Results on MT-groups obtained by descent

10. Representability of $D(G)$ when G is a twisted constant or MT-group

Firstly, we want to complete the proof of Propositions 4.4 and 4.5. In both cases, we have a contravariant functor F : (Schemes/S) $\to$ (Groups) given by $T \mapsto \mathrm{Hom}_{T\text{-Gr}}(G_T, I_T)$, where $I = \mathbb{G}_{m,S}$ and G is the given S-group scheme (either twisted constant or of multiplicative type). More generally, let be given S-schemes X, Y and consider the functor

$$F = \underline{\mathrm{Hom}}_S(X, Y) : \quad (\text{Schemes}/S) \to (\text{Sets}), \qquad T \mapsto \mathrm{Hom}_{T\text{-Sch}}(X_T, Y_T).$$

We want to give conditions ensuring that F is representable by a S-scheme.

Firstly, this F has the following property. Let T be a S-scheme and (U_i) a covering of T by open subsets; one has $U_i \cap U_j = U_i \times_T U_j$, denote it by U_{ij}. To give a morphism of T-schemes $X_T \to Y_T$ is the same thing as giving morphisms $f_i : X_{U_i} \to Y_{U_i}$ which agree on the intersections $X_{U_i} \cap X_{U_j} = X \times_T U_{ij}$, i.e. such that $\mathrm{pr}_1^*(f_i) = \mathrm{pr}_2^*(f_j)$ for all i, j, where $\mathrm{pr}_1, \mathrm{pr}_2$ denote the projections from $U_i \times_T U_j$ to the first and second factor respectively. Thus we have an exact diagram of sets:

$$(10.1) \qquad \begin{tikzcd} \mathrm{Hom}_{T\text{-Sch}}(X_T, Y_T) \arrow[r] \arrow[d, equal] & \prod_i \mathrm{Hom}_{U_i\text{-Sch}}(X_{U_i}, Y_{U_i}) \arrow[r, shift left] \arrow[r, shift right] \arrow[d, equal] & \prod_{i,j} \mathrm{Hom}_{U_{ij}\text{-Sch}}(X_{U_{ij}}, Y_{U_{ij}}) \arrow[d, equal] \\ F(T) \arrow[r] & \prod_i F(U_i) \arrow[r, shift left, "\mathrm{pr}_1^*"] \arrow[r, shift right, "\mathrm{pr}_2^*"'] & \prod_{i,j} F(U_i \times_T U_j). \end{tikzcd}$$

Definition 10.1. Let $\mathscr{C}$ denote the category of S-schemes and $\widehat{\mathscr{C}}$ that of contravariant functors $\mathscr{C} \to$ (Sets). One says that a functor $F \in \widehat{\mathscr{C}}$ having the previous property is a **local functor**, or a **sheaf for the Zariski topology**.

Remark. Setting $T' = \coprod_i U_i$, the second line of (10.1) can be written as $F(T) \longrightarrow F(T') \overset{\mathrm{pr}_1^*}{\underset{\mathrm{pr}_2^*}{\rightrightarrows}} F(T' \times_T T')$.

To illustrate, let us give here the following lemma (a more general result will be proved later).

Lemma 10.2. *Let F be a* **local functor** (Schemes/S) $\to$ (Sets)*. Suppose there exists an open covering (S_i) of S such that $F_i = F \times_S S_i$ be representable by a S_i-scheme X_i. Then F is representable by a S-scheme X.*

Proof. Both $X_i \times_S S_j$ and $X_j \times_S S_i$ represent the restriction of F to $S_{ij} = S_i \times_S S_j$ hence, by Yoneda lemma, there exists a unique isomorphism of S_{ij}-schemes

$$\varphi_{ji} : \quad X_i \times_S S_j \xrightarrow{\sim} X_j \times_S S_i.$$

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Then one has isomorphisms of schemes over $S_{ijk} = S_i \times_S S_j \times S_k$:

