LECTURE 18

Survey of SGA3, Exp. XXII to XXV: structural results, automorphims, and a view towards classification

Let S be a base scheme and G a reductive S-group scheme. In the previous lectures we explained that G is étale-locally splittable, that is, each $s \in S$ has an étale neighborhood U such that $G|_U$ has a maximal torus T which is split, i.e. $T = D_U(M)$ for some finite free $\mathbb{Z}$-module M, there exist finite subsets $R \subset M$ and $R^\vee \subset M^\vee = \mathrm{Hom}_{\mathbb{Z}}(M, \mathbb{Z})$ such that $\mathcal{R} = (M, R, M^\vee, R^\vee)$ is a root datum and, setting $\mathfrak{g} = \mathrm{Lie}(G)$ and $\mathfrak{t} = \mathrm{Lie}(T)$, one has $\mathfrak{g} = \mathfrak{t} \oplus \bigoplus_{\alpha \in R} \mathfrak{g}_\alpha$ with $\mathfrak{g}_\alpha \simeq \mathcal{O}_S$ for each α.

Beware that this important result should not hide the fact that the actual classification of reductive, or even semi-simple, groups over an arbitrary field, let alone an arbitrary base S, is very complicated and some questions remain open. However, in order to study arbitrary (i.e. not necessarily split) reductive groups, it is useful to obtain more information about their internal structure, using descent from the split case. In this lecture, we give a summary of these results and refer to the relevant paragraphs of [**SGA3**$_3$] for the proofs.

45. Some structural results

45.1. Recollections about root systems and root data. First, let us recall the following, see [**Hu72**] or [**BLie68**].

Definition 45.1. Let R be a root system in a real vector space V. Let $f \in V^*$ be a linear form such that $f(\alpha) \neq 0$ for all $\alpha \in R$. Then $R^+ = \{\alpha \in R \mid f(\alpha) > 0\}$ is a set of *positive roots*. The set $R^- = -R^+$ is another one.

The set Δ of elements $\alpha \in R^+$ which are *indecomposable* (i.e. cannot be written as $\alpha = \beta + \gamma$ with $\beta, \gamma \in R^+$) is called a *base* of R. The elements of Δ are linearly independent and each $\beta \in R^+$ can be written uniquely $\beta = \sum_{\alpha \in \Delta} n_{\beta,\alpha}\, \alpha$, with $n_{\beta,\alpha} \in \mathbb{N}$.

The Weyl group W acts simply transitively on the set of sets of positive roots (or, equivalently, of bases) of R. The unique element of W swapping R^+ and $-R^+$ is denoted by $w_{0,\Delta}$ or simply w_0.[1]

Next, let us give the following definition ([**SGA3**$_3$], Exp. XXI, Def. 1.1.7 and 6.2.6).

Definition 45.2. Let $\mathcal{R} = (M, R, M^\vee, R^\vee)$ be a root datum. One says that $\mathcal{R}$ is semi-simple if R generates $V = M \otimes_{\mathbb{Z}} \mathbb{R}$, which is equivalent to $R^\vee$ generating the dual vector space $V^* \simeq M^\vee \otimes_{\mathbb{Z}} \mathbb{R}$.

Further, one says that $\mathcal{R}$ is *of adjoint type* if $\mathbb{Z}R = M$ and, at the opposite, that $\mathcal{R}$ is simply-connected (or of simply connected type) if M is a large as possible, that is, if $M^\vee = \mathbb{Z}R^\vee$.

Examples 45.3. For $n \geq 2$, the split semi-simple groups SL_n and PGL_n, say over a field k, are simply connected and adjoint, respectively.

Definition 45.4. If $\mathcal{R} = (M, R, M^\vee, R^\vee)$ is a root datum and Δ is a base of R, we say that $\mathcal{R}$ together with Δ is a *based root datum*.

[0] version of Aug. 27, 2024.

[1] Note that w_0 depends on Δ: for example, if $\Delta = \{\alpha, \beta\}$ is a base of a root system of type A_2 then $w_{0,\Delta} = s_\alpha s_\beta s_\alpha = s_\gamma$, where $\gamma = \alpha + \beta$.

