LECTURE 2

Constant and twisted constant groups, biduality

3. Constant groups and biduality. Character groups

In this section we fix a base scheme S. For the sake of brevity, we will sometimes write D-group (resp. MT-group) over S instead of diagonalisable group over S (resp. S-group of multiplicative type).

Definition 3.1. To every non-empty set M one associates the S-scheme M_S which is the direct sum of a family $(S_m)_{m\in M}$ of copies of S indexed by M. It is étale over S, and is finite over S if and only M is finite. Such a scheme is called a **constant scheme** over S.

The sections $\Gamma(M_S/S)$ of the projection $M_S \to S$ are the locally constant functions from the topological space S to M, denoted by $\mathrm{Loc}(S, M)$. For any S-scheme S', one has $M_S \times_S S' \simeq M_{S'}$ hence:

$$\mathrm{Hom}_{S\text{-Sch}}(S', M_S) = \mathrm{Hom}_{S'\text{-Sch}}(S', M_{S'}) = \mathrm{Loc}(S', M). \tag{3.1}$$

Thus, M_S represents the functor which associates $\mathrm{Loc}(S', M)$ to every S-scheme S'.

On the other hand, to give a morphism from M_S to a S-scheme H is the same as giving, for each $m \in M$, a morphism of S-schemes $S \to H$, i.e. an element of $H(S)$; thus one has:

$$\mathrm{Hom}_{S\text{-Sch}}(M_S, H) = \mathrm{Hom}_{\mathrm{Sets}}(M, H(S)). \tag{3.2}$$

If $u : M \to N$ is a map of sets, it induces a morphism of S-schemes $u_S : M_S \to N_S$. One therefore obtains a functor $M \mapsto M_S$ from the category of non-empty sets to that of S-schemes. It commutes with products, i.e. one has

$$M_S \times_S N_S \simeq (M \times N)_S.$$

Thus, if M is a group one obtains that M_S is a group scheme, called a **constant group scheme** over S. If $u : M \to N$ is a morphism of groups, then $u_S : M_S \to N_S$ is a morphism of S-group schemes. Thus, $M \to M_S$ is a functor from the category of groups to that of S-group schemes. Further, as in (3.2), for every S-group scheme H one has:

$$\mathrm{Hom}_{S\text{-Gr}}(M_S, H) = \mathrm{Hom}_{\mathrm{grps}}(M, H(S)). \tag{3.3}$$

Definition 3.2. Let G, H be S-group schemes. **The functor** $\boxed{\underline{\mathrm{Hom}}_{S\text{-Gr}}(G, H)}$ is defined as follows: its value on a S-scheme T is the set $\mathrm{Hom}_{T\text{-Gr}}(G_T, H_T)$ of morphisms of T-group schemes from G_T to H_T. This is a *contravariant* functor from the category of S-group schemes to the category of sets. This functor is clearly "compatible with base change", that is, for any S-scheme T one has:

$$\underline{\mathrm{Hom}}_{S\text{-Gr}}(G, H) \times_S T = \underline{\mathrm{Hom}}_{T\text{-Gr}}(G_T, H_T) \tag{3.4}$$

as both sides send any T-scheme U to $\mathrm{Hom}_{U\text{-Gr}}(G_U, H_U)$.

Further, if the group law of H is commutative, it induces on $\underline{\mathrm{Hom}}_{S\text{-Gr}}(G, H)$ the structure of a **commutative group functor**. In particular, the group functor $\underline{\mathrm{Hom}}_{S\text{-Gr}}(G, \mathbb{G}_{m,S})$ is denoted by $D(G)$. For future use, let us record (3.4) in this case as:

$$D(G)_T = D(G_T) \tag{3.5}$$

[0]version of August 16, 2023, after the lecture.

In the rest of this section, M denotes a finitely generated abelian group.

Remark 3.3. Suppose that $G = M_S$. Then, for every S-scheme T one has, by the previous definition and (1.2):

$$D(M_S)(T) = \mathrm{Hom}_{T\text{-Gr}}(M_T, \mathbb{G}_{m,T}) = \mathrm{Hom}_{\mathrm{grps}}(M, \mathcal{O}_T(T)^\times) = D(M)_S(T).$$

Thus the functor $D(M_S)$ is represented by the diagonalisable S-group scheme $D(M)_S$.