$$\begin{tikzcd}
X_i \times_S S_j \times S_k \arrow[r, "\varphi_{ji} \times \mathrm{id}_{S_k}"] \arrow[d, equal] & X_j \times_S S_i \times_S S_k \arrow[r, equal] & X_j \times_S S_k \times_S S_i \arrow[d, "\varphi_{kj} \times \mathrm{id}_{S_i}"] \\
X_i \times_S S_k \times_S S_j \arrow[r, "\varphi_{ki} \times \mathrm{id}_{S_j}"] & X_k \times_S S_i \times_S S_j \arrow[r, equal] & X_k \times_S S_j \times_S S_i
\end{tikzcd}$$

and as all these objects represent the restriction of F to S_{ijk}, this diagram commutes, i.e. the φ_{ji} satisfy the cocyle condition $\varphi_{kj} \circ \varphi_{ji} = \varphi_{ki}$. Therefore the X_i glue together to give a S-scheme X such that $X \times_S S_i = X_i$ for each i. It remains to prove that X represents F.

For every scheme Y over S_i, one has

$$(*) \qquad F(Y) = F_i(Y) = \mathrm{Hom}_{S_i}(Y, X \times_S S_i) = \mathrm{Hom}_S(Y, X) = h_X(Y).$$

Next, for every scheme Y over S, the $Y_i = Y \times_S S_i$ form an open covering of Y; set $Y_{ij} = Y_i \times_Y Y_j = Y \times_S S_{ij}$. As F (resp. h_X) is a local functor then, taking $(*)$ into account, both $F(Y)$ and $h_X(Y)$ identify with the equalizer of the double-arrow:

$$\begin{tikzcd}
\prod_i F(Y_i) \arrow[r, shift left] \arrow[r, shift right] \arrow[d, equal] & \prod_{i,j} F(Y_{ij}) \arrow[d, equal] \\
\prod_i h_X(Y_i) \arrow[r, shift left] \arrow[r, shift right] & \prod_{i,j} h_X(Y_{ij}) \,.
\end{tikzcd}$$

This proves that X represents F. ☐

Moreover, our functor $F = \underline{\mathrm{Hom}}_S(X, Y)$ has the following additional property.

Proposition 10.3. *Let $p : T' \to T$ be faithfully flat and quasi-compact. Denoting by* $\mathrm{pr}_1, \mathrm{pr}_2$ *the two projections from $T'' = T' \times_T T'$ to T', the following diagram is exact:*

$$(10.2) \qquad \begin{tikzcd}
\mathrm{Hom}_{T\text{-Sch}}(X_T, Y_T) \arrow[r] \arrow[d, equal] & \mathrm{Hom}_{T'\text{-Sch}}(X_{T'}, Y_{T'}) \arrow[r, shift left] \arrow[r, shift right] \arrow[d, equal] & \mathrm{Hom}_{T''\text{-Sch}}(X_{T''}, Y_{T''}) \arrow[d, equal] \\
F(T) \arrow[r] & F(T') \arrow[r, shift left, "\mathrm{pr}_1^*"] \arrow[r, shift right, "\mathrm{pr}_2^*"'] & F(T' \times_T T').
\end{tikzcd}$$

Proof. As we saw in the proof of Prop. 8.7, the first line of (10.2) identifies with the diagram:

$$\begin{tikzcd}
\mathrm{Hom}_S(X_T, Y) \arrow[r, "p^*"] & \mathrm{Hom}_S(X_{T'}, Y) \arrow[r, shift left, "\mathrm{pr}_1^*"] \arrow[r, shift right, "\mathrm{pr}_2^*"'] & \mathrm{Hom}_S(X_{T''}, Y)
\end{tikzcd}$$

which is exact since p is a universal effective epimorphism. ☐

Definition 10.4. A functor $F \in \widehat{\mathscr{C}}$ is called a **sheaf for the fpqc topology** if it is local and satisfies the conclusion of Proposition 10.3. For the sake of brevity; we will simply say fpqc-sheaf.

Remark 10.5. If X, Y are S-group schemes, one obtains similarly that $\underline{\mathrm{Hom}}_{S\text{-Gr}}(X, Y)$ is a sheaf for the fpqc topology.