DEFINITION 45.5. Let $\mathcal{R} = (M, R, M^\vee, R^\vee)$ and $\mathcal{R}' = (M', R', M'^\vee, R'^\vee)$ be root data. A *morphism of root data* $h : \mathcal{R}' \to \mathcal{R}$ is a $\mathbb{Z}$-linear map $h : M' \to M$ such that there exists a bijection $b : R \xrightarrow{\sim} R'$ such that $h(b(\alpha)) = \alpha$ and ${}^t h(\alpha^\vee) = b(\alpha)^\vee$ for all $\alpha \in R$.

If $\mathcal{R}$ and $\mathcal{R}'$ are endowed with bases Δ and Δ', one says that h is a morphism of based root data if one further has $b(\Delta) = \Delta'$.

For the rest of this section, let G be a reductive S-group scheme.

45.2. Weyl group. Assume that T is a maximal torus of G. Recall from Lect. 11, Cor. 26.3 that the normaliser $\underline{\mathrm{Norm}}_G(T)$ is represented by a closed subgroup scheme of G, and that $\underline{\mathrm{Cent}}_G(T) = T$ (see the proof of Th. 33.2 in Lect. 13).

From [**SGA3**$_3$], XIX, Th. 2.5 and Prop. 6.3, and XXII, Prop. 3.4, one has the following:

DEFINITION 45.6. As T is a group of multiplicative type, and a normal subgroup of $\underline{\mathrm{Norm}}_G(T)$, the quotient $\underline{\mathrm{Norm}}_G(T)/T$ is represented by a S-group scheme, denoted by $W_G(T)$ and called the Weyl group of (G, T).

PROPOSITION 45.7. (a) *$W_G(T)$ is a finite étale S-group scheme.*

(b) *If (G, T) is split, with root datum $(M, R, M^\vee, R^\vee)$, and if W is the usual Weyl group of R, there is a natural isomorphism $W_S \xrightarrow{\sim} W_G(T)$.*

45.3. Borel and parabolic subgroups. One has the following definition (see [**SGA3**$_3$], Exp. XXII, 5.2.3).

DEFINITION 45.8. Let H be a smooth, closed subgroup of G, with connected fibers. One says that H is a Borel subgroup (resp. a parabolic subgroup) of G is for each $s \in S$, the geometric fiber $H_{\bar{s}}$ is (resp. contains) a Borel subgroup of $G_{\bar{s}}$.

From Exp. XXII, Prop. 5.5.1 and Cor. 5.6.5, one obtains the following:

PROPOSITION 45.9. *Suppose that (G, T) is split and let R^+ be a set of positive roots. There exists a closed subgroup $U^+ = U(R^+)$ of G, characterised by the property that for any total ordering $\{\beta_1, \dots, \beta_N\}$ of R^+, the multiplication map*

$$U_{\beta_1} \times_S \cdots \times_S U_{\beta_N} \to G$$

is an isomorphism of S-schemes between the left-hand side and U^+. Further, U^+ is normalised by T and the multiplication map induces an isomorphism of S-schemes between $T \times_S U^+$ and a Borel subgroup $B^+ = B(R^+)$.

Similarly, for the set $R^- = -R^+$ one obtains a closed subgroup U^- and Borel subgroup $B^- = B(R^-)$.

For later use, let us record here the following proposition and corollary, see Exp. XXII, Prop. 4.1.2, Cor. 4.1.6 and 4.1.7, and Prop. 4.3.5:

PROPOSITION 45.10. *Suppose that (G, T) is split and let R^+ be a set of positive roots. There exists an open subset $\Omega = \Omega_{R^+}$ of G, universally schematically dense relative to S, characterised by the property that for any total ordering of R^+, the multiplication map*

$$\left(\prod_{\beta \in R^+} U_{-\beta}\right) \times_S T \times_S \left(\prod_{\beta \in R^+} U_{-\beta}\right) \to G$$

is an isomorphism of S-schemes between the left-hand side and Ω_{R^+}.