Next, let $G = D(M)_S$. For every S-scheme T, both $G_T = D(M)_T$ and $\mathbb{G}_{m,T}$ are affine over T and correspond to the quasi-coherent $\mathcal{O}_T$-Hopf algebras $\mathcal{O}_T[M]$ and $\mathcal{O}_T[X, X^{-1}]$ (the comultiplication of the latter being $\Delta(X) = X \otimes X$). Hence, one has:

$$D(D(M)_S)(T) = \mathrm{Hom}_{T\text{-Gr}}(D(M)_T, \mathbb{G}_{m,T}) = \mathrm{Hom}_{\mathcal{O}_T\text{-Hopf}}(\mathcal{O}_T[X, X^{-1}], \mathcal{O}_T[M]).$$

Note that any locally constant function $\phi : T \to M$ defines a partition of S into the open and closed subschemes S_m on which ϕ takes the value m, and on S_m this defines the Hopf algebra morphism given by $X \mapsto m$. This defines a monomorphism $M_S \to D(D(M)_S)$. Further, one has the following **biduality theorem**:

Theorem 3.4. *The natural morphism $M_S \to D(D(M)_S)$ is an isomorphism.*

Proof. We must prove that every morphim of $\mathcal{O}_T$-Hopf algebras $\psi : \mathcal{O}_T[X, X^{-1}] \to \mathcal{O}_T[M]$ is obtained as above. But $\mathcal{O}_T$ has a natural structure of $\mathcal{O}_T[X, X^{-1}]$-comodule, given by $\mu_{\mathbb{G}_m}(f) = f \otimes X$ for any local section of $\mathcal{O}_T$ (this corresponds to the natural action of $\mathcal{O}_{T'}(T')^\times$ on $\mathcal{O}_{T'}(T')$ for any T-scheme T').

Therefore, ψ makes $\mathcal{F} = \mathcal{O}_T$ into an $\mathcal{O}_T[M]$-comodule, with coaction μ_G given by

$$(*) \qquad \mu_G(f) = f \otimes \psi(X).$$

for any local section of $\mathcal{O}_T$. Hence, by Proposition 2.3, for each $t \in T$, the local ring $\mathcal{O}_{T,t}$ is the direct sum of the stalks $\mathcal{F}_{m,t}$, for $m \in M$, which are therefore projective $\mathcal{O}_{T,t}$-modules of rank 0 or 1. It follows that for each $t \in T$ there exists a unique $m \in M$ such that $\mathcal{F}_m \neq 0$ on some open neighbourhood of t, and one has $\mathcal{F}_m = \mathcal{O}_T$ on this neighbourhood.

This gives a partition of T into open and closed subschemes T_m, hence a locally constant function $\phi : T \to M$. Further, over each T_m one has $\mathcal{O}_T = \mathcal{F}_m$ hence for any local section f of $\mathcal{O}_{T_m}$ one has $\mu_G(f) = f \otimes m$.

Comparing with $(*)$ above, we see that on T_m the Hopf algebra morphism ψ is given by $X \mapsto m$, hence ψ is the Hopf algebra morphism corresponding to the locally constant function ϕ. This proves the theorem. □

Definition 3.5. The constant group scheme M_S is called the *character group* of the diagonalisable group $D(M)_S$.

4. Twisted constant groups. Anti-equivalence with groups of multiplicative type

Definition 4.1. A group scheme E over S is said to be a **twisted constant group** of type M if for each $s \in S$ there exists an affine open neighbourhood U of s and a surjective flat morphism $U' \to U$, with U' affine, such that $E \times_S U' \simeq M_{U'}$. Further, one says that E is :

- **quasi-isotrivial** if one may choose the maps $U' \to U$ to be étale;
- **isotrivial** if there exists a surjective **finite** étale map $S' \to S$ such that $E \times_S S' \simeq M_{S'}$.
- locally isotrivial (resp. locally trivial) if each $s \in S$ admits an affine open neighbourhood U such that $E \times_S U$ is isotrivial (resp. constant).