Remarks 10.6. (1) For each $X \in \mathscr{C}$ one has the functor $h_X \in \widehat{\mathscr{C}}$ defined by $h_X(Y) = \mathrm{Hom}_{\mathscr{C}}(Y, X)$. By Yoneda lemma, for each $F \in \widehat{\mathscr{C}}$ and $X \in \mathscr{C}$ there is a natural isomorphism $\mathrm{Hom}_{\widehat{\mathscr{C}}}(h_X, F) = F(X)$. In particular, for $X, Y \in \mathscr{C}$ one has $\mathrm{Hom}_{\widehat{\mathscr{C}}}(h_X, h_Y) = h_Y(X) = \mathrm{Hom}_{\mathscr{C}}(X, Y)$. This shows that $\mathscr{C}$ identifies with a full subcategory of $\widehat{\mathscr{C}}$.

(2) Noting that $\underline{\mathrm{Hom}}_S(S, X) = h_X$ one obtains that each $X \in \mathscr{C}$ (identified with h_X) is a sheaf for the fpqc topology.

(3) The *larger* categories of sheaves for the fppf, étale, finite étale topologies consist of all local functors which satisfy the condition of Def. 10.4 only for faithfully flat morphisms f which are of finite presentation, resp. étale, resp. finite étale. Note that the finer the topology is (i.e. the less restrictions on f there are), the closer the resulting sheaves get to actual schemes.

Then, one has the following important result, which is "well-known to the experts" but not easy to find in the literature in this explicit form (see however [**SGA3**$_1$], IV, Prop. 3.5.2). We have included it as Lemma 1.7.2 in Exp. VIII of the new edition of [**SGA3**$_2$].

PROPOSITION 10.7. *Let* $F : (\mathrm{Sch}_{/S}) \to (\mathrm{Sets})$ *be a* fpqc *sheaf. Assume there exists* $S' \to S$ *faithfully flat and quasi-compact such that* $F' = F \times_S S'$ *is representable by a* S'*-scheme* X'.

(1) *Then* X' *is endowed with a descent datum with respect to* $S' \to S$.
(2) *If this descent datum is effective, that is, if* X' *comes by base change from a* S*-scheme* X*, then* X *represents* F.
(3) *The descent datum is always effective is* X' *is* quasi-affine *over* S'*; more generally if* X' *is covered by saturated open subsets which are quasi-affine over* S'.

PROOF. (1) We use the notation S_i'' and S_i''' introduced in 8.5. The hypotheses imply that $F_i'' = F' \times_{S'} S_i''$ is represented by $X_i'' = X' \times_{S'} S_i''$. But $F_1'' = F \times_S S'' = F_2''$. Hence, by uniqueness of the representing scheme (Yoneda lemma), there exists a unique S''-isomorphism $\varphi : X_1'' \xrightarrow{\sim} X_2''$.

For $i < j$ in $\{1, 2, 3\}$, denote by $\mathrm{pr}_{ji} : S''' \to S''$ the projection to the factors i and j. Then, set $X_i''' = X' \times_{S'} S_i'''$ and denote by $\mathrm{pr}_{ji}^*(\varphi) : X_i''' \xrightarrow{\sim} X_j'''$ the isomorphism of S'''-schemes obtained from φ by base change. Then, one obtains a diagram of isomorphisms of S'''-schemes:

$$
\begin{tikzcd}
X_1''' \arrow[r, "\mathrm{pr}_{21}^*(\varphi)"] \arrow[dr, "\mathrm{pr}_{31}^*(\varphi)"'] & X_2''' \arrow[d, "\mathrm{pr}_{32}^*(\varphi)"] \\
& X_3'''
\end{tikzcd}
$$

and as these schemes represent the restriction of F to S''' they are uniquely isomorphic, hence one has the cocyle relation $\mathrm{pr}_{31}^*(\varphi) = \mathrm{pr}_{32}^*(\varphi) \circ \mathrm{pr}_{21}^*(\varphi)$, that is, φ is a descent datum on X' relative to $S' \to S$.

