LECTURE 16

Extension of partially defined morphims of group schemes

The goal of the next lecture is to explain the proof of Demazure's crucial theorem 2.1 in Exp. XX. As the proof uses, at some points, Artin's proposition 2.3 from Exp. XVIII, which is interesting in its own right, we start by explaining that result.

42. Schematic density and extension of partially defined morphisms

All the results in this section are taken from [**SGA3**$_2$], Exp. XVIII by M. Artin.

42.1. Schematic density. We start with the following definition, see [**SGA3**$_2$], IX, Def. 4.1 or [**EGA**] IV$_3$, Prop. 11.10.1.

Definition 42.1. Let X be a S-scheme. One says that an open subscheme U is *schematically dense* if for every open subscheme V of X, the following equivalent conditions are satisfied:
- (a) Every section $f \in \mathcal{O}_X(V)$ which vanishes on $V \cap U$ is zero.
- (b) If $V \cap U$ is a subscheme of a closed subscheme Y of V, then $Y = V$.
- (c) For any scheme Y *separated* over S, if $f, g : V \to Y$ are S-morphisms which coincide on $V \cap U$, then $u = v$.

Remark 42.2. This condition is slightly stronger than topological density. For example, let k be a field, $A = k[x, y]$ and $X = \operatorname{Spec}(A/I)$ where $I = (x^2, xy)$. Note that the primary decomposition of I is $I = (x) \cap \mathfrak{m}^2$, where $\mathfrak{m} = (x, y)$, so that $\mathfrak{m}$ is an embedded associated prime of A/I. Then X is irreducible (but not reduced) and the open subset U of X defined by $y \neq 0$ is dense in X, but not schematically dense: it is contained in the closed subscheme $\operatorname{Spec} A/(x)$.

In fact, one has the following proposition, see [**EGA**] IV$_2$, Prop. 3.1.8 (a) $\Leftrightarrow$ (b), or [**StaPr**], Lemma 31.4.5 (Tag 083P).

Proposition 42.3. *Let X be a locally noetherian scheme and U an open subscheme. The following are equivalent:*
- (1) *U is schematically dense in X.*
- (2) *U contains all associated points of X.*
- (3) *U is topologically dense in X and contains all embedded associated points of X.*

We will also need the following stronger definition, see [**SGA3**$_2$], XVIII, Definition before Prop. 1.1 or [**EGA**] IV$_3$, Def. 11.10.8.

Definition 42.4. Let X be a S-scheme. One says that an open subscheme U is *universally schematically dense* in X relative to S if for any base change $S' \to S$, the open subscheme $U \times_S S'$ is schematically dense in $X \times_S S'$.

One has the following easy lemma ([**SGA3**$_2$], XVIII, Prop. 1.1).

Lemma 42.5. *Let X be a S-scheme.*
- (i) *If U, V are universally schematically dense in X relative to S, so is $U \cap V$.*

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(ii) *Let $V \subset W \subset X$ be open imbeddings. Then V is universally schematically dense in X relative to S iff it is so in W and W is so in X.*

(iii) *Suppose U is universally schematically dense in X relative to S and let $T \to S$. Then U_T is universally schematically dense in X_T relative to T, and also relative to S.*

(iv) *Let Y be a S-scheme. If $U \subset X$ and $V \subset Y$ are universally schematically dense relative to S then so is $U \times_S V$ in $X \times_S Y$.*

PROOF. (i) and (ii) are easy, as well as the first part of (iii). The second part follows since, for any $S' \to S$, one has $X_T \times_S S' \simeq X_T \times_T (T \times_S S')$ and similarly for U_T.

Then (iv) follows because, by (iii), $U \times_S Y$ and $X \times_S V$ are universally schematically dense in $X \times_S Y$ relative to S, hence so is their intersection $U \times_S V$, by (i). □

For the sake of completeness, let us also record the following result, see [**EGA**] IV_3, Th. 11.9.10 and §11.10:

THEOREM 42.6. *Let X be a S-scheme, U open in X, and $X' \to X$ a flat morphism locally of finite presentation. Then U is universally schematically dense in X relative to S if and only if $U \times_X X'$ in so in X'.*

Next, one has the following useful criterion, see [**SGA3**$_2$], IX, Cor. 4.6 (repeated in XVIII, Prop. 1.2 and Cor. 1.3), or [**EGA**] IV_3, Prop. 11.10.10:

PROPOSITION 42.7. *Let $X \to S$ be flat and locally of finite presentation and let U be an open subscheme of X.*

(i) *U is universally schematically dense relative to S if and only if, for each $s \in S$, U_s is schematically dense in X_s.*
(ii) *The latter condition is satisfied iff U_s contains all associated points of X_s. In. particular, if X_s has no embedded associated points, for example if X_s is reduced, it suffices that U_s be dense in X_s.*

REMARK 42.8. If G is a group scheme of finite type over a field then, by [**SGA1**], VI_A, Prop. 1.1.1, each local ring of G is Cohen-Macaulay and hence G has no embedded associated points.