Corollary 45.11. (a) *Suppose that* (G, T) *is split and let* Δ *be the base of* R *corresponding to* R^+. *Then the center* $Z(G)$ *of* G *is represented by the closed, diagonalisable subgroup* $\bigcap_{\beta\in R^+}\mathrm{Ker}(\beta) = \bigcap_{\alpha\in\Delta}\mathrm{Ker}(\alpha)$. *Further, the quotient* $G/Z(G)$ *exists and is a split semi-simple group of* adjoint type.

(b) *In general one obtains by faithfully flat descent that* $Z(G)$ *is a closed subgroup of multiplicative type and that the quotient* $G/Z(G)$ *exists and is a semi-simple group of adjoint type, i.e. all its geometric fibers are semi-simple groups of adjoint type.*

Coming back to Proposition 45.9, it shows, in particular, that Borel subgroups exists étale-locally. Further, by Exp. XXII, Corollaries 5.3.12, 5.3.15, 5.3.5 and 5.8.5, one has:

Proposition 45.12. (a) *For each parabolic subgroup* P *one has* $P = \underline{\mathrm{Norm}}_G(P)$.

(b) *The Borel subgroups and the maximal tori of* G *are conjugate étale locally.*

(c) *So are the pairs* (B, T) *where* T *is a maximal torus of* G *and* B *a Borel subgroup containing* T. *Further, the normaliser of such a pair is* $B \cap \underline{\mathrm{Norm}}_G(T) = T$.

(d) *If* (G, T) *is split and if* B *is a Borel subgroup of* G *containing* T, *then for each* $s \in S$ *there exists a clopen neighborhood* V *of* s *and a set of positive roots* R^+ *such that* $B = B(R^+)$ *over* V.

Remark 45.13. In (d) above, one has $V = S$ if S is connected. Otherwise, one can have the following situation: let S be the disjoint union of two copies S_1, S_2 of $\mathrm{Spec}(\mathbb{C})$ and let (G, T) be the split S-group SL_2 with T the subgroup of diagonal matrices. Then $M = \mathbb{Z}$ and the root system is $\{\pm 2\}$. There are exactly two sets of positive roots: $R^+ = \{2\}$ and $R^- = \{-2\}$. But on the non-connected base S we may consider the Borel subgroup B which is B^+ over S_1 and B^- over S_2. It equals neither $B(R^+)$ nor $B(R^-)$ (but equals the former over S_1 and the latter over S_2). So in this case we see that there are four Borel subgroups containing T.

Remark 45.14. Beware that in $G = \mathrm{SL}_{2,\mathbb{R}}$ the subgroup T of diagonal matrices is a maximal torus, as well as the subgroup K defined for every $S' \to S$ by:

$$K(S') = \left\{ \begin{pmatrix} x & -y \\ y & x \end{pmatrix} \middle| \, x^2 + y^2 = 1 \right\}.$$

Note that T and K are *not* conjugate by an element of $G(\mathbb{R})$, but in $G_{\mathbb{C}}$ they become conjugate by an element of $G_{\mathbb{C}}(\mathbb{C})$.

Finally, one has the following theorem, taken from [**Co14**], Th. 2.3.6 (the fact that G/P is a projective scheme is proved in [**SGA3**$_3$], Exp. XXII, Cor. 5.8.3 and 5.8.5, whereas the explicit description of a S-ample line bundle on G/P is proved by Raynaud in [**SGA3**$_2$], Exp. XVI, proof of Cor. 2.4.)

Theorem 45.15. *Let* G *be a smooth affine* S*-group scheme with connected fibers and* P *a smooth closed subgroup with connected fibers such that* $P = \underline{\mathrm{Norm}}_G(P)$. *Then*

(1) *The quotient* G/P *is represented by a smooth* S*-scheme, which is quasi-projective Zariski-locally over* S.

(2) *For every* $S' \to S$, *each* S*-morphism* $S' \to G/P$ *corresponds to a closed* S'*-subgroup* H *of* $G_{S'}$ *which is conjugate to* $P_{S'}$ *étale-locally on* S'. *Applying this to the identity morphism of* $Y = G/P$, *one obtains a "canonical" closed* Y*-subgroup* $\mathbf{H}$ *of* G_Y.

