LECTURE 4

Faithfully flat descent

8. Faithfully flat descent

We give an overview of descent theory, trying to emphasize the main ideas and to avoid unnecessary formalism. The two main notions are that of **universal effective epimorphisms** and **morphisms of effective descent**.[1] Beware that *effective* does not have the same meaning in both expressions: we will see below that a universal effective epimorphism is the same thing as a morphism of descent, whereas the effectiveness of descent is a further property.

In this section, $\mathscr{C}$ denotes a category with fiber products. (In our applications, it will be the category of schemes.)

Definition 8.1. Consider a morphism $p : X \to Y$ in $\mathscr{C}$. One says that:

(1) p is an **epimorphism** if for every morphism $Y \to Z$, the induced map $\mathrm{Hom}(Y, Z) \to \mathrm{Hom}(X, Z)$ is injective.[2]

(2) p is a **universal epimorphism** if for every morphism $T \to Y$ in $\mathscr{C}$ the morphism $p_T : X \times_Y T \to T$ obtained by base change is an epimorphism (in this case each p_T is again a universal epimorphism).

(3) p is an **effective epimorphism** if, denoting by p_1, p_2 the two projections from $X \times_Y X$ to X, the following diagram in $\mathscr{C}$:

$$X \times_Y X \underset{p_2}{\overset{p_1}{\rightrightarrows}} X \xrightarrow{p} Y \tag{8.1}$$

is **exact**, which means that for every object Z of $\mathscr{C}$ the following diagram of sets:

$$\mathrm{Hom}(Y, Z) \xrightarrow{p^*} \mathrm{Hom}(X, Z) \underset{p_2^*}{\overset{p_1^*}{\rightrightarrows}} \mathrm{Hom}(X \times_Y X, Z) \tag{8.2}$$

is **exact**, which in turn means that the map $f \mapsto f \circ p$ is a bijection from $\mathrm{Hom}(Y, Z)$ onto the set $\{g \in \mathrm{Hom}(X, Z) \mid g \circ p_1 = g \circ p_2\}$.

(4) p is a **universal effective epimorphism** if for every morphism $T \to Y$ in $\mathscr{C}$ the morphism $p_T : X \times_Y T \to T$ obtained by base change is an effective epimorphism (in this case each p_T is a universal effective epimorphism).

Firstly, let us record here the following easy remark and lemma.

Remark 8.2. It is clear that if $p : X \to Y$ and $q : Y \to Z$ are epimorphisms (resp. universal epimorphisms), so is $q \circ p$. Further, if p and $p' : X' \to Y'$ are universal epimorphisms, so is $p \times p' : X \times X' \to Y \times Y'$, since it is the composition of the two morphisms obtained by base change: $X \times X' \xrightarrow[\mathrm{id}_X \times p']{} X \times Y' \xrightarrow[p \times \mathrm{id}_{Y'}]{} Y \times Y'$.

Lemma 8.3. *If $p : X \to Y$ admits a section σ, then p is a universal effective epimorphism.*

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[1] And the companion notions of *equivalence relations* and *descent data*.

[2] For example, an epimorphism in the category of sets is just a surjective map.

PROOF. Since having a section is preserved by base change, it suffices to prove that p is an effective epimorphism. It is clear that p^* is injective, since $\sigma^* \circ p^* = \mathrm{id}$. Let τ be the morphism $X \to X \times_Y X$, $x \mapsto (x, \sigma p(x))$. Let $g : X \to Z$ such that $g \circ p_1 = g \circ p_2$, then $g = g \circ p_1 \circ \tau = g \circ p_2 \circ \tau = g \circ \sigma \circ p$, i.e. $g = p^*(f)$ with $f = g \circ \sigma$. □

To be concrete, let us enunciate immediately the following fundamental result (see [**SGA1**], VIII, Th. 5.2 or [**BLR**], §6.1, Th. 6).

THEOREM 8.4. *In the category of schemes, every faithfully flat, quasi-compact morphism is a universal effective epimorphism.*

Now, let us fix a morphism $p : S' \to S$ in our category $\mathscr{C}$. We want to study the base-change functor $\mathscr{C}_{/S} \to \mathscr{C}_{/S'}$ which sends every object X over S to the object $X \times_S S'$ over S'. The goal of *descent theory* is to give conditions on an arbitrary S'-object X' which would ensure that $X' \simeq X \times_S S'$ for some S-object X.

