proof. $\square$

Notes for this Lecture

Prop. 18.1 and Th. 18.3 are respectively Prop. 2.1 and Cor. 2.3 of Exp. X.