(3) *If all geometric fibers* $(G/P)_{\overline{s}} = G_{\overline{s}}/P_{\overline{s}}$ *are* projective, *the morphism* $G/P \to S$ *is proper and the line bundle* $\det\big(\mathrm{Lie}(\mathbf{H})\big)^*$ *is* S*-ample. In particular,* $G/P \to S$ *is projective Zariski-locally over* S.

Remarks 45.16. In Exp. XXII, §5 Demazure proves these result in a more general setting. He introduces in 5.1.1 and 5.1.6, respectively, S-group schemes of type (RR) and (RA), satisfying certain conditions with respect to a maximal torus, and observes in 5.1.3 and 5.17 that reductive S-groups, respectively semi-simple groups of adjoint type, are of type (RR) and (RA) respectively.[2] Note that a parabolic subgroup P of a reductive S-group G is also a group of type (RR), which is perhaps the reason for this extra generality. Next, in 5.2.1 Demazure introduces the subgroups of type (R) of a smooth affine S-group G with connected fibers: they are the smooth, closed subgroups H with connected fibers such that each geometric fiber $H_{\overline{s}}$ contains a Cartan subgroup of $G_{\overline{s}}$. He observes in 5.2.2 that in this case one has $H = \underline{\mathrm{Norm}}_G(\mathrm{Lie}(H))^0$, a condition which will play an important role later. Indeed, he proves that if G is of type (RA) (for example semi-simple adjoint) and contains a maximal torus T, then the subgroups of type (R) containing T are determined by their Lie algebra (Cor. 5.3.4 and 5.3.5). He uses this to prove Th. 45.15 above — except the assertion about the S-ample line bundle $\det\big(\mathrm{Lie}(\mathbf{H})\big)^*$ — by realizing G/H as a subscheme of the Grassmannian $\underline{\mathrm{Grass}}_r(\mathfrak{g})$, where $\mathfrak{g} = \mathrm{Lie}(G)$ and r is the relative dimension of H over S.

45.4. The derived subgroup. For the sake of completeness, let us record here the following theorem (Exp. XXII, Th. 6.2.1).

THEOREM 45.17. *Let G be a reductive S-group scheme. For simplicity, assume that all geometric fibers $G_{\overline{s}}$ have the same root datum $\mathcal{R} = (M, R, M^\vee, R^\vee)$.*

(i) $\underline{\mathrm{Hom}}_{S\text{-Gr}}(G, \mathbb{G}_{m,S})$ *is representable by a twisted constant S-group $D_S(G)$, whose type is $\mathbb{Z}^r$ with $r = \operatorname{rank} M - \operatorname{rank} \mathbb{Z}R$. Thus, $D_S(D_S(G))$ is a torus of dimension r.*

(ii) *Let* $\mathrm{der}(G)$ *be the kernel of the biduality homomorphism $f_0 : G \to D_S(D_S(G))$. Then* $\mathrm{der}(G)$ *is a closed subgroup scheme of G and is a semi-simple S-group scheme, called the* derived subgroup *of G. On each geometric fiber,* $\mathrm{der}(G)_{\overline{s}}$ *is the derived subgroup of G in the usual sense. If G is semi-simple, one has* $\mathrm{der}(G) = G$.

IDEA OF PROOF. All assertions are local for the étale topology so by descent we may assume that G is split, with split maximal torus $T \simeq D_S(M)$. Let N be the subgroup of M consisting of those $\lambda \in M$ such that $\langle \lambda, \alpha^\vee \rangle = 0$ for all $\alpha^\vee \in R^\vee$. Then M/N is torsion-free and $T' = D_S(M/N)$ is a subtorus of T. (It is the image of the morphism $\mathbb{G}_{m,S}^R \to T$ given by $(z_\alpha)_{\alpha \in R} \mapsto \prod_{\alpha \in R} \alpha^\vee(z_\alpha)$.) Let R^+ be a set of positive roots and $\Omega = U(R^-) \times_S T \times_S U(R^+)$ the corresponding big cell. Using the schematic density of Ω in G, one constructs a morphism of S-group schemes $f : G \to T/T'$, such that $\mathrm{Ker}(f)$ is a smooth, closed subgroup of G with connected fibers, and such that f induces an isomorphism of S-group schemes

$$D_S(G) \simeq D_S(T/T') = \underline{\mathrm{Hom}}_{S\text{-Gr}}(T/T', \mathbb{G}_{m,S}) \simeq N.$$