NOTATION 8.5. Denote by S''_1 (resp. S''_2) the scheme $S'' = S' \times_S S'$ regarded as a S'-scheme via the first projection p_1 (resp. second projection p_2).

Further,[3] for $i = 1, 2, 3$, denote by S'''_i the scheme $S''' = S' \times_S S' \times_S S'$ regarded as a S'-scheme via the projection to the i-th factor. For $i < j$ in $\{1, 2, 3\}$, denote by $\mathrm{pr}_{ji} : S''' \to S''$ the projection to the factors i and j. Further, for every S'-object X', set $X''_i = X' \times_{S'} S''_i$ for $i = 1, 2$ and $X'''_i = X' \times_{S'} S'''_i$ for $i = 1, 2, 3$.

DEFINITION 8.6. One says that p is a **morphism of descent** if the following property is satisfied: for all objects X, Y over S, if we set $X' = X \times_S S'$ and $X'' = X \times_S S''$ and define Y', Y'' similarly, then the following diagram of sets is exact:

$$\mathrm{Hom}_S(X, Y) \xrightarrow{p^*} \mathrm{Hom}_{S'}(X', Y') \overset{p_1^*}{\underset{p_2^*}{\rightrightarrows}} \mathrm{Hom}_{S''}(X'', Y''). \tag{8.3}$$

PROPOSITION 8.7. *$p : S' \to S$ is a morphism of descent if and only if it is a universal effective epimorphism.*[4]

PROOF. Let X, Y be arbitrary S-objects. The diagram (8.3) identifies with the diagram:

$$\mathrm{Hom}_S(X, Y) \xrightarrow{p^*} \mathrm{Hom}_{S}(X', Y) \overset{p_1^*}{\underset{p_2^*}{\rightrightarrows}} \mathrm{Hom}_{S}(X'', Y). \tag{8.4}$$

Since $X'' = X \times_S S''$ identifies with the fiber product $X' \times_X X'$ where $p_X : X' \to X$ is obtained from p by the base change $X \to S$, we see that the exactness of the second diagram, for all Y and each given X, means that p_X is an effective epimorphism. Thus we see that p is a morphism of descent if and only if p is a universal effective epimorphism. □

Before we introduce the notion of *descent datum*, we need to introduce that of *equivalence relation.*

DEFINITION 8.8. An **equivalence relation**[5] on an object X is a subfunctor of $X \times X$, which is represented by an object R (equivalently, one is given a monomorphism $R \hookrightarrow X \times X$) such that, for every object T of $\mathscr{C}$, the set $R(T)$ is the graph of an equivalence relation on $X(T) \times X(T)$.
In this case, one denotes by p_1, p_2 the restrictions to R of the two projections from $X \times X$ to X.

Here are two important examples.

[3]This is not used in the definition of morphism of descent, but this will be used later.

[4]One may wonder why introducing a new name for an already existing notion. The reason will appear later.

[5]By hypothesis, $\mathscr{C}$ has a final object e and the unadorned fiber product $\times$ is taken over e. When $\mathscr{C} = (\mathrm{Sch}_{/S})$ we will write explicitly $\times_S$ and we say that R is an equivalence relation on X "over S".

EXAMPLE 8.9. To every morphism $f : X \to Y$ in $\mathscr{C}$ is associated the equivalence relation $R_f = X \times_Y X$ on X. In this case, $R_f \to X \times X$ is an *immersion*.[6] For every object T, one has $R_f(T) = \{(x, x') \in X(T) \times X(T) \mid f(T)(x) = f(T)(x')\}$.

DEFINITION 8.10 (**Free actions**). Let X be an object of $\mathscr{C}$ and H a group-object in $\mathscr{C}$ acting on X, say on the right. One says that H acts *freely* if for every object T, the group $H(T)$ acts freely on $X(T)$. In this case, the morphism $X \times H \to X \times X$ defined on arbitrary T-points by $(x, h) \mapsto (x, xh)$ is a monomorphism and is an equivalence relation R_H on X.[7]

Now, let us study more closely the base change functor $\mathscr{C}_{/S} \to \mathscr{C}_{/S'}$. Observe that for each S-object X the S'-object $X' = X \times_S S'$ has the following three properties:

(1) There is a canonical isomorphism of S''-schemes $\varphi : X''_1 \simeq X \times_S S'' \simeq X''_2$.

