LECTURE 7

Infinitesimal liftings and Hochschild cohomology: the beauty of SGA3, Exp. III

15. Group cohomology and extensions

Remark 15.1. For simplicity, we have written this section in the category $\mathscr{C}$ = (Sets), so that a $\mathscr{C}$-group G is just a usual group. However, the results remain valid if we replace (Sets) by an arbitrary category $\mathscr{C}$ with fiber products, G by a group-object in $\mathscr{C}$, and the abelian group V by a contravariant functor $F : \mathscr{C} \to$ (Abelian groups) on which G acts linearly, **provided that** the set-theoretic sections considered in the proof of Lemma 15.4 exist as morphisms in $\mathscr{C}$; see [**SGA3**$_1$], Exp. III, Section 1.

Definition 15.2. Firstly, let G be an abstract group and V a G-module. The cohomology groups $H^i(G,V)$ are the cohomology groups of the following complex, where Hom denotes maps of sets:

$$0 \longrightarrow V \xrightarrow{d^0} \mathrm{Hom}(G,V) \xrightarrow{d^1} \mathrm{Hom}(G^2,V) \xrightarrow{d^2} \mathrm{Hom}(G^3,V) \xrightarrow{d^3} \cdots \tag{15.1}$$

where $d^0(v)$ is the map $g \mapsto gv - v$, then, given $f : G \to V$, $d^1(f)$ is the map $(g_1,g_2) \mapsto g_1 f(g_2) - f(g_1g_2) + f(g_1)$, then, given $f : G^2 \to V$, $d^2 f$ is the map

$$(g_1,g_2,g_3) \mapsto g_1 f(g_2,g_3) - f(g_1g_2,g_3) + f(g_1,g_2g_3) - f(g_1,g_2).$$

More generally, for $n \geq 2$ and $f : G^{n-1} \to V$, $d^{n-1}(f)$ is the map

$$d^{n-1} f(g_1,\dots,g_n) = g_1 f(g_2,\dots,g_n) + \sum_{i=1}^{n-1} (-1)^i f(g_1,\dots,g_i g_{i+1},\dots,g_n) + (-1)^n f(g_1,\dots,g_{n-1}). \tag{15.2}$$

Clearly, $H^0(G,V) = V^G$ is the submodule of invariants. Then $H^1(G,V)$ is the quotient of the $\mathbb{Z}$-module $Z^1(G,M) = \{f : G \to V \mid f(g_1g_2) = g_1 f(g_2) + f(g_1)\}$ of 1-cocycles by the submodule $B^1(G,V) = \{d^0(v) \mid v \in V\}$ of 1-coboundaries. Consider the semi-direct product $E_0 = V \rtimes G$ and, for each $f \in Z^1(G,V)$, denote by σ_f the automorphism of E_0 defined by $\sigma_f(u,g) = (u - f(g), g)$.

Lemma 15.3. *Then $f \mapsto \sigma_f$ is a group isomorphism between $Z^1(G,V)$ and the group of automorphisms of E_0 which restrict to the identity on V and on the quotient E/V; and for each $v \in V$ the coboundary $d^0(v)$ corresponds under this isomorphism to the inner automorphism $(u,g) \mapsto v(u,g)v^{-1} = (u + v - gv, g)$.*

Proof. The proof is easy and left to the reader. ☐

In the next lemma, V is just an abelian group, without a given structure of G-module.

Lemma 15.4. *Each exact sequence of groups:*

$$1 \longrightarrow V \longrightarrow E \xrightarrow{\pi} G \longrightarrow 1 \tag{15.3}$$

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makes V into a G-module and defines a class $c(E) \in H^2(G,V)$.

This class is zero if and only if there exists a morphism of groups $\tau : G \to E$ such that $\pi \circ \tau = \mathrm{id}_G$, that is, if and only if E is a semi-direct product of G and V.

In this case, the set of all such τ' is $\tau + Z^1(G,V)$, and the set of all such τ' up to conjugacy by the elements of V is $\tau + H^1(G,V)$.

Proof. Let s be a set-theoretic section of π. For each $g \in G$, consider the automorphism $c(g)$ of V defined $c(g)(v) = s(g)\, v\, s(g)^{-1}$. Since V is *abelian*, one sees that:

(1) Any other section s' of π defines, for each $g \in G$, the same automorphism $c(g)$ of V.