Example 4.2. Let $S' \to S$ be a finite étale Galois covering with Galois group Γ and let $\Gamma \to \mathrm{Aut}(M)$ be a morphism of groups. Then recall (see e.g. [**SGA1**], Exp. V) the following facts:

(1) One has $S' \times_S S' \simeq S' \times \Gamma$ (a disjoint sum of copies of S').
(2) For each subgroup G of Γ, there exists a scheme Y, étale over S, which is the quotient S'/G, that is, for every S-scheme Z one has
$$\mathrm{Hom}_S(S'/G, Z) = \mathrm{Hom}_S(S', Z)^G$$
where the right-hand side denotes the G-equivariant S-morphims $f : S' \to Z$, that is, $f \circ \gamma = f$ for all $\gamma \in \Gamma$ (note that the action of G on Z is trivial).[1]
(3) For any S-scheme T, one has $(S'/G) \times_S T \simeq (S' \times_S T)/G$.
(4) In particular, one has $(S'/G) \times_S S' \simeq (S' \times \Gamma)/G \simeq S' \times (\Gamma/G)$.

Now Γ acts on $M_{S'} = \coprod_{m\in M} S'_m$ by sending the m-th copy of S' to the $\gamma(m)$-th copy via the automorphism γ of S'. Denote by M/Γ the set of Γ-orbits in M, choose a representative m in each orbit and denote by Γ_m its stabilizer. Consider the étale S-scheme
$$E = \coprod_{m\in M/\Gamma} S'/\Gamma_m$$
and denote it by $M_{S'}/\Gamma$. As we will see in a later lecture, this is indeed the quotient of $M_{S'}$ by Γ, in the sense that for any S-scheme Z, we have natural identifications
$$\mathrm{Hom}_S(E, Z) = \prod_{m\in M/\Gamma} \mathrm{Hom}_S(S'/\Gamma_m, Z) = \prod_{m\in M/\Gamma} \mathrm{Hom}_S(S', Z)^{\Gamma_m} = \prod_{m\in M} \mathrm{Hom}_S(M_{S'}, Z)^{\Gamma}.$$
Further, applying Fact (4) above to each S'/Γ_m, one obtains that $\boxed{E \times_S S' \simeq M_{S'}.}$

On the other hand, for any S-scheme T, one has natural identifications
$$\begin{aligned}\mathrm{Hom}_S(T, E) = \mathrm{Hom}_S(T \times_S S', E)^{\Gamma} &= \mathrm{Hom}_{S'}(T \times_S S', E_{S'})^{\Gamma} \\ &= \mathrm{Hom}_{S'}(T \times_S S', M_{S'})^{\Gamma} = \mathrm{Loc}(T \times_S S', M)^{\Gamma}.\end{aligned}$$
Therefore, E represents the group functor $T \to \mathrm{Loc}(T \times_S S', M)^{\Gamma}$. Since $E_{S'} \simeq M_{S'}$, it an isotrivial twisted constant group of type M, which splits over S'.

EXAMPLE 4.3. Let $S = \mathrm{Spec}\,\mathbb{R}$ and $S' = \mathrm{Spec}\,\mathbb{C}$, with Galois group $\Gamma = \{\mathrm{id}, \tau\}$ acting on $M = \mathbb{Z}$ by $\tau(n) = -n$. Then τ acts on $\mathbb{Z}_{S'} = \coprod_{n\in\mathbb{Z}}(\mathrm{Spec}\,\mathbb{C})_n$ by swapping $(\mathrm{Spec}\,\mathbb{C})_n$ and $\mathrm{Spec}(\mathbb{C}_{-n})$, the comorphism being $\tau : \mathbb{C} \to \mathbb{C}$. The quotient scheme $E = \mathbb{Z}_{S'}/\Gamma$ is the sum of $\mathrm{Spec}(\mathbb{R})_0$ and a sum of copies of $\mathrm{Spec}\,\mathbb{C}$ indexed by $(\mathbb{Z} - \{0\})/\Gamma$. This is an isotrivial twisted constant group of type $\mathbb{Z}$ over $\mathrm{Spec}\,\mathbb{R}$, which splits over $\mathrm{Spec}\,\mathbb{C}$.

Now, we have the following three results. The complete proofs rely on the powerful technique of faithfully flat descent, to be discussed in another lecture.