This proves (i). Then one obtains that f_0 identifies with f, hence $\mathrm{der}(G)$ is a smooth, closed subgroup of G with connected fibers. Finally, one checks the claim about the geometric fibers of $\mathrm{der}(G)$; it follows that $\mathrm{der}(G)$ is a semi-simple S-group scheme. □

REMARK 45.18. If (G, T) is split, then $D_S(G) = \mathbb{Z}\chi_1 \oplus \cdots \oplus \mathbb{Z}\chi_r$ for some characters $\chi_1, \ldots, \chi_r$ and f_0 is simply the morphism of S-groups $G \to \mathbb{G}_{m,S}^r$, $g \mapsto (\chi_1(g), \ldots, \chi_r(g))$. For example, if $G = \mathrm{GL}_{n,S}$ and $T \simeq D_S(M)$ is the subgroup of diagonal matrices, with $M \simeq \mathbb{Z}^n$, then N is the subgroup $\mathbb{Z} \det$ of M and f_0 is simply the morphism $\det : \mathrm{GL}_{n,S} \to \mathbb{G}_{m,S}$. Its kernel is $\mathrm{SL}_{n,S}$.

46. Pinnings and automorphisms

46.1. Pinnings of a split reductive group scheme. Let G be a split reductive S-group scheme, with maximal torus $T = D_S(M)$ and root datum $\mathcal{R} = (M, R, M^\vee, R^\vee)$.

DEFINITION 46.1. A *pinning*[3] of G is the choice of a base Δ of R and, for each $\alpha \in \Delta$, of a nowhere vanishing section $X_\alpha \in \Gamma(S, \mathfrak{g}_\alpha)^\times$. The latter gives an element $u_\alpha = \exp_\alpha(X_\alpha) \in U_\alpha(S)^\times$.

[2]So, one may think that in (RR) and (RA) the initial R stands for "Roots" whereas the final R (resp. A) stands for "Reductive" and "Adjoint" respectively.

[3]Demazure reports that this terminology (épinglage in French) was introduced to him by Grothendieck, with the image of a butterfly pinned to a frame. The fact that Laurent Schwartz had a large collection of butterflies may have played a role in this.

Recall from §45.1 that $(\mathcal{R}, \Delta)$ is then called a based root datum.

For the sake of simplicity, let us give only the simplified definition below (and refer to Exp. XXII, Def. 4.2.1 and Exp. XXIII, Def. 1.3 for the true definition).

Definition 46.2. Let $(G, T, \mathcal{R}, \Delta)$ and $(G', T', \mathcal{R}', \Delta')$ be two split reductive S-groups with pinnings. A morphism of S-group schemes $f : G \to G'$ is called a morphism of pinned groups if:

(1) there exists a morphism of (h, b) of based root data $(\mathcal{R}', \Delta') \to (\mathcal{R}, \Delta)$. (See Def. 45.5.)

(2) f maps T to T' and the resulting morphism $f_T : T \to T'$ coincides with $D_S(h)$,

(3) for each $\alpha \in \Delta$, one has $f(u_\alpha) = u'_{b(\alpha)}$.

In this case, h is unique and one sets $h = \mathcal{R}(f)$.

Then, one has the following theorem (Exp. XXIII, Th. 4.1). Note that the second assertion follows from the first, because if the root datum is the same we can choose the same base Δ.

Theorem 46.3. *Let $(G, T, \mathcal{R}, \Delta)$ and $(G', T', \mathcal{R}', \Delta')$ be two pinned reductive S-groups. For every morphism h of based root data $(\mathcal{R}', \Delta') \to (\mathcal{R}, \Delta)$, there exists a unique morphism of pinned groups $f : G \to G'$ such that $\mathcal{R}(f) = h$.*

In particular, if G and G' are split reductive groups with the same root datum, they are isomorphic.

The uniqueness statement in the second assertion above is complemented by the following existence result, see [**SGA3**$_3$], Exp. XXV, Cor. 1.2 and the comments given in footnotes (1) and (2) there.