(2) The resulting map $c : G \to \mathrm{Aut}(V)$ is a morphism of group. Indeed, for $g_1, g_2 \in G$, one has $\pi(s(g_1 g_2)) = g_1 g_2 = \pi(s(g_1)s(g_2))$ hence the element

$$\widetilde{s}(g_1, g_2) = s(g_1 g_2) s(g_2)^{-1} s(g_1)^{-1} \tag{15.4}$$

is in V and hence for any $v \in V$ one has $c(g_1 g_2)(v) = c(g_1)\big(c(g_2)(v)\big)$. Thus, setting $gv = c(g)(v)$, one obtains that V is a G-module.

Now, to any set-theoretic section s of π one associates the function $\widetilde{s} : G^2 \to V$ defined in (15.4) above. One checks easily that $\widetilde{s}$ is a 2-cocycle, i.e. it belongs to the $\mathbb{Z}$-module:

$$Z^2(G,M) = \{f : G^2 \to V \mid g_1 f(g_2, g_3) + f(g_1, g_2 g_3) - f(g_1, g_2) = 0\}.$$

Indeed, in E one has the equality:

$$\begin{aligned} d^2\widetilde{s}(x,y,z) &= (x \cdot \widetilde{s}(y,z))\, \widetilde{s}(x,y)^{-1}\, \widetilde{s}(xy,z)^{-1}\, \widetilde{s}(x,yz) \\ &= s(x)s(yz)s(z^{-1})s(y^{-1})s(x^{-1})s(x)s(y)s(xy)^{-1}s(xy)s(z)s(xyz)^{-1}s(xyz)s(yz)^{-1}s(x^{-1}) = e. \end{aligned}$$

Next, denote by $B^2(G,V) = \mathrm{Im}(d^1)$ the submodule of 2-coboundaries. If s, s' are two set-theoretic sections of π, there exists $f : G \to V$ such that $s'(g) = f(g)^{-1} s(g)$ for all g, and one has in E the equalities:

$$\begin{aligned} \widetilde{s'}(x,y) &= s'(xy)s'(y^{-1})s'(x^{-1}) = f(xy)^{-1}s(xy)s(y^{-1})f(y)s(x^{-1})f(x) \\ &= f(xy)^{-1}\widetilde{s}(x,y)s(x)f(y)s(x^{-1})f(x) = \widetilde{s}(x,y)f(xy)^{-1}(x \cdot f(y))f(x), \end{aligned}$$

where in the last equality we used that V is a normal abelian subgroup of E. Writing additively the group law of V, one has $f(xy)^{-1}(x \cdot f(y))f(x) = xf(y) - f(y) + f(x) = d^1 f(x,y)$. This shows that $\widetilde{s'} = \widetilde{s} + d^1 f$. This proves two things:

(1) The image of $\widetilde{s}$ in $H^2(G,V) = Z^2(G,V)/B^2(G,V)$ does not depend on the choice of the set-theoretic section s; it it the class $c(E)$ of the extension.

(2) A section s' is a group homomorphism if and only if $\widetilde{s'} = 0$. Since, with the notation above, $\widetilde{s'} = \widetilde{s} + d^1 f$ for some f, this is the case if and only if $c(E) = 0$.

Now, assume $c(E) = 0$ and let τ be a section of π which is a group homomorphism. By the above, any other such τ' has the form $f\tau$, with $f \in Z^1(G,V)$. Further, for any $v \in V$ one has:

$$v^{-1}\tau'(g)v = v^{-1}\big(\tau'(g)v\tau'(g)^{-1}\big)\tau'(g) = (d^0 v)(g)\tau'(g).$$

Thus the set of τ' up to conjugacy by the elements of V identifies with $\tau + H^1(G,V)$. □

Now, let $\phi : Y \to G$ be a morphism of groups. Using ϕ we can form the group $E_\phi = E \times_G Y$ and pull-back the exact sequence (15.3) to obtain the following exact sequence of groups:

$$1 \longrightarrow V \longrightarrow E_\phi \xrightarrow{\pi_\phi} Y \longrightarrow 1 \tag{15.5}$$

where V is sent into E_ϕ via the inclusion into E and via the unit morphism to Y. Note that the resulting action of Y on V is the same as the one derived from the G-action through the morphism $\phi : Y \to G$.