(2) Assume further that this descent datum is effective, i.e. that there exists a S-scheme X such that $X' \simeq X \times_S S'$. We prove that X represents F, as in Lemma 10.2: for every $Y \to S'$, one has

$$(**) \qquad F(Y) = F'(Y) = \mathrm{Hom}_{S'}(Y, X \times_S S') = \mathrm{Hom}_S(Y, X) = h_X(Y).$$

Next, for every $Y \to S$ set $Y' = Y \times_S S'$ and $Y'' = Y' \times_Y Y' \simeq Y \times_S S''$. Then $Y' \to Y$ is faithfully flat and quasi-compact, since $S' \to S$ is so. As F and h_X are sheaves for the fpqc topology, one deduces from $(**)$ that $F(Y)$ and $h_X(Y)$ both identify to the equalizer of the double arrow:

$$
\begin{tikzcd}
F(Y') \arrow[r, shift left] \arrow[r, shift right] \arrow[d, equal] & F(Y'') \arrow[d, equal] \\
h_X(Y') \arrow[r, shift left] \arrow[r, shift right] & h_X(Y'') .
\end{tikzcd}
$$

This proves that X represents F.

(3) This follows from Th. 8.18 and Lemma 8.21. □

We can now complete the proof of propositions 4.4 and 4.5. Recall the hypotheses: M is a finitely generated abelian group and E, resp. H is a twisted constant group, resp. MT-group, of type M over S. Further, one assumes that H is quasi-isotrivial. We have to prove that the functors $D(E)$ and $D(H)$ are representable, respectively, by a MT-group and a twisted constant group of type M.

Proof. By Lemma 10.2 and the previous proposition, we only have to prove that if S is affine and if $p : S' \to S$ is a flat surjective morphism, with S' affine, such that the functor $D(E)_{S'}$ (resp. $D(H)_{S'}$) is represented by $X' = D(M)_{S'}$ (resp. $X' = M_{S'}$), then the descent datum on X' is effective. In the first case, this follows immediately from the previous proposition, since $D(M)_{S'}$ is affine.

In the second case, using the hypothesis that H is quasi-isotrivial, we may assume that the above morphism $p : S' \to S$ is étale, hence locally of finite presentation. Further, in this case, $X' = M_{S'}$ is étale over S' hence separated, locally of finite presentation and locally quasi-finite over S'. So we conclude by Prop. 8.23 that the descent datum on X' is effective. $\square$

Example 10.8. Illustrate this when S is the nodal cubic curve of Remark 4.10 by constructing H and E over S which become trivial over the principal $\mathbb{Z}$-bundle $P \to S$, but are not isotrivial over S. (To be done during the lecture).

11. Isotriviality over a locally noetherian normal base

We fix a base scheme S and an abelian group M. For the sake of completeness, let us record here the following theorem ([**SGA3**$_2$], X, Cor. 4.5).[1]

Theorem 11.1. *Let H be a* MT*-group over S of type M. If M is finitely generated, then H is quasi-isotrivial.*

From now on, we assume that M is **finitely generated** and that S is **locally noetherian**.

Lemma 11.2. *Let P be a quasi-isotrivial twisted constant scheme over S. Let Z be an open and closed subset of P.*

(1) *Let U be the set of those $s \in S$ such that the fiber Z_s is finite. Then U is open and closed in S and the map $Z_U \to U$ is finite.*
(2) *In particular, if S is connected and U non-empty, then $Z \to S$ is finite.*

Proof. By assumption, there exists a surjective étale map $f : S' \to S$ such that $P_{S'} = I_{S'}$ for some set I. We have a cartesian diagram:

$$\begin{tikzcd} Z' \arrow[r] \arrow[d, hook] & Z \arrow[d, hook] \\ I_{S'} \arrow[r] \arrow[d] & P \arrow[d] \\ S' \arrow[r, "f"] & S \end{tikzcd}$$

and, since f is étale, the inverse image $f^{-1}(U)$ equals the set U' of $s' \in S'$ such that the fiber $Z'_{s'}$ is finite.[2] Further, since f is étale, S' is still locally noetherian, hence its connected components are open and closed. If C is such a component, the C_i are the connected components of I_C, hence Z'_C equals J_C for some subset $J = J(C)$ of I, and one sees that the points of C belong to U' if and only if $J(C)$ is finite, in which case the map $Z'_C \to C$ is finite.

