LECTURE 3

Duality between twisted constant and MT-groups. Exactness of the functor D

6. More on the duality functor D: reflexive groups

In this section we fix a base scheme S. As in SGA3 VIII §1, we denote by I the commutative group scheme $\mathbb{G}_{m,S}$. Let G be a group scheme over S, and assume that the functor $D(G) = \underline{\mathrm{Hom}}_{S\text{-Gr}}(G, I)$ is representable.[1] For every S-scheme T, one has:

$$\mathrm{Hom}_{S\text{-Sch}}(T, D(G)) = D(G)(T) = \mathrm{Hom}_{T\text{-Gr}}(G_T, I_T) = \mathrm{Hom}_{S\text{-Gr}}(G \times_S T, I).$$

This is the subset of morphisms of S-schemes $G \times_S T \to I$ which are "*multiplicative with respect to the first argument*". If $T = G'$ is another S-group scheme, we may consider the subset $\mathrm{Hom}_{S\text{-Gr}}(G', D(G))$; it is the subset of morphims of S-schemes $G \times_S G' \to I$ which are *bimultiplicative*, that is, multiplicative with respect to both arguments. As here G and G' play symmetric roles, we obtain the first assertion of the following proposition:

Proposition 6.1. *Let G, G' be S-group schemes, and assume that $D(G)$ is representable. Then one has the first equality below, and also the second if $D(G')$ is representable:*

$$\mathrm{Hom}_{S\text{-Gr}}(G', D(G)) = D(G')(G) = \mathrm{Hom}_{S\text{-Gr}}(G, D(G')). \tag{6.1}$$

This is compatible with any base change $T \to S$, i.e. if $f : G' \to D(G)$ is a morphism of S-group schemes corresponding to $g : G \to D(G')$, then the morphism $f_T : G'_T \to D(G)_T = D(G_T)$ corresponds to $g_T : G_T \to D(G')_T = D(G'_T)$.

The second assertion follows since f and g correspond to a given bimultiplicative morphism $\phi : G \times_S G' \to I$; by base change it defines a bimultiplicative map $\phi_T : (G \times_S G')_T = G_T \times_T G'_T \to I_T$ which gives rise to f_T on the one hand and to g_T on the other hand.

Definition 6.2. Let G be an S-group scheme. We say that G is **reflexive** if $D(G)$ is representable and the canonical morphism $G \to D(D(G))$ is an isomorphism.[2] Note that this implies that G is commutative.

In this case, for any S-group scheme G' such that $D(G')$ is representable, (6.1) gives:

$$\mathrm{Hom}_{S\text{-Gr}}(G', G) = \mathrm{Hom}_{S\text{-Gr}}(D(G), D(G')). \tag{6.2}$$

Corollary 6.3. *The functor D induces an anti-equivalence of categories from the category of reflexive S-group schemes to itself.*

In view of this corollary, we see that Theorem 4.6 follows from Propositions 4.4 and4.5.

We will prove in the next lecture assertion (1) of both propositions. We take this for granted for the moment and we prove assertions (2) and (3), firstly in the case of 4.4.

[0] version of August 16, 2023, after the 2nd lecture.

[1] This is not really needed, see SGA3 VIII §1.

[2] This is more restrictive than Exp. VIII, Def. 1.0.1, which does not require that $D(G)$ be representable, but this suffices for our purposes.

Proof. Let us prove assertion (2). Since $H = D(E)$, Proposition 6.1 gives us a S-morphism $u : E \to D(H)$. The assertion that u is an S-isomorphism is local on the base so we may assume that S is affine and that there exists a surjective flat morphism $S' \to S$, with S' affine, such that $E_{S'} \simeq M_{S'}$. Then $H_{S'} = D(E)_{S'} = D(E_{S'}) \simeq D(M_{S'})$.

Further, the morphism $u_{S'} : E_{S'} \to D(H)_{S'} = D(H_{S'})$ obtained by base-change corresponds to the bimultiplicative map

$$E_{S'} \times_{S'} H_{S'} \simeq M_{S'} \times_{S'} D(M_{S'}) \to \mathbb{G}_{m,S'},$$

hence $u_{S'}$ is an isomorphism since $M_{S'}$ is reflexive. Then one can invoke again [**EGA**] IV$_2$, Prop. 2.7.1, which says that u is an isomorphim.[3] This proves assertion (2).

Now, over any $S' \to S$, if $E_{S'}$ is constant then $D(E)_{S'} = D(E_{S'})$ is diagonalisable, and the converse is true by the biduality theorem 3.4. This proves assertion (3). The proof is completely similar in the case of 4.5. □

7. Exactness of the functor D

In this section, we fix a base scheme S. Before we can speak of kernels and quotients in Proposition 7.4, we need to introduce some definitions. Consider a morphism of S-group schemes $\phi : G \to Y$.