Further, any map of sets $\psi : Y \to E$ lifting ϕ defines a set-theoretic section of π_ϕ. Denoting by pr_1 the first projection $E_\phi \to E$, one sees easily that the map $\tau \mapsto \mathrm{pr}_1 \circ \tau$ is a bijection from the set of sections of π_ϕ to the set of liftings of ϕ; further, under this bijection the sections and liftings which are group homomorphisms correspond to each other.

Therefore, the question of lifting ϕ to a group homomorphism $\psi : H \to E$ is reduced to the question of finding a splitting of (15.5). By the previous discussion with G replaced by Y, we obtain the:

COROLLARY 15.5. *Consider the exact sequence* (15.3) *and let $\phi : Y \to G$ be a morphism of groups.*

(1) *Suppose that $H^2(Y, V) = 0$. Then ϕ lifts to a morphism of groups $\psi : Y \to G$.*
(2) *Suppose further that $H^1(Y, V) = 0$. Then any two such lifting are conjugate by an element of V, that is, by an element of E whose image in G is the identity.*

The following lemma will be useful in the next section.

LEMMA 15.6. *Let N be an abelian group. Let* $\mathrm{Ind}(N) = \mathrm{Hom}(G, N)$ *be the* induced G-*module, where G acts on a function $\phi : G \to N$ by $(gf)(g') = \phi(g'g)$ for all $g, g' \in G$. Then* $\mathrm{Ind}(N)$ *is* acyclic*, i.e. $H^i(G, \mathrm{Ind}(N)) = 0$ for all $i > 0$.*

PROOF. Set $P = \mathrm{Ind}(N)$. For $n > 0$ let $s^n : \mathrm{Hom}(G^{n+1}, P) \to \mathrm{Hom}(G^n, P)$ be given by

$$s^n(f)(g_1, \ldots, g_n)(g) = f(g, g_1, \ldots, g_n)(e).$$

Then, for $f \in \mathrm{Hom}(G^n, P)$ one has

$$\begin{aligned} s^n d^n f(g_1, \ldots, g_n)(g) = d^n f(g, g_1, \ldots, g_n)(e) &= f(g_1, \ldots, g_n)(g) - f(gg_1, g_2, \ldots, g_n)(e) \\ &+ \sum_{i=1}^{n-1} (-1)^{i+1} f(g, \ldots, g_i g_{i+1}, \ldots, g_n)(e) + (-1)^{n+1} f(g, g_1, \ldots, g_{n-1})(e), \end{aligned}$$

$$\begin{aligned} d^{n-1} s^{n-1} f(g_1, \ldots, g_n)(g) = f(gg_1, g_2, \ldots, g_n)(e) &+ \sum_{i=1}^{n-1} (-1)^i f(g, \ldots, g_i g_{i+1}, \ldots, g_n)(e) \\ &+ (-1)^n f(g, g_1, \ldots, g_{n-1})(e) \end{aligned}$$

and hence $f = s^n d^n f + d^{n-1} s^{n-1} f$. Thus, if $d^n f = 0$ then $f = d^{n-1} s^{n-1} f$, hence $H^n(G, P) = 0$ for $n > 0$. ☐

REMARK 15.7. For simplicity, we have worked with G-modules, i.e. modules over the group ring $\mathbb{Z}[G]$. If Λ is any commutative ring, the same discussion applies to $\Lambda[G]$-modules V, and then the cohomology groups $H^i(G, V)$ are Λ-modules.

16. Hochschild cohomology

In this section, Λ is a ring and G is a flat[1] affine group scheme over $S = \mathrm{Spec}\,\Lambda$, given by the Λ-Hopf algebra A. Denote by $c : A \to A \otimes A$ its comultiplication (we write $\otimes$ instead of $\otimes_\Lambda$) and by ε it counit. Let L be a Λ-module endowed with a structure of G-module, that is, we are given a Λ-linear coaction $\mu_L : L \to L \otimes A$ satisfying the conditions seen in Def. 2.1.