Thus, U' is the union of those connected components C of S' such that $J(C)$ is finite, hence is open and closed in S', and the map $Z'_{U'} \to U'$ is finite. Since the topology of S is the quotient of that of S', one obtains that U is open and closed in S. Further, the map $Z'_{U'} \to U'$ is the

[1] This theorem is one of the reasons why we restricted to finitely generated abelian groups M.

[2] Setting $s = f(s')$, one has $Z'_{s'} \simeq Z_s \otimes_{\kappa(s)} \kappa(s')$, where $\kappa(s)$ denotes the residue field of s.

pull-back via f of $Z_U \to U$ and, since the former is finite, so is the later (use e.g. Lemma 8.22 and [**EGA**], IV_2, Prop. 2.7.1). This proves assertion (1), and assertion (2) follows immediately. $\square$

Definition 11.3. Recall that S is supposed locally noetherian. Then one says that S is **geometrically unibranch** (see [**EGA**] IV_2, 6.15.1 and paragraph before 6.15.14) if the normalization map $\widetilde{S} \to S_{\mathrm{red}}$ is radicial (hence a universal homeomorphism).[3]

The important fact is that, in this case, the connected components of S are irreducible.[4] Further, if a morphism $P \to S$ is étale, then P is also locally noetherian and geometrically unibranch (see [**EGA**], IV_4, Prop. 17.7.5).

Proposition 11.4. *Suppose that S is locally noetherian and geometrically unibranch. Let $p : P \to S$ be a quasi-isotrivial twisted constant scheme over S. Then the connected components of P are* finite *over S.*[5]

Proof. Since S and P are locally noetherian and geometrically unibranch, their connected components are open and closed, and are irreducible. In particular, replacing S by one of its connected components, we may assume S irreducible; let η be its generic point. Let C be a connected component of P; it is irreducible, denote by ξ its generic point. As p is flat, one has $p(\xi) = \eta$. As C is the closure of ξ in P, it follows that $C \cap p^{-1}(\eta)$ is the closure of ξ in $p^{-1}(\eta)$. But since p is étale, hence locally quasi-finite, the fiber $p^{-1}(\eta)$ is discrete. Thus, for the open and closed subset C of P one has $C_\eta = \{\xi\}$, which is finite. Hence, by the previous lemma, C is finite over S. $\square$

Theorem 11.5. *Suppose that S is locally noetherian and geometrically unibranch. Let H be a* MT-*group of type M, which is quasi-isotrivial.*[6] *Then H is in fact isotrivial.*

Proof. Set $G = D(M)_S$ and denote the functors $\underline{\mathrm{Hom}}_{S\text{-}\mathrm{Gr}}(G,H)$ and $\underline{\mathrm{Isom}}_{S\text{-}\mathrm{Gr}}(G,H)$ by E and I respectively.

Let $S' \to S$ be an étale map such that $H_{S'} \simeq D(M)_{S'}$. Since M is finitely generated we obtain, by Prop. 7.5, that

$$E_{S'} = \underline{\mathrm{Hom}}_{S'\text{-}\mathrm{Gr}}(D(M)_{S'}, D(M)_{S'}) = \underline{\mathrm{Hom}}_{S'\text{-}\mathrm{Gr}}(M_{S'}, M_{S'})$$

is represented by the constant scheme $\mathrm{End}(M)_{S'}$, and then that $I_{S'}$ is represented by the constant scheme $\mathrm{Aut}(M)_{S'}$. By the effectiveness result of Prop. 8.23, I is represented by a twisted constant scheme P over S.

