LECTURE 6

A first look at maximal tori and Lie algebras

13. Motivation: tori in reductive group schemes

Definition 13.1. Let k be an algebraically closed field an G a connected affine algebraic group over k, that is, a connected affine smooth group scheme over k.

One knows that all maximal tori T of G have the same dimension; in fact they are all conjugate under $G(k)$. Their common dimension is called the *reductive rank* of G and denoted by $\mathrm{rk}_{\mathrm{red}}(G)$.

One also knows that there exists a largest *normal* smooth connected solvable (resp. unipotent) subgroup of G, it is called the *radical* (resp. *unipotent radical*) of G and is denoted by $\mathrm{rad}(G)$ (resp. $\mathrm{rad}^u(G)$).

One says that G is *reductive* (resp. *semi-simple*) if $\mathrm{rad}^u(G) = \{e\}$ (resp. $\mathrm{rad}(G) = \{e\}$). In this case, if K is a larger algebraically closed field, G_K is reductive (resp. semi-simple).

Definition 13.2. Let S be a base scheme. One says that a S-group scheme G is **reductive** (resp. **semi-simple**) if:

(1) G is affine and smooth, with connected fibers.
(2) Its geometric fibers are reductive (resp. semi-simple), that is, for every $s \in S$, denoting by $\overline{s}$ the spectrum of an algebraic closure $\overline{\kappa}(s)$ of the residue field $\kappa(s)$, the $\overline{\kappa}(s)$-group $G_{\overline{s}}$ is reductive (resp. semi-simple).

Definition 13.3. Let G be a smooth affine S-group scheme. A **maximal torus** of G is a closed subgroup scheme[1] T such that:

(1) T is a torus (in the sense of Def. 1.3).
(2) For every geometric point $\overline{s}$ of S, the subgroup $T_{\overline{s}}$ is a maximal torus of $G_{\overline{s}}$

Remark 13.4. (1) In particular, one will obtain that a smooth affine S-group scheme H such that all its fibers are tori, is itself a torus. This is far from obvious!

(2) Let k be an algebraically closed field of characteristic $\neq 2$ and let $S = \mathbb{A}^1_k$. The constant S-group $\{\pm 1\}_S$ is smooth and isomorphic with $\mu_{2,S}$. Let H be the open subgroup obtained by removing the non-neutral point over $s = 0$. Then H is a smooth affine S-group scheme and all its fibers are MT-groups, but H is not a MT-group. This shows that the assumption that the fibers are connected is important. (However, Cor. 4.8 of Exp. X proves that if H is a flat S-group scheme of finite presentation, such that all fibers H_s are MT-groups and their type (i.e. the corresponding abelian group) is locally constant function of s, then H is a MT-group over S.)

A fundamental point in the study of reductive group schemes is to prove that maximal tori exist locally in the étale topology. The proof consists in showing that certain functors F are representable and *formally smooth*, so that for every $s \in S$ there exists some étale neighbourhood S' of s such that $F(S') \neq \emptyset$ (Hensel Lemma).

[0] corrected version of Nov. 4, 2023

[1] If H is a MT-group over S, every monomorphism of S-groups $H \to G$ is closed immersion, see Cor. 21.2 in Lect. 9.

DEFINITION 13.5. One says that a contravariant functor $F : (\mathrm{Sch}) \to (\mathrm{Sets})$ is **formally smooth** if for every nilpotent ideal I in a ring A, the map $F(\operatorname{Spec} A) \to F(\operatorname{Spec}(A/I))$ is surjective.

14. A first glimpse of Lie algebras

In this section and later ones, we consider a square-zero ideal I in a ring Λ and denote by $S \supset S_0$ the spectra of Λ and Λ/I; they have the same underlying topological space. For every S-scheme X we denote by X_0 its pullback over S_0.

DEFINITION 14.1. **(1)** Let $S' \to S$ be a morphism of schemes and let X' be a S'-scheme. One denotes by $\prod_{S'/S} X'$ the functor sending each S-scheme T to $X'(T \times_S S')$. It is called the *Weil restriction* of X' from S' to S.