DEFINITION 16.1. The Hochschild complex of L is

$$0 \longrightarrow L \xrightarrow{d^0} L \otimes A \xrightarrow{d^1} L \otimes A \otimes A \xrightarrow{d^2} L \otimes A \otimes A \otimes A \xrightarrow{d^3} \cdots \tag{16.1}$$

[1]Flatness ensures that the category of A-comodules is abelian, see e.g. [**SGA3**$_1$], Exp. I, Cor. 4.7.2.1.

where $d^0(v) = \mu_L(v) - v \otimes 1$ and, for $n \geq 1$:

$$d^n(v \otimes a_1 \otimes \cdots \otimes a_n) = \mu_L(v) \otimes a_1 \otimes \cdots \otimes_n + \sum_{i=1}^{n} (-1)^i v \otimes a_1 \otimes \cdots c(a_i) \otimes \cdots \otimes a_n + (-1)^{n+1} v \otimes a_1 \otimes \cdots \otimes a_n \otimes 1. \tag{16.2}$$

Its cohomology groups are denoted by $H^i(G, L)$.

REMARK 16.2. Let $\mathscr{C}$ denote the category of affine schemes over S and let $W(L)$ be the functor such that $W(L)(T) = L \otimes \mathcal{O}(T)$ for any object T of $\mathscr{C}$. Then $G(T)$ acts linearly on $W(L)(T)$: for any $g \in G(T)$, i.e. any Λ-algebra morphism $g : A \to \mathcal{O}(T)$ and $x \otimes 1 \in L \otimes \mathcal{O}(T)$, one has $g(x \otimes 1) = (\mathrm{id}_L \otimes g)\mu_L(x)$.

Further, one has $\mathrm{Hom}_{\mathscr{C}}(G^n, W(L)) = L \otimes A^{\otimes n}$. Thus we see that Hochschild cohomology can be viewed as the cohomology of groups in $\mathscr{C}$. The results of Section 15 about lifting of homomorphims remain valid, provided that there exist morphisms of schemes that replace the set-theoretic sections used in Section 15. This will be ensured by the hypothesis that the group-scheme G is *smooth* and that we are concerned with *infinitesimal liftings*.

Analogously to Lemma 15.6, one has:

LEMMA 16.3. *For a Λ-module N, let $\mathrm{Ind}(N) = N \otimes A$ regarded as A-comodule via $\mathrm{id}_N \otimes c : N \otimes A \to N \otimes A \otimes A$. Then $H^i(G, \mathrm{Ind}(N)) = 0$ for all $i > 0$.*

PROPOSITION 16.4. *Suppose that $G = D(M)_S$ is a diagonalisable group. Then $H^i(G, L) = 0$ for $i > 0$ and any G-Λ-module L.*

PROOF. [2] Let L a G-module. Recall that $L = \bigoplus_{m \in M} L_m$, where $L_m = \{x \in L \mid \mu_L(x) = x \otimes m\}$. Let $(p_m)_{m \in M}$ denote the corresponding family of projectors. Note first that the map $\mu_L : L \to \mathrm{Ind}(L)$ is a morphism of A-comodules: indeed, by one of the defining properties of comodules, the diagram below is commutative:

$$\begin{tikzcd} L \arrow[r, "\mu_L"] \arrow[d, "\mu_L"'] & \mathrm{Ind}(L) \arrow[d, "\mathrm{id}_L \otimes c"] \\ L \otimes A \arrow[r, "\mu_L \otimes \mathrm{id}_A"] & \mathrm{Ind}(L) \otimes A. \end{tikzcd}$$

Now the map $r : \mathrm{Ind}(L) \to L$ sending each finite sum $\sum_m x_m \otimes m$ to $\sum_m p_m(x_m)$ is a retraction of μ_L, because writing $x = \sum_m p_m(x)$ we have $\mu_L(x) = \sum_m p_m(x) \otimes m$ and hence $r(\mu_L(x)) = x$. Further, r is a morphism of A-comodules, that is the diagram below is commutative:

$$\begin{tikzcd} \mathrm{Ind}(L) \arrow[r, "r"] \arrow[d, "\mathrm{id}_V \otimes c"'] & L \arrow[d, "\mu_L"] \\ \mathrm{Ind}(L) \otimes A \arrow[r, "r \otimes \mathrm{id}_A"] & L \otimes A. \end{tikzcd}$$

Indeed, an element $y = \sum_m x_m \otimes m$ of $\mathrm{Ind}(L)$ is sent by $\mathrm{id}_V \otimes c$ to $\sum_m x_m \otimes m \otimes m$ which goes by $r \otimes \mathrm{id}_A$ to $\sum_m p_m(x_m) \otimes m$, which is also $\mu_L(r(y))$. Thus L is a G-module direct summand of $\mathrm{Ind}(L)$ and since the latter is acyclic, so is L. $\square$