Let C be a connected component of P. It is étale over S and, by the previous proposition, finite. Hence $p(C)$ is open and closed in S, hence $p(C) = S$ since S is connected. Thus $p : C \to S$ is étale, surjective and finite. Further, the diagonal map $C \to C \times_S C$ produces a section over C of $P_C = \mathrm{Isom}_{C\text{-}\mathrm{Gr}}(G_C, H_C)$, hence H_C is isomorphic with $D(M)_C$. This proves that H is isotrivial. $\square$

12. Classification of isotrivial groups of multiplicative type

In this section, we assume that the base scheme S is **connected**.

Definition 12.1. (1) An **étale covering** of S is a morphism $\pi : S' \to S$ which is étale, surjective and finite (in particular, affine). Then $\pi_* \mathcal{O}_{S'}$ is a locally free $\mathcal{O}_S$-algebra of rank n, and n is called the *degree* of the covering.

(2) The group Γ of S-automorphisms of S' is finite, of cardinality $\leq n$. If S' is connected and $|\Gamma| = n$, one says that $S' \to S$ is a **Galois covering** with group Γ.

[3]For example, this is the case if S is normal or if S is a cuspidal curve.

[4]Beware that without the locally noetherian hypothesis, there exists connected normal schemes which are not irreducible, see [**StaPr**], Tag 033O or Exercise 2.4.12 in [**Co14**].

[5]Contrast this with the connected principal $\mathbb{Z}$-bundle over a nodal curve of Remark 4.10.

[6]As we suppose that M is finitely generated, H is automatically quasi-isotrivial, by Th. 11.1.

REMARK 12.2. Let $\pi : E \to S$ be an étale covering. One knows that: [7]

(1) E has finitely many connected components $C_1, \dots, C_r$, each open and closed.

Then each morphism $\pi_i : C_i \to S$ is still finite and étale; further, since S is connected and π is open and closed (being étale and finite), each π_i is surjective. So each $\pi_i : C_i \to S$ is a *connected* étale covering.

(2) Every *connected* étale covering $p : C \to S$ is dominated by a Galois covering, that is, there exists a Galois covering $\pi : S' \to S$, with Galois group Γ, and a S-morphism $q : S' \to C$ such that $\pi = p \circ q$.

Now, if H is an isotrivial S-group scheme of multiplicative type, there exists an étale covering $E \to S$ such that $H_E \simeq D(M)_E$ for some (finitely generated) abelian group M. By the previous remark, we may replace E by a Galois covering $S' \to S$ with Galois group Γ. For the sake of simplicity, let us further assume that $S = \operatorname{Spec} R$ is affine. Then $S' = \operatorname{Spec} R'$ for some Galois covering $R \to R'$ with group Γ.

Consider now the category $\mathrm{MT}(S'/S)$ of all S-groups H of multiplicative type which become diagonalisable over S'. It is anti-equivalent to the category of R-Hopf algebras A such that $A \otimes_R R'$ is isomorphic with $R'[M]$, for some finitely generated abelian group M. In this case, we have an action of Γ on $B = R'[M]$ by semi-linear automorphisms of R-Hopf algebra. This induces an action of Γ on M by group automorphisms because, by the proof of the biduality theorem 3.4 and the fact that S' is connected, we have: [8]

$$D(M_{S'})(S') = \operatorname{Hom}_{R'\text{-Hopf}}(R'[X, X^{-1}], R'[M]) = \operatorname{Loc}(S', M) = M.$$

Thus, base change from S to S' is a contravariant functor from $\mathrm{MT}(S'/S)$ to the category of finitely generated Γ-modules. Now, the gist of Galois descent theory is contained in Example 1.5 above, namely that a quasi-inverse is given by the functor taking such a Γ-module M to $H = \operatorname{Spec} R'[M]^\Gamma$.