When $X' = X \times_S S'$ for some S-scheme X, one has for every S-scheme T:

$$\Big(\prod_{S'/S} X_{S'}\Big)(T) = \mathrm{Hom}_{S'}(T_{S'}, X_{S'}) = \mathrm{Hom}_S(T_{S'}, X) = X(T \times_S S').$$

(2) Now, let G be a smooth affine S-group scheme. We denote by G^+ the functor $\prod_{S_0/S} G_0$; that is, for any $T \to S$ a T-point of G^+ is a morphism of S-schemes $\phi : T_0 \to G$. Clearly, G^+ is a group functor and there is a canonical morphism of group functors $G \to G^+$, which sends an arbitrary point $T \to G$ to the point $T_0 \to G$ of G^+. Denote by[2] L'_G its kernel. Note that since G is smooth and I nilpotent, any morphism of schemes $f_0 : T_0 \to G$ lifts to a morphism of S-*schemes* $f : T \to G$, so we have an exact sequence of group functors:

$$1 \longrightarrow L'_G \longrightarrow G \xrightarrow{\pi} G^+ \longrightarrow 1. \tag{14.1}$$

If Y is a MT-group over S and $u_0 : Y_0 \to G$ is a morphism of S_0-group schemes, an important result in the sequel is that u_0 can be lifted to a morphism of S-**group schemes** $u : Y \to G$. We are going to describe the functor L'_G in order to prove this result.

Before going into this, let us illustrate some results with the following example.

EXAMPLE 14.2. Let k be a ring (if one wants, an algebraically closed fied) and consider the group scheme $G = \mathrm{GL}_{n,k}$ over $\operatorname{Spec} k$. Its Lie algebra $\mathrm{Lie}(G)$ is the free k-module $M_n(k)$; we define the functor $W(\mathrm{Lie}(G))$ on the category of k-algebras by $W(\mathrm{Lie}(G))(R) = \mathrm{Lie}(G) \otimes_k R$. Denoting the dual k-module $\mathrm{Lie}(G)^*$ by $\omega_{G/S}$, one can also say that $W(\mathrm{Lie}(G))$ is represented by the spectrum $\mathbb{V}(\omega_{G/S})$ of the symmetric algebra over k of $\omega_{G/S}$. .

Now, let ε be a square-zero variable and set $TG = \prod_{k[\varepsilon]/k} G_{k[\varepsilon]}$. The projection $k[\varepsilon] \to k$ sending ε to 0 induces a short exact sequence of group functors:

$$1 \longrightarrow W(\mathrm{Lie}(G)) \longrightarrow TG \xrightarrow{\pi} G \longrightarrow 1 \tag{14.2}$$

that is, for every k-algebra R, one has an exact sequence of groups:

$$1 \longrightarrow \mathrm{Lie}(G) \otimes_k R \longrightarrow G(R[\varepsilon]) \xrightarrow{\pi} G(R) \longrightarrow 1.$$

Here, the inclusion $k \hookrightarrow k[\varepsilon]$ is a section of $k[\varepsilon] \to k$ hence induces a morphism of group functors $G \to TG$ which is a section of π. Hence TG (which is the tangent bundle to G) is the semi-direct product of $\mathbb{V}(\omega_{G/S})$ by G, where G acts on $\omega_{G/S}$ via the so-called *coadjoint action*; in particular, TG is representable.

The point of this example is two-fold:

1. The additive group law on $\mathrm{Lie}(G)$ coming from its structure of k-module coincides with the group law on the kernel H of the morphism $TG \to G$.

[2] We use the notation L'_G as in Exp. III, where L_G denotes another functor.

(2) The action of G on H by conjugation coincides, under the previous identification, with the adjoint action of G on $\mathrm{Lie}(G)$.

Indeed, these assertions are easily verified in this case: a R-point of H is a matrix of the form $I_n + \varepsilon A$, for some $A \in M_n(R)$. The product of two such elements is:

$$(I_n + \varepsilon A_1)(I_n + \varepsilon A_2) = I_n + \varepsilon(A_1 + A_2).$$

Further, for any $B \in G(R) = \mathrm{GL}_n(R)$, one has $B(I_n + \varepsilon A)B^{-1} = I_n + \varepsilon BAB^{-1}$. These facts will remain true in the more general case consider below.

Notes for this Lecture

Reductive (or semi-simple) groups over an algebraically closed field and reductive (or semi-simple) group schemes are defined in Exp. XIX, 1.6.1 and 2.7.

Maximal tori are defined in Exp. XII, Def. 1.3 and studied in Exp. XII–XIV.

Weil restriction of scalars is defined in Exp. II, §1.

The functors G^+ and L'_G are defined in Exp. III, under more general hypotheses in Def. 0.1.1 and remarks 0.4–0.5, and then put together, in the simpler case where G is a S-group scheme, in Cor. 0.9