Definition 7.1. Its **kernel** $K = \operatorname{Ker}\phi$ is the S-group scheme defined as the fiber product:

$$\begin{tikzcd} K \arrow[r, dotted] \arrow[d, dotted] & G \arrow[d, "\phi"] \\ S \arrow[r, "e"] & Y \end{tikzcd}$$

where $e : S \to Y$ denotes the unit section. For any S-scheme T, one has $K(T) = \operatorname{Ker}\phi(T)$, which is a normal subgroup of $G(T)$; thus K is a *normal* subgroup scheme of G.[4] If e is a closed immersion[5], so is $K \to G$.

Definition 7.2. Note that the morphism $G \times_Y G \to G \times_S G$ is an *immersion*[6], see [**EGA**], I 5.3.10 together with the correction III$_2$, Err$_{10}$ to I 5.3.9. The multiplication of G induces a morphism of S-schemes $G \times_S K \to G \times_S G$, given on arbitrary T-points by $(g, k) \mapsto (g, gk)$, and this morphism induces an isomorphism of S-schemes:

$$G \times_S K \to G \times_Y G. \tag{7.1}$$

Set $R_\phi = G \times_Y G$ and denote by p_1, p_2 the two projections from R_ϕ to G.

Given a S-scheme Z, we say that a morphism of S-schemes $u : G \to Z$ is *K-invariant* if $p_1^*(u) = p_2^*(u)$; this is equivalent to saying that for an arbitrary S-scheme T and any $x \in G(T)$ and $h \in K(T)$, one has $u(T)(xh) = u(T)(x)$.

Next, we say that the morphism $\phi : G \to Y$ "is **the quotient scheme** G/K" if the map ϕ^* induces, for every S-scheme Z, a bijection:

$$\operatorname{Hom}_{S\text{-Sch}}(Y, Z) \xrightarrow[\sim]{\phi^*} \{K\text{-invariant } S\text{-morphisms } X \to Z\}. \tag{7.2}$$

[3] In *loc. cit.* this is buried as one among 17 cases of fpqc descent of properties of morphisms, but in fact this follows from first principles of descent theory for fpqc morphisms ([**SGA1**], VIII, Cor. 5.3 combined with [**SGA3**$_1$], IV, Prop. 2.4 a)), as we shall explain in another lecture.

[4] This can be expressed in terms of morphisms by saying that the morphism $G \times_S K \to G$ "given on arbitrary T-points by $(g, k) \mapsto gkg^{-1}$" factors through K.

[5] This is the case if Y is separated (in particular, affine) over S, because e is the pull-back of the diagonal map $G \to G \times_S G$ by the map $\mathrm{id}_G \times e$.

[6] That is, a closed immersion followed by an open immersion.

Using the language of the next lecture, (7.2) means that $\phi : G \to Y$ is an **effective epimorphism**.

REMARK 7.3. In the classical theory, if k is an algebraically closed field and $H \subset G$ are algebraic groups over k (that is, reduced k-group schemes of finite type), one shows that the set $G(k)/H(k)$ is the set of k-points of some algebraic variety W, namely the orbit $G(k) \cdot [v]$ in $\mathbb{P}(V)$, where V is a representation of G and $[v]$ is a line for which the isotropy group is $H(k)$ *and the isotropy Lie algebra* inside $\mathrm{Lie}(G)$ is $\mathrm{Lie}(H)$ (which ensures that H is actually the schematic stabiliser of $[v]$), and one defines G/H as W. Then it is proved, somehow as a side result, that W satisfies the universal property (7.2) above, with K replaced by H.

Over a general base scheme S, the situation is completely different: quotients G/H *do not always exist*! For this reason, one has to characterize the quotients G/H (if they exist) by the universal property (7.2) above (with K replaced by H), and then look for some S-scheme Y satisfying this property. By Yoneda lemma, such a Y is unique up to *unique isomorphism*; due to this strong uniqueness property, Y may be constructed *locally*, that is, if we find an open cover (U_i) of S and a scheme Y_i over U_i, then the Y_i glue to a scheme Y over S which is the sought-for quotient G/H. More on this in the next lecture.

In the rest of this section, given an abelian group M we write $D_S(M)$ instead of $D(M)_S$.