Theorem 46.4. *Let $\mathcal{R}$ be a root datum.*

(i) *There exists a split reductive group $\mathcal{G}(\mathcal{R})$ over $\mathbb{Z}$ with root datum $\mathcal{R}$, which is unique up to isomorphism.*

(ii) *For any base scheme S, any split reductive S-group G of type $\mathcal{R}$ is isomorphic to $\mathcal{G}(\mathcal{R})_S = \mathcal{G}(\mathcal{R}) \times S$.*

46.2. Automorphisms. In order to study the reductive S-group schemes G which become isomorphic to $\mathcal{G}(\mathcal{R})_{S'}$ after an appropriate base change $S' \to S$, we need to study the group functor of automorphisms of G. We start with the following proposition (Exp. XXI, Prop. 6.7.2 and Cor. 7.4.5):

Proposition 46.5. *Let $\mathcal{R}$ be a root datum,* $\mathrm{Aut}(\mathcal{R})$ *its group of automorphisms, Δ a base of the root system, W the Weyl group.*

(i) $\mathrm{Aut}(\mathcal{R})$ *is the semi-direct product of W by $A_\Delta(\mathcal{R}) = \{u \in \mathrm{Aut}(\mathcal{R}) \mid u(\Delta) = \Delta\}$.*

(ii) *If $\mathcal{R}$ is semi-simple and simply-connected or adjoint, then $A_\Delta(\mathcal{R})$ identifies with the group $A(\mathcal{D}) = \mathrm{Aut}(\mathcal{D})$ of automorphisms of the Dynkin diagram $\mathcal{D}$.*[4]

Example 46.6. Consider the root system R of type $A_1 \times A_1$ (which is also D_2). The Dynkin diagram $\mathcal{D}$ has two vertices and no edge, and $\mathrm{Aut}(\mathcal{D}) \simeq \mathbb{Z}/2\mathbb{Z}$. For simplicity, take $S = \mathrm{Spec}(\mathbb{C})$. The semi-simple group over $\mathbb{C}$ of type R which is simply-connected (resp. adjoint) is $\mathrm{SL}_2 \times \mathrm{SL}_2$ (resp. $\mathrm{PGL}_2 \times \mathrm{PGL}_2$) and we see that the non-trivial diagram automorphism τ acts on G by swapping the two factors. However, if $G = \mathrm{SL}_2 \times \mathrm{PGL}_2$, which is neither adjoint nor simply-connected, τ does not lift to an automorphism of G. A similar example exists for R of type D_{2n} for each $n \geq 2$. This explains the above assumption that G be adjoint or simply-connected.

Corollary 46.7. *Let $(G, T, \mathcal{R}, \Delta)$ be a pinned semi-simple S-group of adjoint type or simply-connected. Then the group of automorphisms of G which preserve the pinning identifies with the group* $\mathrm{Aut}(\mathcal{D})$ *of automorphisms of the Dynkin diagram $\mathcal{D}$.*

[4]In general, if $\mathcal{R}$ is semi-simple, $A_\Delta(\mathcal{R})$ identifies with a subgroup of $\mathrm{Aut}(\mathcal{D})$.

One also has the following proposition and corollary, see Exp. XXIV, Lemma 1.5 and Th. 1.3:

PROPOSITION 46.8. *Let $(G, T, \mathcal{R}, (X_\alpha)_{\alpha\in\Delta})$ be a pinned reductive S-group. The adjoint group* $\mathrm{ad}(G) = G/Z(G)$ *acts simply transitively on the set of pinnings of* G.

PROOF. Let E be the given pinning. Let us first show that its stabiliser in G is $Z(G)$. Observe that if $g \in G(S')$ stabilises T and Δ, it stabilises T and the Borel subgroup $B(R^+)$, hence $g = t$ belongs to $T(S')$ since $B = \underline{\mathrm{Norm}}_G(B)$ and $B \cap \underline{\mathrm{Norm}}_G(T) = T$. Further, for each $\alpha \in \Delta$, we have $X_\alpha = \alpha(t)X_\alpha$, so $\alpha(t) = 1$ for all $\alpha \in \Delta$, and hence $t \in Z(G)(S')$.