So far, we have assumed $S = \operatorname{Spec} R$ for simplicity, so that $H_{S'} = \operatorname{Spec} B$, where $B = R'[M]$, in which case we know that the quotient of $H_{S'}$ by Γ exists and is $\operatorname{Spec} B^\Gamma$. But it is known in general (see [**SGA1**], V, Cor. 1.8) that if $\pi : X \to S$ is affine and Γ is a finite group of S-automorphisms of X, then the quotient $Y = X/\Gamma$ exists, and over any open affine subset $U = \operatorname{Spec} R$ of S one has $Y_U = \operatorname{Spec} B^\Gamma$, if one denotes by $\operatorname{Spec} B$ the affine scheme $\pi^{-1}(U)$. So we have obtained the:

THEOREM 12.3. *Let S be a* connected *base scheme and $S' \to S$ a Galois covering with group Γ. The category $\mathrm{MT}(S'/S)$ of S-groups of multiplicative type which split over S' is anti-equivalent to the category of finitely generated Γ-modules.*

REMARKS 12.4. Let $H \in \mathrm{MT}(S'/S)$ correspond to a Γ-module M. Denote by M^Γ and M_Γ the sets of invariants and coinvariants, that is, the largest submodule (resp. quotient module) on which the action of Γ is trivial; one has $M_\Gamma = M/N$, where N is the submodule generated by the elements $m - \gamma(m)$, for $m \in M$ and $\gamma \in \Gamma$. Then:

(1) H is diagonalisable if and only if the action of Γ on M is trivial.
(2) Regarding by $\mathbb{Z}$ as a Γ-module with trivial Γ-action, one has:

$$\operatorname{Hom}_{S\text{-Gr}}(H, \mathbb{G}_{m,S}) = \operatorname{Hom}_\Gamma(\mathbb{Z}, M) = M^\Gamma, \tag{12.1}$$

$$\operatorname{Hom}_{S\text{-Gr}}(\mathbb{G}_{m,S}, H) = \operatorname{Hom}_\Gamma(M, \mathbb{Z}) = \operatorname{Hom}(M_\Gamma, \mathbb{Z}). \tag{12.2}$$

[7] In many places, this is buried in the package about Galois categories. It would be nice to give a precise reference or, better, a self-contained proof.

[8] A purely algebraic formulation is that the set of group-like elements in $R'[M]$ is exactly M.

(3) The natural pairing $M^\Gamma \times \mathrm{Hom}_\Gamma(M,\mathbb{Z}) \to \mathbb{Z}$ is not necessarily perfect, even if M is a free $\mathbb{Z}$-module: if M is the permutation representation $\mathbb{Z}[\Gamma]$ one has $M^\Gamma = \mathbb{Z}v$, where $v = \sum_{\gamma\in\Gamma} e_\gamma$, whilst $M \to M_\Gamma = \mathbb{Z}$ is given by $\sum_{\gamma\in\Gamma} n_\gamma e_\gamma \mapsto \sum_{\gamma\in\Gamma} n_\gamma$. Thus the image of the pairing is $d\mathbb{Z}$, where $d = |G|$.

Remark 12.5. To illustrate Remark (3) above, consider the Deligne torus $H = \mathrm{Res}_{\mathbb{R}}^{\mathbb{C}}\, \mathbb{G}_{m,\mathbb{C}}$, which corresponds to the permutation module $M = \mathbb{Z}e_0 \oplus \mathbb{Z}e_1$, where τ swaps e_0 and e_1. One has $M^\Gamma = \mathbb{Z}(e_0+e_1)$, whereas the kernel of $M \to M_\Gamma$ is $\mathbb{Z}(e_0 - e_1)$. The exact sequence $0 \longrightarrow M^\Gamma \longrightarrow M \longrightarrow M/M^\Gamma \longrightarrow 0$ corresponds to the exact sequence

$$1 \longrightarrow \mathbb{S}^1 \longrightarrow H \xrightarrow{N} \mathbb{G}_{m,\mathbb{R}} \longrightarrow 1 \tag{12.3}$$

where N is the norm homorphism. It has no section; indeed $D(M_\Gamma)$ is the largest split subtorus of H and the cokernel of $M^\Gamma \to M_\Gamma$ is $\mathbb{Z}/2\mathbb{Z}$. However, the group of $\mathbb{R}$-points $H(\mathbb{R})$ splits as $\mathbb{S}^1(\mathbb{R}) \times \mathbb{R}_+^\times$.