PROPOSITION 7.4. *Let* $0 \longrightarrow P \xrightarrow{u} M \xrightarrow{v} N \longrightarrow 0$ *be an exact sequence of abelian groups. Set* $G = D_S(M)$ *and* $Y = D_S(P)$. *Then:*

(1) $D_S(v)$ *is an isomorphism from* $K = D_S(N)$ *to* $\mathrm{Ker}\, D_S(u)$ *and is a closed immersion.*
(2) $D_S(u) : G \to Y$ *is affine and faithfully flat.*
(3) Y *is the quotient* G/K.
(4) The formation of this quotient commutes with base change, *i.e. for any* S*-scheme* T, Y_T *is the quotient* G_T/K_T.

PROOF. (1) Let us prove that the morphism $D_S(v) : D_S(N) \to K$ is an isomorphism. It suffices to prove that, for any S-scheme T, the map $D_S(N)(T) \to K(T)$ is bijective. But $K(T)$ is the set of group morphisms $f : M \to \mathcal{O}_T(T)^\times$ such that $f \circ u$ is the trivial morphism, which is the same as $\mathrm{Hom}_{\mathrm{grp}}(M/P, \mathcal{O}_T(T)^\times) = D_S(N)(T)$. This proves the first assertion. Further, the map $\mathcal{O}_S[M] \to \mathcal{O}_S[N]$ is surjective, hence $D_S(v)$ is a closed immersion.[7]

(2) Let (U_i) be a covering of S by affine open subschemes $U_i = \mathrm{Spec}\, A_i$. Then $Y = D_S(P)$ is covered by the affine open subschemes $Y_{U_i} = \mathrm{Spec}\, A_i[P]$ and $G_{U_i} = \mathrm{Spec}\, A_i[M]$ is affine over Y_{U_i}. Further, denoting by $\tau : N \to M$ a set-theoretic section of the projection $M \to N$, one sees that $A_i[M]$ is free over $A_i[P]$ with basis $(\tau(n))_{n \in N}$. It follows that G is affine and faithfully flat.

Assertion (3) follows since any faithfully flat quasi-compact morphism is an **effective epimorphism** (see Def. 7.2), as we shall see in the next lecture.

As for (4), we could invoke the general fact that a faithfully flat quasi-compact morphism remains so after base change. But here (4) follows directly since $G_T = D_T(M)$ and $K_T = D_T(N)$, hence by (3) applied to T instead of S one has $G_T/K_T = D_T(P) = D_S(P) \times_S T = Y_T$. ☐

PROPOSITION 7.5. *Let* M, N *be abelian groups. Set* $E = \mathrm{Hom}_{\mathrm{grp}}(M, N)$.

(1) *There is a natural monomorphism* $E_S \to \underline{\mathrm{Hom}}_{S\text{-}\mathrm{Gr}}(M_S, N_S)$.
(2) *If* M *is finitely generated, this monomorphism is an isomorphism.*

PROOF. Set $F = \underline{\mathrm{Hom}}_{S\text{-}\mathrm{Gr}}(M_S, N_S)$. Let T be a S-scheme. Then $E_S(T) = \mathrm{Loc}(T, E)$ identifies with the set of maps $f : M \times T \to N$ which are additive in the first variable and "uniformly locally constant" in the second variable, i.e. each $t \in T$ admits a neighbourhood U such that $f(m, t) = f(m, u)$ for all $u \in U$ and $m \in M$, whereas $F(T)$ is the larger set of all maps $g : M \times T \to N$ which are additive in the first variable and such that for each $m \in M$ and $t \in T$, there exists a neighbourhood U_m of t such that the function $g_m : u \mapsto f(m, u)$ is constant on U_m.

[7] One could also say that $D_S(M) \to S$ is affine, hence separated, and apply a general result.

Note that, by additivity, $g_{m'}$ is constant on U_m for all m' in the subgroup generated by m. Therefore, if M is generated by elements $m_1, \dots, m_r$, then all g_m are constant on the open neighbourhood $\bigcap_{i=1}^r U_{m_i}$ of t. This proves that g belongs to the subset $\mathrm{Loc}(T, E)$.[8] $\square$

From now on, we assume again that all abelian groups (resp. S-group schemes of multiplicative type) under consideration are **finitely generated**. Recall that a S-group scheme G is called *locally diagonalisable* if each $s \in S$ admits an open neighbourhood U such that $G_U \simeq D(M)_U$ for some abelian group M (uniquely defined by s since $M = \mathrm{Loc}(\mathrm{Spec}\,\mathcal{O}_{S,s}, M_U)$). Thus, one obtains a partition of S into open and closed subsets over which G is diagonalisable.

PROPOSITION 7.6. *Let $u : G \to G'$ be a morphism of locally diagonalisable S-group schemes and let $K = \mathrm{Ker}\, u$. Then:*

(1) *K is locally diagonalisable and $K \to G$ is a closed immersion.*
(2) *The quotient $Y = G/K$ exists and is a locally diagonalizable S-group.*
(3) *One has $u = i \circ p$, where $p : G \to Y$ is affine and faithfully flat, and i is a closed immersion.*
(4) *Setting $H = i(Y)$, the quotient G'/H exists, is locally diagonalisable, and is a cokernel of u.*

Therefore, the category of locally diagonalisable S-groups is abelian.