Then, let $E' = (T', \mathcal{R}', (X'_\beta)_{\beta\in\Delta'})$ be another pinning. We have to prove that, locally for the (fppf) topology, it is of the form $\mathrm{int}(g)(E)$ for some $g \in G(S')$. Since T and T' are conjugate étale-locally, we may assume that $T \simeq T'$ and, shrinking S' if necessary, that this isomorphism comes from an isomorphism $h : \mathcal{R}' \to \mathcal{R}$ of root data. So we can assume that $M' = M$ and $R' = R$. Since the bases of R are conjugate under the Weyl group W, we may further assume that we have $\Delta' = \Delta$. Then, for each $\alpha \in \Delta$, there exists $\alpha \in \mathbb{G}_m(S')$ such that $X'_\alpha = z_\alpha X_\alpha$. Since the morphism $T \to \mathbb{G}^\Delta_{m,S'}$ is faithfully flat (it corresponds to the inclusion $\mathbb{Z}\Delta \subset M$), there exists $S'' \to S'$ faithfully flat and $t \in T(S'')$ such that $\alpha(t) = z_\alpha$ for all $\alpha \in \Delta$. This completes the proof of the proposition. $\square$

COROLLARY 46.9. *Let G be a semi-simple S-group scheme of adjoint type. For simplicity, assume that all geometric fibers $G_{\overline{s}}$ have the same root system R. Let $\mathcal{D}$ be the Dynkin diagram of R.*

(i) *The functor $\underline{\mathrm{Aut}}_{S\text{-}\mathrm{Gr}}(G)$ is representable by a S group scheme and one has an exact sequence of S-group schemes:*

$$1 \longrightarrow G \longrightarrow \underline{\mathrm{Aut}}_{S\text{-}\mathrm{Gr}}(G) \longrightarrow A(G) \longrightarrow 1 \tag{46.1}$$

where $A(G)$ is a twisted constant group of type $A(\mathcal{D})$.

(ii) *If G is split, $A(G)$ is the constant S-group $A(\mathcal{D})_S$ and the exact sequence above splits.*

Let us introduce the following definition, see Exp. XXIV, Def. 1.16

DEFINITION 46.10. Let G be a reductive S-group scheme, say of constant type $\mathcal{R}$ over each geometric fiber of S. Another reductive S-group scheme G' is called a *form* of G over S if it is locally isomorphic to G for the (fppf) topology. By results in Lect. 13, this condition is equivalent to each of the following ones:

(a) G' is locally isomorphic to G for the étale topology.

(b) Each geometric fiber of G' is of type $\mathcal{R}$.

Then, by general results, one obtains the following theorem (see Exp. XXIV, Cor. 1.17). For the sake of simplicity, we restrict to the case of semi-simple adjoint groups.

THEOREM 46.11. *Let S be a scheme, $\mathcal{R}$ a semi-simple root datum of adjoint type, $\mathcal{D}$ its Dynkin diagram, $G = \mathcal{G}(\mathcal{R})_S$ the corresponding split semi-simple group over S. Then:*

(i) *$\underline{\mathrm{Aut}}_{S\text{-}\mathrm{Gr}}(G)$ is the semi-direct product $G \rtimes A(\mathcal{D})$, which is affine over S.*

(ii) *The functor $G' \mapsto \underline{\mathrm{Isom}}_{S\text{-}\mathrm{Gr}}(G, G')$ is an equivalence between the category of forms of G over S and that of principal bundles[5] under $\underline{\mathrm{Aut}}_{S\text{-}\mathrm{Gr}}(G)$ over S. Moreover, any such bundle is locally trivial for the étale topology.*

(iii) *If $S' \to S$ if a covering morphism for the* (fppf) *topology, forms G' of G trivialised by S' correspond to principal bundles trivialised by S', i.e. they are classified by the pointed set $H^1_{\text{ét}}(S'/S, G \rtimes A(\mathcal{D}))$.*

[5]In general, one should consider (fppf) sheaves which are principal homogeneous bundles under $\underline{\mathrm{Aut}}_{S\text{-}\mathrm{Gr}}(G)$, but since the latter is affine over S any such sheaf is in fact representable by a scheme, which is a principal bundle under $\underline{\mathrm{Aut}}_{S\text{-}\mathrm{Gr}}(G)$.