REMARK 16.5. Let $\Lambda \to \Lambda'$ be a flat map, set $L' = L \otimes \Lambda'$ and let $S' = \mathrm{Spec}\,\Lambda'$ and $G' = G_{S'}$. Let $C^\bullet(G, L)$ denote the Hochschild complex of L. Then $C^\bullet(G, L) \otimes \Lambda'$ identifies with the Hochschild complex $C^\bullet(G', L')$ hence, since $\Lambda \to \Lambda'$ is flat, one has $H^i(G, L) \otimes \Lambda' \simeq H^i(G', L')$ for all $i \geq 0$.

[2] This neat proof is taken from [**DG70**], §II.3, Prop. 4.2.

Corollary 16.6. *Let G be a* MT*-group of type M over S and L a G-Λ-module. Then $H^i(G, L) = 0$ for $i > 0$.*

Proof. By hypothesis, there exist faithfully flat morphisms $U'_i \to U_i$, where the U_i and U'_i are affine and the U_i form an open covering of $S = \operatorname{Spec}\Lambda$, such that $G_{U'_i} \simeq D(M)_{U'_i}$. Since S is affine, hence quasi-compact, it is covered by finitely many of the U_i. Then the disjoint sum S' of the corresponding U'_i is affine and faithfully flat over S, and $G_{S'} \simeq D(M)_{S'}$. Set $\Lambda' = \mathcal{O}(S')$ and $L' = L \otimes \Lambda'$. By the previous remark and proposition, one has $0 = H^i(G_{S'}, L') \simeq H^i(G, L) \otimes \Lambda'$. Since Λ' is faithfully flat over Λ, it follows that $H^i(G, L) = 0$ for $i > 0$. □

17. Infinitesimal liftings and Hochschild cohomology

In this section, we fix a nilpotent ideal I in a ring Λ and denote by $S \supset S_0$ the spectra of Λ and Λ/I; they have the same underlying topological space. For every S-scheme X we denote by X_0 its pullback over S_0.

Further, we fix **affine S-group schemes G and Y, with G smooth and Y flat and such that Y_0 is of multiplicative type**. The goal of this section is to prove the following theorem, where $\operatorname{int}(g)$ denote the automorphism of conjugation by g.

Theorem 17.1. *Let $u_0 : Y_0 \to G_0$ be a morphism of S_0-group schemes. Then:*

(1) *There exists a morphism of S-group schemes $u : Y \to G$ that lifts u_0.*
(2) *If v is another such morphism, there exists $g \in \operatorname{Ker}(G(S) \to G(S_0))$ such that $v = \operatorname{int}(g) \circ u$.*
(3) *More generally, if $v : Y \to G$ is a morphism of S-group schemes and $g_0 \in G(S_0)$ is such that $v_0 = \operatorname{int}(g_0) \circ u_0$, there exists a lifting g of g_0 such that $v = \operatorname{int}(g) \circ u$.*

Let n be the smallest positive integer such that $I^n = 0$. Assume first that $n = 2$ and that assertions (1) and (2) are proved for $n = 2$. Let $v : Y \to G$ be a morphism of S-group schemes and $g_0 \in G(S_0)$ such that $v_0 = \operatorname{int}(g_0) \circ u_0$. Since G is smooth, g_0 lifts to an element $g' \in G(S)$; set $u' = \operatorname{int}(g') \circ u$. Then $v_0 = u'_0$ hence, by (2), there exists $g'' \in \operatorname{Ker}(G(S) \to G(S_0))$ such that $v = \operatorname{int}(g'') \circ u'$. Then $v = \operatorname{int}(g''g') \circ u$, and $g''g'$ is a lifting of g_0. This proves (3) for $n = 2$.

Now, let $n \geq 2$, assume the theorem proved for all ideals J such that $J^n = 0$ and let I be such that $I^{n+1} = 0$. Set $J = I^2$ and $S_J = \operatorname{Spec}(\Lambda/J)$, then $J^n = 0$ and the image $\overline{I}$ of I in Λ/J satisfies $\overline{I}^2 = 0$. By the case $n = 2$ and the induction hypothesis, u_0 lifts to a morphism of S_J-group schemes $u_J : Y_J \to G_J$ and u_J lifts to a morphism of S-group schemes $u : Y \to G$. This proves (1).