PROOF. Since all assertions are local on S, we may assume that $G = D_S(M)$ and $G' = D_S(M')$, for some finitely generated abelian groups M, M'. Then, by the biduality theorem 3.4 and Cor. 6.3, combined with Prop. 7.5, we have

$$\mathrm{Hom}_{S\text{-Gr}}(G, G') = \mathrm{Hom}_{S\text{-Gr}}(M'_S, M_S) = \mathrm{Loc}(S, \mathrm{Hom}_{\mathrm{grp}}(M', M)).$$

Then, again, S is partitioned into open and closed subsets over which u comes from a morphism of groups $f : M' \to M$.[9] Setting $P' = \mathrm{Ker}\, f$ and $N' = P = f(M')$ and $N = \mathrm{Coker}\, f$, we have exact sequences

$$0 \longrightarrow P' \longrightarrow M' \longrightarrow N' \longrightarrow 0,$$

$$0 \longrightarrow P \longrightarrow M \longrightarrow N \longrightarrow 0.$$

Then the result follows from Prop. 7.4, with $K = D_S(N)$, $Y = D_S(P) = D_S(N') = H$ and $G'/H = D_S(P')$. $\square$

REMARK 7.7. The enlargement, when S is not connected, of the category of diagonalisable S-groups to that of *locally* diagonalisable S-groups was necessary in order to obtain an abelian category. For example, if k_1, k_2 are fields, $S_i = \mathrm{Spec}\, k_i$ and S is the sum of S_1 and S_2, one may consider the morphism $f : \mathbb{G}_{m,S} \to \mathbb{G}_{m,S}$ which is the identity on S_1 and the trivial morphism on S_2, then $\mathrm{Ker}\, f$ is the locally diagonalisable group which is the trivial group over S_1 and $\mathbb{G}_m$ over S_2. It is not diagonalisable.

Using the same technique of faithfully flat descent as the one needed for the proof of assertion (1) of propositions 4.4 and 4.5 (see the next lecture), one can extend the previous proposition to the case of S-groups of multiplicative type. For this, we need to adopt the definition of [**SGA3**$_2$], X, Def. 1.1:

DEFINITION 7.8. A group scheme H over S is said to be **of multiplicative type** if for each $s \in S$ there exists an affine open neighbourhood U of s, a surjective flat morphism $U' \to U$, with U' affine, and a (finitely generated) abelian group M such that $H \times_S U' \simeq D(M)_{U'}$. Further, one says that H is :

- **quasi-isotrivial** if one may choose the maps $U' \to U$ to be étale;

[8] This explanation, nicer than the one in N.D.E. (3) of [**SGA3**$_2$], Exp. VIII, was given orally by Joseph Oesterlé in his lectures [**Oes14**].

[9] This reduction was omitted in [**SGA3**$_2$], VIII, Cor. 3.4.

- **isotrivial** if there exists a surjective **finite** étale map $S' \to S$ such that $H \times_S S' \simeq D(M)_{S'}$.
- locally isotrivial (resp. locally trivial) if each $s \in S$ admits an affine open neighbourhood U such that $H_U = H \times_S U$ is isotrivial (resp. diagonalisable).

REMARK 7.9. Then there is a partition of S into open and closed subschemes over which the type of H is constant. So, in most results we can restrict ourselves to groups of multiplicative type *of a given type* M, and the more general definition only brings complications in the statements or hypotheses. However when the base S is not connected this generality is needed to ensure that the category groups of multiplicative type has kernels (and is in fact abelian).

PROPOSITION 7.10. *Let $u : G \to G'$ be a morphism of S-group schemes of multiplicative type and let $K = \operatorname{Ker} u$. Then:*

(1) *K is of multiplicative type and $K \to G$ is a closed immersion.*
(2) *The quotient $Y = G/K$ exists and is of multiplicative type.*
(3) *One has $u = i \circ p$, where $p : G \to Y$ is affine and faithfully flat, and i is a closed immersion.*
(4) *Setting $H = i(Y)$, the quotient G'/H exists, is of multiplicative type, and is a cokernel of u.*

Therefore, the category of S-groups of multiplicative type is abelian.

Notes for this Lecture

The exactness of the functor D and the fact that the category is abelian are proved in Exp. VIII, Th. 3.1 and Prop. 3.4 in the (locally) diagonalisable case, and in Exp. IX, Prop. 2.7 in the general case; see also [**Oes14**], 5.3 and 6.5.