PROPOSITION 4.4. *Let E be a twisted constant S-group scheme of type M. Then:*
(1) *$D(E)$ is representable by a S-group scheme H of multiplicative type of type M.*
(2) *One has $E \simeq D(H)$. Thus E is reflexive.*
(3) *E is constant if and only if H is diagonalisable, and E is quasi-isotrivial (resp. isotrivial, resp. locally isotrivial, resp. locally trivial) if and only if H is so.*

PROPOSITION 4.5. *Let H be a S-group scheme of multiplicative type,* quasi-isotrivial *of type M. Then:*
(1) *$D(H)$ is representable by a quasi-isotrival twisted constant group E of type M.*
(2) *One has $H \simeq D(E)$. Thus H is reflexive.*
(3) *H is diagonalisable if and only if E is constant, and is isotrivial (resp. locally isotrivial, resp. locally trivial) if and only if E is so.*

THEOREM 4.6. *Fix a base scheme S.*

[1] If $S = \mathrm{Spec}\,R$ and $S' = \mathrm{Spec}\,R'$, then $S'/G = \mathrm{Spec}\,R'^G$.

(1) *The functors $E \mapsto D(E)$ and $H \mapsto D(H)$ are type-preserving anti-equivalences, quasi-inverse one to another, between the category of twisted constant finitely generated abelian groups E, quasi-isotrivial over S, and the category of finitely presented groups of multiplicative type, quasi-isotrivial over S.*
(2) *These functors induce anti-equivalences, quasi-inverse one to another, between the subcategories of groups which are isotrivial, resp. locally isotrivial, resp. locally trivial.*

We will begin the proof of these results in the next section, and complete it in the next lecture. Before that, let us describe everything explicitly in the Galoisian case: for the rest of this section, $S' \to S$ is a Galois covering with group Γ, and Γ acts by group automorphisms on M.

EXAMPLE 4.7. Let $E = M_{S'}/\Gamma$ be as in example 4.2. For every S-scheme T one has

$$E \times_S T = (M_{S'} \times_S T)/\Gamma = M_{S' \times_S T}/\Gamma,$$

by Fact (3) of 4.2, and hence one has natural identifications

$$D(E)(T) = \mathrm{Hom}_{T\text{-Gr}}(E_T, \mathbb{G}_{m,T}) = \mathrm{Hom}_{T\text{-Gr}}(M_{S' \times_S T}, \mathbb{G}_{m,T})^\Gamma = \mathrm{Hom}_{\mathrm{grp}}\big(M, \mathbb{G}_m(S' \times_S T)\big)^\Gamma.$$

Combined with the discussion in Example 1.5, this shows that $D(E)$ is represented by the S-group of multiplicative type $H = \operatorname{Spec} R'[M]^\Gamma$, assuming for simplicity that $S = \operatorname{Spec} R$ and $S' = \operatorname{Spec} R'$.

EXAMPLE 4.8. Conversely, if $H = \operatorname{Spec} R'[M]^\Gamma$, it follows from the reflexivity part of Prop. 4.4 that $D(H) = M_{S'}/\Gamma$. This can also be seen directly, as follows. Let T be a S-scheme. Firstly, one has

$$H_T \times_S S' = H \times_S T \times_S S' = H_{S'} \times_S T = D(M)_{S'} \times_S T = D(M)_{S' \times_S T}$$

and hence

$$\begin{aligned}\mathrm{Hom}_{T\text{-Gr}}(H_T \times_S S', \mathbb{G}_{m,T}) &= \mathrm{Hom}_{T\text{-Gr}}(D(M)_{S' \times_S T}, \mathbb{G}_{m,T})\\ &= \mathrm{Hom}_{T \times_S S'\text{-Gr}}(D(M)_{S' \times_S T}, \mathbb{G}_{m,T \times_S S'}) = \mathrm{Loc}(T \times_S S', M).\end{aligned}$$

Therefore, one has

$$D(H)(T) = \mathrm{Hom}_{T\text{-Gr}}(H_T, \mathbb{G}_{m,T}) = \mathrm{Hom}_{T\text{-Gr}}(H_T \times_S S', \mathbb{G}_{m,T})^\Gamma = \mathrm{Loc}(T \times_S S', M)^\Gamma$$

and it follows that $D(H)$ is represented by $M_{S'}/\Gamma$.