Further, let $v : Y \to G$ be a morphism of S-group schemes and $g_0 \in G(S_0)$ such that $v_0 = \operatorname{int}(g_0) \circ u_0$. By the case $n = 2$ and the induction hypothesis, g_0 lifts to an element $g_J \in G(S_J)$ such that $v_J = \operatorname{int}(g_J) \circ u_J$ and g_J lifts to an element $g \in G(S)$ such that $v = \operatorname{int}(g) \circ u$. This proves (3), and of course (2) is the special case $g_0 = e$. Thus, it suffices to prove the following proposition. From now on we assume that $I^2 = 0$.

Proposition 17.2. *Suppose that $I^2 = 0$ and let $u_0 : Y_0 \to G_0$ be a morphism of S_0-group schemes. Then:*

(1) *There exists a morphism of S-group schemes $u : Y \to G$ that lifts u_0.*
(2) *If v is another such morphism, there exists $g \in \operatorname{Ker}(G(S) \to G(S_0))$ such that $v = \operatorname{int}(g) \circ u$.*

Recall now the notation introduced in Definition 14.1. In particular, we have an exact sequence of group functors:

$$1 \longrightarrow L'_G \longrightarrow G \xrightarrow{\pi} G^+ \longrightarrow 1. \qquad (17.1)$$

We are going to describe the functor L'_G in order to prove Proposition 17.2.

We have the smooth affine group-scheme $G = \operatorname{Spec}(A)$, where A is a Λ-Hopf algebra. Let $c : A \to A \otimes A$ and $\varepsilon : A \to \Lambda$ be the comultiplication and augmentation (or counit) maps, which correspond to the multiplication of G and to the unit section $S \to G$. Recall that $\Lambda_0 = \Lambda/I$ and that the pull-back of a S-scheme X to $S_0 = \operatorname{Spec} \Lambda_0$ is denoted by X_0.

DEFINITION 17.3. Let $\mathfrak{m} = \operatorname{Ker} \varepsilon$; since G is smooth over S, $\mathfrak{m}/\mathfrak{m}^2$ is a locally free Λ-module of finite rank (equal to the relative dimension d of G over S). Further, one has $(\mathfrak{m}/\mathfrak{m}^2) \otimes \Lambda_0 \simeq \mathfrak{m}_0/\mathfrak{m}_0^2$ with obvious notation.

By definition, $\operatorname{Lie}(G/S)$ is the Zariski tangent space to G along the unit section, i.e. it is the Λ-module $\operatorname{Lie}(G/S) = \operatorname{Hom}_\Lambda(\mathfrak{m}/\mathfrak{m}^2, \Lambda)$. It is locally free of rank d. Similarly for $\operatorname{Lie}(G_0/S_0)$ over Λ_0.

DEFINITION 17.4. The left action of G on itself by inner automorphisms, that is, $\operatorname{int}(g)(g') = gg'g^{-1}$, induces a structure of left A-comodule $\mu : V \to A \otimes V$ on $V = A$, which corresponds to the linear right action of G on A given $(\phi \cdot g)(g') = \phi(gg'g^{-1})$ for $\phi \in A$ and arbitrary R-points $g, g' \in G(R)$. Clearly, $\mathfrak{m} = \operatorname{Ker} \varepsilon$ is stable by this G-action, as well as $\mathfrak{m}^2$, hence there is a natural right action of G on the cotangent space $\mathfrak{m}/\mathfrak{m}^2$ which is called the *coadjoint action.* The induced action on the dual space $\operatorname{Lie}(G/S)$ is called the *adjoint action.*[3]

LEMMA 17.5. *Let $T = \operatorname{Spec} B$ for some Λ-algebra B. Consider the B_0-module $F(T) = \operatorname{Hom}_{\Lambda_0}(\mathfrak{m}_0/\mathfrak{m}_0^2, IB)$.*

(1) *$L'_G(T)$ is the set of Λ-algebras morphisms of the form $\phi = \varepsilon + D$, with $D \in F(T)$.*
(2) *The resulting identification $L'_G(T) = F(T)$ respects the group laws and the conjugation action of G on $L'_G(T)$ corresponds to the action on $F(T)$ induced by the coadjoint action on $\mathfrak{m}/\mathfrak{m}^2$.*
(3) *If T is* **flat** *over S, setting* $\boxed{L_0 = \operatorname{Hom}_{\Lambda_0}(\mathfrak{m}_0/\mathfrak{m}_0^2, I)}$ *one has* $\boxed{F(T) = L_0 \otimes_{\Lambda_0} B_0.}$