(2) The pull-back of φ via the diagonal map $\delta : S' \to S''$ is $\mathrm{id}_{X'}$.

(3) The pull-backs $\mathrm{pr}^*_{ji}(\varphi)$ all identify with the canonical isomorphisms $X'''_i \simeq X \times_S S'' \simeq X'''_j$; in particular they satisfy the cocyle condition:

$$\mathrm{pr}^*_{31}(\varphi) = \mathrm{pr}^*_{32}(\varphi) \circ \mathrm{pr}^*_{21}(\varphi). \tag{8.5}$$

Thus, we see that these conditions are necessary for X' to come from a S-object. This motivates the following definition.

DEFINITION 8.11. **(1) A descent datum** on a S'-object X' relative to $S' \to S$ is an S''-isomorphism $\varphi : X''_1 \xrightarrow{\sim} X''_2$ which satisfies the cocycle condition (8.5). (This implies that $\delta^*(\varphi) = \mathrm{id}_{X'}$, see below.)

A more intuitive way to formulate this is as follows (see [**TDTE1**], §A.1 (c), p. 190-05 or [**BLR**], §6.1, p. 133). For any S-object T and any S-morphism $(t_1, t_2) : T \to S''$, we have a T-isomorphism

$$\varphi_{t_2,t_1} : X'_{t_1} \xrightarrow{\sim} X'_{t_2} \tag{8.6}$$

where $X'_{t_i} = X' \times_{S',t_i} T$ (i.e. T is over S' via t_i), and these isomorphisms are subject to the transitivity condition below, when t_1, t_2, t_3 are S-morphisms $T \to S'$:

$$\varphi_{t_3,t_1} = \varphi_{t_3,t_2} \circ \varphi_{t_2,t_1}. \tag{8.7}$$

In particular, for $t_1 = t_2 = t_3 = t$ one obtains $\varphi_{t,t} = \varphi_{t,t} \circ \varphi_{t,t}$ hence (since $\varphi_{t,t}$ is an isomorphism) $\varphi_{t,t} = \mathrm{id}_{X_t}$. This implies that $\varphi_{t_1,t_2} = \varphi^{-1}_{t_2,t_1}$. Further, applying this to the identity morphism $S' \to S'$, one obtains that $\delta^*(\varphi) = \mathrm{id}_{X'}$.

(2) Moreover, let q_1 be the first projection from $X''_1 = X' \times_{S'} S''_1$ to X' and q_2 the composition of φ and the first projection of X''_2. Then the morphism

$$X''_1 \xrightarrow{(q_1,q_2)} X' \times_S X'$$

is a monomorphism, since its composition with $X' \to S'$ is the isomorphism $X''_1 \simeq X' \times_S S'$, and the above interpretation of the cocycle condition shows that (q_1, q_2) is an **equivalence relation** on X' over S.[8] Namely, for any S-scheme T and pair of points $(x_1, x_2) \in \mathrm{Hom}_S(T, X' \times_S X')$ mapping to the pair $(t_1, t_2) \in \mathrm{Hom}_S(T, S' \times_S S')$, one has $x_1 \sim x_2$ if and only if $x_2 = \varphi_{t_2,t_1}(x_1)$. This is reflexive since $\varphi_{t,t} = \mathrm{id}$, symmetric since $\varphi_{t_1,t_2} = \varphi^{-1}_{t_2,t_1}$, and transitive since $\varphi_{t_3,t_1} = \varphi_{t_3,t_2} \circ \varphi_{t_2,t_1}$.

[6] That is, a closed immersion followed by an open immersion, see [**EGA**], I 5.3.10 and the correction III_2, Err_{10} to I 5.3.9.

[7] If there exists a quotient $p : X \to Y = G/H$ then $R_H = R_p = X \times_Y X$, as we shall see later.

[8] See [**StaPr**], Tag 024E (Lemma 30.1 in Chapter Simplicial Spaces) for a proof using only the cocycle condition and cartesian diagrams.

(3) There is an obvious notion of morphism of S'-objects with descent data relative to $S' \to S$. So we may introduce the category $\mathrm{Desc}(S'/S)$ of S'-objects with descent data.