Proposition 12.6. *Let k be a field and M a finitely generated abelian group. Then any k-group H of multiplicative type of type M is isotrivial, i.e. there exists a finite separable extension k' of k such that $H_{k'} \simeq D(M)_{k'}$.*

Proof. Set $H = \operatorname{Spec}\Lambda$. By hypothesis, there exists a k-algebra A and an isomorphism of Hopf algebras $\phi : A[M] \xrightarrow{\sim} A \otimes_k \Lambda$. We reduce first to the case where A is a finitely generated k-algebra.

Let $m_1, \dots, m_s$ be a set of generators of M, write firstly $\phi(m_i) = \sum_j a_{ij} \otimes r_{ij}$ and then $\phi^{-1}(r_{ij}) = \sum_{m\in M} \alpha_{ijm} m$ (all sums being finite), and let Λ_1 (resp. B) be the k-subalgebra of Λ (resp. of A) generated by the r_{ij}'s (resp. the a_{ij} and α_{ijm}'s). Using that k is a field, one obtains firstly that $\Lambda_1 = \Lambda$ (because $A \otimes_k (\Lambda/\Lambda_1) = 0$) and secondly that $B \otimes_k \Lambda$ is a subalgebra of $A \otimes_k \Lambda$. Clearly, ϕ maps $B[M]$ into $B \otimes_k \Lambda$ and ϕ^{-1} maps $B \otimes_k \Lambda = B \otimes_k \Lambda_1$ into $B[M]$. It follows that ϕ induces an isomorphism $B[M] \xrightarrow{\sim} B \otimes_k \Lambda$.

Next, let $\mathfrak{m}$ be a maximal ideal of B and $K = B/\mathfrak{m}$. On the one hand, $K[M] \xrightarrow{\sim} K \otimes_k \Lambda$. On the other hand, by the Nullstellensatz, K is a finite extension of k. Let k' be the separable closure of k in K and set $S = \operatorname{Spec} k$ and $S' = \operatorname{Spec} k'$. Set also $R = \operatorname{Spec} K$.

Now, consider the twisted constant group $E = D(H)_S$, it is étale over S. Denote by E' and E_R its pull-backs over S' and R. To emphasize the idea, we can now invoke the general result that since $R \to S'$ is radicial, the base change from étale S'-schemes to étale R-schemes is *fully faithful* (and even an equivalence of categories), see e.g. [**SGA1**], IX, Cor. 3.4 (and Th. 4.10). Since $E_R \simeq M_R$, one has $E' \simeq M_{S'}$ and hence, by Proposition 4.4, $H_{S'} \simeq D_{S'}(E') = D(M)_{S'}$.

In our simple case we can give a direct proof of the key step. Let C be a connected component of E'. Then $C = \operatorname{Spec} L$ for a field L finite and separable over k', hence $C_R = \operatorname{Spec}(L \otimes_k K)$ is a sum of finitely many $\operatorname{Spec}(K_i)$, where each K_i is a field separable over K. Further, each K_i equals K, since each connected components of E_R is equal to $R = \operatorname{Spec} K$. On the other hand, as K/k' is purely inseparable, C_R is irreducible (see e.g. [**EGA**] IV_2, Prop. 4.3.2). It follows that $L \otimes_k K = K$ and hence $L = k$. This proves that E' is trivial over S'. □

Notes for this Lecture

The representability of $D(R)$ (resp. $D(H)$) is proved in Exp. X, Prop. 5.3 (resp. Cor. 5.7).

Theorem 11.5 is proved in Exp. X, Th. 5.16.

The classification of isotrivial groups of multiplicative type is given in Exp. X, Prop. 1.1.

The fact that a group of multiplicative type over a field is isotrivial is proved in Exp. X, Prop. 1.4, but the proof there uses radicial descent for groups of multiplicative type, proved using cohomology in a much more general setting in Exp. IX, Cor. 5.4. That proof has been much simplified by Oesterlé ([**Oes14**], §§12–13).

Theorem 11.1, that we gave without proof, is proved in Exp. X, Cor. 4.5 as a corollary of the spreading theorem Th. 4.4. It also uses in an essential manner the algebrisation theorem IX, Th. 7.1.