PROOF. (1) By definition, $L'_G(T)$ is the set of algebra morphisms $\phi : A \to B$ which reduce to ε modulo I. Thus, for any $a \in A$, we can write $\phi(a) = \varepsilon(a) + D(a)$, with $D(a) \in IB$. One has $A = \Lambda 1 \oplus \mathfrak{m}$ and $\phi(1) = 1 = \varepsilon(1)$, so we may consider D as a Λ-linear map $\mathfrak{m} \to IB$. Since $(IB)^2 = 0$, the condition that ϕ be a morphim of algebras becomes:

$$\varepsilon(a_1 a_2) + D(a_1 a_2) = \phi(a_1 a_2) = \phi(a_1)\phi(a_2) = \varepsilon(a_1 a_2) + \varepsilon(a_1)D(a_2) + \varepsilon(a_2)D(a_1),$$

which is equivalent to

$$D(a_1 a_2) = \varepsilon(a_1)D(a_2) + \varepsilon(a_2)D(a_1). \tag{17.2}$$

One expresses this equality by saying that D is an ε-derivation $A \to IB$. This implies that D vanishes on $\mathfrak{m}^2$. Conversely, one sees that any Λ-linear map $\mathfrak{m}/\mathfrak{m}^2 \to IB$ defines a map D as above. This proves the first equality below, and the second follows since IB is annihilated by I:

$$L'_G(T) = \operatorname{Hom}_\Lambda(\mathfrak{m}/\mathfrak{m}^2, IB) = \operatorname{Hom}_{\Lambda_0}\big((\mathfrak{m}/\mathfrak{m}^2) \otimes \Lambda_0, IB\big).$$

Finally, since $(\mathfrak{m}/\mathfrak{m}^2) \otimes \Lambda_0 \simeq \mathfrak{m}_0/\mathfrak{m}_0^2$, one obtains assertion (1).

(2) Let $\phi \in \mathfrak{m}$. Since $A = \Lambda 1 \oplus \mathfrak{m}$ one can write uniquely:

$$c(\phi) = \lambda 1 \otimes 1 + \phi_1 \otimes 1 + 1 \otimes \phi_2 + \sum_i \psi_i \otimes \theta_i \tag{17.3}$$

with ϕ_1, ϕ_2 and the ψ_i, θ_i in $\mathfrak{m}$. Since $\phi = (\operatorname{id} \otimes \varepsilon)c(\phi) = (\varepsilon \otimes \operatorname{id})c(\phi)$, and $\varepsilon(\phi) = 0$, one obtains successively that $\lambda = 0$ and $\phi_1 = \phi = \phi_2$.

[3] Over a base field, the adjoint action is considered as the primary object and the coadjoint action is its dual, but over an arbitrary base one has to note that the action on $\mathfrak{m}/\mathfrak{m}^2$ (which is $\varepsilon^*(\Omega_{G/S})$) comes first.

Now, let $g_1, g_2 \in L'_G(T)$ and write $g_i = \varepsilon + D_i$ for $i = 1, 2$. Recall that the product $g_1 g_2$ is $m_B \circ (g_1 \otimes g_2) \otimes c$, where m_B is the multiplication of B. It follows from (17.3) that for any $\phi \in \mathfrak{m}$ one has

$$(g_1 g_2)(\phi) = D_1(\phi) + D_2(\phi) + \sum_i D_1(\psi_i) D_2(\theta_i) = (D_1 + D_2)(\phi)$$

where in the second equality we have used that $(IB)^2 = 0$. This proves the first part of (2), i.e. that under the identification $g_i \leftrightarrow D_i$ the group law of $L'_G(T)$ transforms into the addition law of the B_0-module $F(T)$.