Then, comparing with Def. 8.6 we obtain the following:

COROLLARY 8.12. *The morphism $p : S' \to S$ is a morphism of descent if and only if the base change functor $p^* : \mathscr{C}_{/S} \to \mathrm{Desc}(S'/S)$ is* **fully faithful**, *i.e. induces bijections between the* Hom-*sets.*

To illustrate the concept of descent data, consider the following example in the category of schemes.

EXAMPLE 8.13. Let $S' \to S$ be a Galois covering, with group Γ. This means, assuming for convenience that Γ acts on the right, that the morphism $\mu : S' \times \Gamma \to S'' = S' \times_S S'$ given by $\mu(s', \gamma) = (s', s'\gamma)$, for every $T \to S$ and $s' \in S'(T)$, is an isomorphism. Then we have isomorphisms:

$$\begin{array}{ccccc} S' \times \Gamma \times \Gamma & \xrightarrow{\sim} & S' \times_S S' \times \Gamma & \xrightarrow{\sim} & S' \times_S S' \times_S S' \\ (s', \gamma_1, \gamma_2) & \mapsto & (s', s'\gamma_1, \gamma_2) & \mapsto & (s', s'\gamma_1, s'\gamma_1\gamma_2). \end{array}$$

Thus, any T-point of S''' can be written uniquely as $(s', s'\gamma_1, s'\gamma_1\gamma_2)$.

Now, let $f : X' \to S'$ and assume given a right action of Γ on X' compatible with its action on S': this means that the obvious diagram is commutative, and at the level of arbitrary T-points this is expressed by $f(x'\gamma) = f(x')\gamma$. Then we have the S''-isomorphism:

$$\varphi : \quad X \times_{S'} S''_1 \xrightarrow{\sim} X' \times_{S'} S''_2, \qquad (x', s', s'\gamma) \mapsto (x'\gamma, s', s'\gamma)$$

and its pull-back $\mathrm{pr}_{21}^*(\varphi) : X' \times_{S'} S'''_1 \xrightarrow{\sim} X' \times_{S'} S'''_1$, as well as the two other pull-backs, are given in terms of arbitrary T-points by the diagram below:

$$\begin{tikzcd} (x', s', s'\gamma_1, s'\gamma_1\gamma_2) \arrow[r, mapsto, "\mathrm{pr}_{21}^*(\varphi)"] \arrow[dr, mapsto, "\mathrm{pr}_{31}^*(\varphi)"'] & (x'\gamma_1, s', s'\gamma_1, s'\gamma_1\gamma_2) \arrow[d, mapsto, "\mathrm{pr}_{32}^*(\varphi)"] \\ & (x'\gamma_1\gamma_2, s', s'\gamma_1, s'\gamma_1\gamma_2). \end{tikzcd}$$

So we see that the cocycle condition follows from (and is in fact equivalent to) the associativity condition $(s'\gamma_1)\gamma_2 = s'(\gamma_1\gamma_2)$.

DEFINITION 8.14. Let $p : S' \to S$ be a morphism of descent.

(1) On a S'-object X', a descent datum relative to p is said to be **effective** if X' (together with its descent datum) comes from a S-object X. (Necessarily unique, since the functor p^* is fully faithful.)

(2) One says that p is a **morphism of effective descent** if every descent datum relative to p on a S'-object X' is effective.

(3) Given a full subcategory of $\mathcal{D}$ of $\mathrm{Desc}(S'/S)$, for example the subcategory $\mathrm{QAff}(S'/S)$ of $\mathrm{Desc}(S'/S)$ consisting of schemes quasi-affine over S', one says that:

p is an **morphism of effective descent for the category $\mathcal{D}$**

if every descent datum relative to p on an object X' of $\mathcal{D}$ is effective. For example, we will see later that "A faithfully flat quasi-compact morphism $S' \to S$ is a morphism of effective descent for $\mathrm{QAff}(S'/S)$".

One has the following important lemma.