EXAMPLE 4.9. Now, consider the case where H corresponds to the permutation representation $M = \mathbb{Z}[\Gamma]$. Then, by the discussion in Example 1.5, one has for every S-scheme T:

$$H(T) = \mathrm{Hom}_{\mathrm{grp}}\big(\mathbb{Z}[\Gamma], \mathbb{G}_m(T \times_S S')\big)^\Gamma = \mathrm{Hom}_{\mathrm{grp}}\big(\mathbb{Z}, \mathbb{G}_m(T \times_S S')\big) = \mathbb{G}_m(T \times_S S').$$

Thus, H is the Weil restriction $\mathrm{Res}_S^{S'} \mathbb{G}_{m,S'}$. (This generalizes Example 1.7.)

REMARK 4.10. To answer a question of Prof. Balaji during the lecture, let us give immediately an example of a MT-group over S which is quasi-isotrivial but not locally isotrivial. Let k be a field, algebraically closed if one wants, and let S be the affine curve obtained by identifying the points 0 and 1 of $\mathbb{A}^1_k$, that is, its ring of functions is $R = \mathcal{O}(S) = \{P \in k[t] \mid P(0) = P(1)\}$. As k-algebra, R is generated by the elements $x = t(t-1)$ and $y = t^2(t-1)$, which satisfy the equation $x^3 = y(y-x)$ and one finds that S is the nodal cubic given by this equation.

Consider the auxiliary curve S obtained by glueing two copies of $\mathbb{A}^1_k$ by identifying 0 of each copy with 1 of the other copy. Then $\Gamma = \mathbb{Z}/22\mathbb{Z}$ acts freely on Q and the quotient is S; thus $Q \to S$ is an étale covering.[2] Then the open subscheme U of Q obtained by removing one of the singular points is still étale over S. Let us say that it is copy 0 of $\mathbb{A}^1_k$ with its point 1 glued to point 0 of the copy number 1 of $\mathbb{A}^1_k$. Then we can glue the point 0 of that copy to the point 0 of a copy number 2 of $\mathbb{A}^1_k$, and then the point 1 of that copy to the point 0 of copy number 3, and so on. We can do the same in the negative direction, that is, glue the point 0 of copy 0 to the

[2]For another proof, see [**Tsi14**], Lect. 7, §5.3.

point 1 of copy -1, and so on. In this way, we obtain a curve P (not quasi-compact!) which is étale over S, is endowed with an obvious action of the constant group ZZ, and is in fact a principal $\mathbb{Z}$-bundle over S in the étale topology; that is, $P \times_S P \simeq P \times \mathbb{Z}$.

Using the group morphism $\mathbb{Z} \to \mathrm{GL}(\mathbb{Z}^2)$ given by $n \mapsto \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix}$, ones obtains an action of $\Gamma = \mathbb{Z}$ on the split 2-dimensional torus $D(\mathbb{Z}^2)_P$, extending the action of P. Using that $D(\mathbb{Z}^2)_P$ is affine over P, one can construct the quotient by Γ (more on this in a later lecture) and one obtains a S-group scheme H such that $H_P \simeq D(\mathbb{Z}^2)_P$, hence H is isotrivial. From the principal $\mathbb{Z}$-bundle P we can construct a $(\mathbb{Z}/n\mathbb{Z})$-bundle P over S, for each integer $n > 1$. (Note that P_2 is the previous auxiliary curve Q.) Clearly, the pull-back of H to P_n is *not* trivial because the given action of $\mathbb{Z}$ on $\mathbb{Z}^2$ does not factor trough any quotient $\mathbb{Z}/n\mathbb{Z}$.

Finally, one can prove that P is a universal cover of S, in the sense that any finite Galois covering $S' \to S$ is dominated by some P_n. (This is implicit in [**SGA1**], Exp. I, §11 a) together with [**SGA3**$_2$], Exp. X, 1.6, and a proof can be found, e.g. in [**Tsi14**], Lect. 7, §5.3.) It follows that H is not isotrivial on any neighbourhood of the singular point s. (However one sees that is is trivial on $S - \{s\}$.)

5. Notes for this Lecture

The biduality theorem (Th. 3.5) is proved in Exp. VIII, Th. 1.2 in a greater generality.

Groups of multiplicative type over S are defined in Exp. IX, Def. 1.1. Their duals, the twisted constant groups, are introduced in Exp. X, Def. 5.1.

The example of the previous remark is discussed in Exp. X, 1.6.