Next, let $g \in G(T)$ and $g_1 = \varepsilon + D_1 \in L'_G(T)$. For any $\phi \in \mathfrak{m}$, let $\overline{\phi}$ denote its image in $\mathfrak{m}/\mathfrak{m}^2$. Then $\operatorname{int}(g)(g_1)$ sends ϕ to

$$\phi\big(c(g)(g_1)\big) = (\phi \cdot g)(g_1) = g_1(\phi \cdot g) = (\varepsilon + D_1)(\overline{\phi} \cdot g) = D_1(\overline{\phi} \cdot g)$$

and on the right-hand side this is the action of G on $F(T)$ induced by the coadjoint action on $\mathfrak{m}/\mathfrak{m}^2$. This completes the proof of (2).

(3) Suppose that B is flat over Λ. Then we have isomorphims $I \otimes B_0 = I \otimes B \xrightarrow{\sim} IB$ and hence, since B_0 is flat over Λ_0 and $\mathfrak{m}_0/\mathfrak{m}_0^2$ is locally free of finite rank, one obtains

$$F(T) \simeq \operatorname{Hom}_{\Lambda_0}(\mathfrak{m}_0/\mathfrak{m}_0^2, I) \otimes_{\Lambda_0} B_0 \simeq \operatorname{Lie}(G_0/S_0) \otimes_{\Lambda_0} I \otimes_{\Lambda_0} B_0.$$

Thus, setting $L_0 = \operatorname{Hom}_{\Lambda_0}(\mathfrak{m}_0/\mathfrak{m}_0^2, I) \simeq \operatorname{Lie}(G_0/S_0) \otimes_{\Lambda_0} I$, one has $F(T) = L_0 \otimes_{\Lambda_0} B_0$. □

We can now prove Proposition 17.2

Proof of proposition 17.2. Let $Y = \operatorname{Spec} B$ be a flat affine group scheme over S, with Y_0 of multiplicative type and suppose given a morphism of S-group functors $\phi : Y \to G^+$, i.e. a morphism of S_0-groups $u_0 : Y_0 \to G_0$. Since G is smooth and I nilpotent, there exists a morphism of *S-schemes* $s : Y \to G$ lifting u_0.

As in Section 15, we can use ϕ to form the S-group functor $E = G \times_{G^+} Y$ and pull-back the short exact sequence (17.1) to obtain the following short exact sequence of S-group functors:

$$1 \longrightarrow L'_G \longrightarrow E \xrightarrow{\pi} Y \longrightarrow 1. \tag{17.4}$$

The morphism of *S-schemes* $s : Y \to G$ is a section of π. Proceeding as in Section 15, we obtain a morphism $\widetilde{s} : Y^2 \to L'_G$ defined for arbitrary points $y_1, y_2 \in Y(T)$ by

$$\widetilde{s}(T)(y_1, y_2) = s(y_1 y_2) s(y_1)^{-1} s(y_2)^{-1}.$$

This is an element of $L'_G(Y^2)$, which equals $L_0 \otimes B_0 \otimes B_0$ since B is flat over Λ. By Yoneda lemma, the fact that $\widetilde{s}(T)$ is a cocycle for group cohomology, for any T, translates into the fact that $\widetilde{s}$ defines a class in the Hochschild cohomology group $H^2(Y_0, L_0)$. But the latter is 0 by Cor. 16.6 since Y_0 is of multiplicative type. Therefore, u_0 can be lifted to a morphism of S-group schemes $u : Y \to G$.

Then, as in Section 15, any other such morphsim v has the form $v = fu$, where f is a morphism $Y \to L'_G$, i.e. an element of $L'_G(Y) = L_0 \otimes B_0$, which is a 1-cocyle. Since $H^1(Y_0, L_0) = 0$, one obtains by Corollary 15.5 that $v = \operatorname{int}(g) \circ u$ for some $g \in \operatorname{Ker}(G(S) \to G(S_0))$. This completes the proof of Proposition 17.2. □

Notes for this Lecture

The cohomology of groups in a category is defined in Exp. I, §5.1. Then Lemma 15.4 and Corollary 15.5 are proved in Exp. III, Prop. 1.2.4 but are standard results in group cohomology, as well as Lemma 15.6.

Hochschild homology is defined in Exp. I, §5.3, where Prop. 16.4 is proved as Th. 5.3.3 (whereas Lemma 16.3 is contained in the proof of 5.3.1.1). The extension to groups of multiplicative type (Cor. 16.6) is Exp. IX, Th. 3.1.

Theorem 17.1 corresponds to Theorems 3.2 and 3.6 of Exp. IX, whose proofs rely on Exp. III, Th. 2.1 and Cor. 2.5.