LEMMA 8.15. *Consider morphisms $U \xrightarrow{v} T \xrightarrow{u} S$.*

(1) *If $u \circ v$ is a universal effective epimorphism, so is u.*

(2) *If $u \circ v$ is a morphism of effective descent, so is u.*
(3) *If $u : U \to T$ and $v : T \to S$ are universal effective epimorphisms, resp. morphisms of effective descent, so is $v \circ u$.*

Proof. (1) Suppose that $u \circ v$ is a universal effective epimorphism and consider the diagram

$$\begin{tikzcd}
S & T \arrow[l, "u"'] & T \times_S T \arrow[l, shift left] \arrow[l, shift right] \\
U \arrow[u, "u \circ v"] & U \times_S T \arrow[l] \arrow[u] & U \times_S T \times_S T \arrow[l, shift left] \arrow[l, shift right] \arrow[u] \\
U \times_S U \arrow[u, shift left] \arrow[u, shift right] & U \times_S U \times_S T \arrow[l] \arrow[u, shift left] \arrow[u, shift right] & U \times_S U \times_S T \times_S T \arrow[l, shift left] \arrow[l, shift right] \arrow[u, shift left] \arrow[u, shift right]
\end{tikzcd}.$$

Columns 1,2,3 are exact since $u \circ v$ is an universal effective epimorphism. Row 2 is exact, since $U \times_S T \to U$ is an effective epimorphism (as it has a section over U, see Lemma 8.3) and so is row 3. Then a diagram-chasing shows that row 1 is exact, hence u is an effective epimorphism. As the hypotheses are stable under any base change $S' \to S$, it follows that u is a universal effective epimorphism.

(2) Now, suppose that $u \circ v$ is a morphism of effective descent. Since a morphism of descent is the same thing as a universal effective epimorphism (by Prop. 8.7), (1) gives already that u is a morphism of descent. Hence, given a T-object Y, we only have to show that any descent datum on Y relative to $u : T \to S$ is effective.

Since $f = u \circ v$ is a morphism of effective descent, there exists a S-object X such that $v^*(Y) \simeq (u \circ v)^*(X)$ as objects of $\mathrm{Desc}(U/S)$. It remains to show that $Y \simeq u^*(X)$ as objects of $\mathrm{Desc}(T/S)$. Since f is a universal effective epimorphism, so is the morphism $f_T : U \times_S T \to T$ obtained by base change. Further, $f_T^*(Y) \simeq f_T^*(X_T)$ as objects of $\mathrm{Desc}(U_T/T)$ (since f_T^* factors through v^*) and it follows that there exists a unique T-isomorphism $Y \simeq u^*(X)$, which respects the descent data. This proves (2).

(3) Suppose that u and v are universal effective epimorphisms and consider the diagram

$$\begin{tikzcd}
S & T \arrow[l, "u"'] & T \times_S T \arrow[l, shift left] \arrow[l, shift right] \\
 & U \arrow[u, "v"] & U \times_S U \arrow[l, shift left] \arrow[l, shift right] \arrow[u, "v \times_S v"] \\
 & U \times_T U \arrow[u, shift left] \arrow[u, shift right] \arrow[ur] &
\end{tikzcd}.$$

By hypothesis, the first line and column are exact, and $v \times_S v$ is an epimorphism by Remark 8.2. The conclusion follows by diagram-chasing: for every object X of $\mathscr{C}$, if an element x_U of $X(U)$ has the same images in $X(U \times_S U)$, it has also the same images in $X(U \times_T U)$, hence it comes from an element x_T of $X(T)$ since the first column is exact. We want to prove that x_T comes from an element of $X(S)$. As the first row is exact, it suffices to see that x_T has the same images in $X(T \times_S T)$ and as $v \times_S v$ is an epimorphism, it suffices to see that x_T has the same images in $X(U \times_S U)$, which is true because these are the images of x_U. This proves the first assertion of (3). Then one sees easily that if u, v are morphisms of effective descent, so is $u \circ v$. $\square$

From now on, we take $\mathscr{C} = (\mathrm{Sch})$. Let us then give another example of morphism of effective descent.

Lemma 8.16. *Let (U_i) be an open cover of a scheme S and let $T = \coprod_i U_i$. Then $\pi : T \to S$ is a morphism of effective descent.*[9]

Proof. Note that $T \times_S T = \coprod_{i,j} U_i \times_S U_j \simeq \coprod_{i,j} U_i \cap U_j$, and the first (resp. second) projection $T \times_S T$ correspond to the inclusion of each $U_i \cap U_j$ into U_i (resp. U_j).

Let Y be a T-scheme endowed with a descent datum relative to $T \to U$. Then the $Y_i = Y \times_T U_i$ are open subschemes of Y, and the descent datum consists of isomorphisms on the intersections, which satisfy the cocyle condition. Hence the Y_i glue together to give a scheme X over S, whose pullback to T is Y. $\square$

[9] It is even a *universal* morphism of effective descent, since for any $S' \to S$, the $U_i' = U_i \times_S S'$ form an open cover of S'.

REMARK 8.17. π is not quasi-compact if there exists a point of S contained in infinitely many U_i's. For example, if k is an algebraically closed field, $S = \mathbb{A}^1_k$ and we take the covering by the $S - \{\lambda\}$, for λ running through the set of closed points of S.

Now, to illustrate the previous basic results, let us sketch the proof of the following theorem, which complements theorem 8.4 (and is in fact used in the proof of the latter).

THEOREM 8.18. *Let $p : S' \to S$ be a faithfully flat, quasi-compact morphism. Let $X' \to S'$ be quasi-affine. Then every descent datum on X' relative to $S' \to S$ is effective.*

PROOF. Firstly, assume the theorem proved when S is affine. For arbitrary S, let (U_i) be a covering of S by affine open subschemes. Consider the following cartesian diagram:

$$\begin{tikzcd} T' = \coprod_i U'_i \arrow[r, "\pi'"] \arrow[d, "p_T"'] & S' \arrow[d, "p"] \\ T = \coprod_i U_i \arrow[r, "\pi"] & S. \end{tikzcd}$$

We have seen (Lemma 8.16) that π is a morphism of effective descent, and by assumption p_T is a morphism of descent, the descent being effective in $\mathrm{QAff}(S'/S)$. Hence, using point (3) and then points (1,2) of Lemma 8.15, we obtain that the same is true for $\pi \circ p_T$ and then for p.

So, it suffices to prove the theorem when S is affine. Then, since p is quasi-compact, S' is covered by a finite number of affine open subsets, their sum is an affine scheme S_1 and the induced morphism $S_1 \to S$ is flat and surjective, hence faithfully flat. As $S_1 \to S$ factors through p, it follows from points (1,2) of Lemma 8.15 that it suffices to prove the theorem when both S and S' are affine.

So, consider a faithfully flat map of rings $A \to A'$ and set $A'' = A' \otimes_A A'$. For any A'-module M', denote by $p_1^*(M')$ the A''-module $A'' \otimes_{A'} M'$, where A'' is regarded as A'-algebra via $a' \mapsto a' \otimes 1$, and define $p_2^*(M')$ similarly. Then, when $X' = \operatorname{Spec} R'$ for some A'-algebra R', the theorem follows from the following proposition, applied to $M' = R'$.

PROPOSITION 8.19. *In the category of A'-modules, every descent datum relative to $A \to A'$ is effective. That is, if M' is a A'-module endowed with an isomorphism of A''-modules $\varphi : p_1^*(M') \xrightarrow{\sim} p_2^*(M')$ satisfying the cocycle condition, then $M = \{x \in M' \mid \varphi(1 \otimes x) = 1 \otimes x\}$ is a A-submodule of M' such that $A' \otimes_A M = M'$.*

For the proof, we refer to [**SGA1**], VIII, 1.4–1.6. For the extension to the case where X' is only quasi-affine over S', we refer to [**SGA1**], VIII, Cor. 7.9 or [**BLR**], §6, Th. 6.1. □

Let us give more criteria for effective descent, that will be used in the sequel. We start with the following:

DEFINITION 8.20. Let $R \underset{q_2}{\overset{q_1}{\rightrightarrows}} X$ be an equivalence relation on X over S. It induces an equivalence relation on the topological space X: two points x_1, x_2 of X are equivalent if there exists a point z of R such that $q_1(z) = q_2(z)$. One says that a subset V of X is **saturated** if it is stable by this equivalence relation. This amounts to saying that $q_1^{-1}(V) = q_2^{-1}(V)$.

For any subset U of X, one sees that $V = q_2(q_1^{-1}(U))$ is the smallest saturated set containing U (hence $V = q_1(q_2^{-1}(U))$ also); one calls it the *saturation* of U.

LEMMA 8.21. *Let $p : S' \to S$ be faithfully flat and quasi-compact. Let X' be an object of $\mathrm{Desc}(S'/S)$. Assume that X' is covered by saturated open subsets (V'_i) such that the descent datum on each V'_i is effective. Then so is the descent datum on X'.*

PROOF. Recall first that if $g : Y' \to Y$ is a faithfully flat quasi-compact morphism, the topology of Y is the quotient of the one of Y', that is, $g(V)$ is open in Y for every *saturated* open subset V of Y'.

Now, by hypothesis, there exist S-schemes (V_i) such that $V_i' \simeq V_i \times_S S'$ in $\mathrm{Desc}(S'/S)$. For all i, the projection $V_i' \to V_i$ is faithfully flat and quasi-compact (being a pull-back of $S' \to S$) hence has the property recalled above. For all i, j, $V_i' \cap V_j'$ is a saturated open subset of V_i' and V_j', hence its images $V_{ij}(i)$ in V_i and $V_{ij}(j)$ in V_j are open.

Since p^* is full and faithful, the glueing data on the V_i' descend and allow us to glue the V_i by identifying the open subsets $V_{ij}(i)$ and $V_{ij}(j)$. This gives a S-scheme X such that $X \times_S S' \simeq X'$ in $\mathrm{Desc}(S'/S)$ (because the descent data relative to $S' \to S$ coincide on each V_i'). $\square$

LEMMA 8.22. *Let $f : S' \to S$ be faithfully flat and locally of finite presentation, with S affine. Then f induces a faithfully flat morphism of finite presentation $S'' \to S$, with S'' affine.*

PROOF. Let $(S_i')_{i \in I}$ be a covering of S' by affine open subsets; each is of finite presentation over S. The hypothesis imply that f is open, hence the $f(S_i')$ form an open covering of S. As S is affine, hence quasi-compact, there exists a finite subset J of I such that S is covered by the $f(S_j')$, for $j \in J$. Then $S'' = \coprod_{j \in J} S_j'$ is affine, of finite presentation over S, and the morphism $S'' \to S$ is flat and surjective, hence faithfully flat. $\square$

PROPOSITION 8.23. *Let $S' \to S$ be faithfully flat and locally of finite presentation and let X' be a S'-scheme such that the morphism $X' \to S'$ is separated, locally of finite presentation and locally quasi-finite. Then every descent datum on X' relative to $S' \to S$ is effective.*

PROOF. As in the proof of Th. 8.18 we may reduce to the case where S is affine. Then, by the previous lemma we may assume that S' is affine, too.

Assume first that X' is quasi-compact. Then the morphism $X' \to S'$ is separated, of finite presentation and quasi-finite hence, by [**EGA**], IV_3, Th. 8.11.2 (or [**SGA1**], VIII, Th. 6.2 when S' is notherian), $X' \to S'$ is quasi-affine, and hence the descent datum is effective by Th. 8.18.

Now, in general, let U' be an affine open subset of X' and let $V' = q_1(q_2^{-1}(U))$ be its saturation. Recall that $q_1 : X' \times_{S'} S_1'' \to X'$ is obtained by base change from the first projection $p_1 : S'' \to S'$. As p is faithfully flat of finite presentation and affine, so is q_1; in particular q_1 is open and affine, and the same is true for q_2. Therefore, the open subscheme $q_2^{-1}(U')$ of X'' is affine, hence quasi-compact, and therefore V' is open and quasi-compact. By the previous argument, it is quasi-affine over S' hence the descent datum on V' is effective. Finally, as X' is covered by the various saturated open subsets V', the descent datum on X' is effective by Lemma 8.21. $\square$

9. Notes for this Lecture

Lemma 8.3 is proved in Exp. IV, Prop. 1.12. Then Proposition 8.7 is Exp. IV, Prop. 2.3, while Corollary 8.12 is taken in Exp. IV, Def. 2.2 as the definition of "morphism of descent", while it is observed there that this depends only on the isomorphism φ in the descent datum, and not on the cocycle condition. (This same definition is given in many places.)

Assertion (1) and the first part of assertion (3) of Lemma 8.15 are proved in Exp. IV, Prop. 1.8 and Lemma 1.7 respectively. Part (2) is mentioned in [**SGA1**], Exp. VIII, proof of Th. 1.1 (top of p. 155), refering to [**Gir64**], but the lecturer has been unable to locate this statement in *loc. cit.*

Lemma 8.16 occurs, for example, in [**SGA1**], VIII, first paragraph of the proof of Th. 1.1, while Lemma 8.21 is [**SGA1**], VIII, Prop. 7.2. Then Lemma 8.22 is contained in [$\mathbf{SGA3}_1$], IV, Prop. 6.3.1 (iv), and Prop. 8.23 is [$\mathbf{SGA3}_2$], X, Lemme 5.4.