In fact, we will mostly use the previous proposition when $X = \mathbb{G}_{a,S}^d$, in which case each fiber is reduced and irreducible, hence any open neighborhood of the unit section of $X \to S$ is universally schematically dense relative to S.

We will need later the following proposition, contained in [**SGA3**$_2$], XVIII, Prop. 1.4:

PROPOSITION 42.9. *Let X be a S-scheme and U an open subscheme of X universally schematically dense relative to S. Let Y be a S-scheme such that each fiber Y_s is separated and let $f, g : X \to Y$ be two S-morphisms. If f, g coincide on U then $f = g$.*

PROOF. The diagonal of $Y \times_S Y$ is a subscheme (not necessarily closed), hence its inverse image $\text{Ker}(f, g)$ by the morphism (f, g) is a subscheme Z of X. Each fiber Z_s equals $\text{Ker}(f_s, g_s)$, which equals X_s because Y_s is separated and U_s is schematically dense in X_s. Hence Z has the some topological space as X, hence is a *closed* subscheme of X. Since $U \subset Z$, it follows that $Z = X$, hence $f = g$. □

Since every group scheme over a field is separated (see [**SGA3**$_1$], VI_A, 0.3, we obtain the:

COROLLARY 42.10. *Let X be a S-scheme and U an open subscheme of X universally schematically dense relative to S. Let H be a S-group scheme and $f, g : X \to H$ be two S-morphisms. If f, g coincide on U then $f = g$.*

42.2. Extension of partially defined morphisms of S-group schemes. The following results are contained in [**SGA3**$_2$], XVIII, §2.

Lemma 42.11. *Let G be a S-group scheme, locally of finite presentation and flat over S, and let $U \subset G$ be an open subscheme universally schematically dense in G relative to S. Then the morphism $m_G|_{U\times_S U} : U \times_S U \to G$ is locally of finite presentation, flat and surjective, hence an universal effective epimorphism.*

Proof. The morphism $\theta : G \times_S G \to G \times_S G$, $(g, g') \mapsto (g, gg')$ is an isomorphism. By base change, the second projection $p_2 : G \times_S G \to G$ is locally of finite presentation and flat, hence so is m_G.

Since U is universally schematically dense in G relative to S, then U_s is schematically dense in G_s for each $s \in S$, and hence $U_s \times_{\kappa(s)} U_s \to G_s$ is surjective, by [**SGA3**$_1$], VI$_A$, Prop. 0.5. The lemma follows. □

Proposition 42.12. *Let G, H be S-group schemes, with G flat and locally of finite presentation over S, and let m_G, m_H be their group laws. Suppose that we have open immersions $U \subset G$ and $V \subset m_G^{-1}(U) \cap (U \times_S U)$, both universally schematically dense relative to S, and that $f : U \to H$ is a S-morphism such that the diagram below commutes:*

$$\begin{array}{ccc} V & \xrightarrow{(f\times f)|_V} & H \times_S H \\ \downarrow{\scriptstyle m_G} & & \downarrow{\scriptstyle m_H} \\ U & \xrightarrow{f} & H. \end{array}$$

Then f extends uniquely to a morphism of S-groups $\phi : G \to H$.

Proof. Note first that, by base change, $U \times_S G$ and $G \times_S U$ are universally schematically dense in $G \times_S G$ relative to S, and so is $m_G^{-1}(U)$, using the isomorphism θ as in the proof of Lemma 42.11. Hence so is their intersection $W = m_G^{-1}(U) \cap U \times_S U$. By hypothesis, V is universally schematically dense in W relative to S, hence it is so in $G \times_S G$ by point (ii) of Lemma 42.5.

Next, Corollary 42.10, we see that if ϕ exists it is unique, and is a morphism of S-groups (because $\phi \circ m_G$ and $m_H \circ (\phi \times \phi)$ will coincide on V). So, it suffices to prove the existence of ϕ.

Recall that, by faithfully flat descent, if $\pi : X \to Y$ a fppf-morphism of S-schemes then, for every S-scheme Z, a morphism of S-schemes $\phi : Y \to Z$ is the same thing as a S-morphism $\psi : X \to Z$ whose two pull-backs to $X \times_Y X$ coincide. Using Lemma 42.11, we can apply this to $X = U \times_S U$ and $Y = G$, with $\pi = m_G|_X$, and to the S-morphism

$$\psi = m_H \circ (f \times f) : U \times_S U \to H$$

in order to get the sought-for S-morphism $\phi : G \to H$.

We must check that for any $S' \to S$ and $a, b, c, d \in U(S')$ such that $ab = cd$, one has $f(a)f(b) = f(c)f(d)$. By hypothesis, this is true if (a, b) and (c, d) are in $V(S')$, because in this case one has

$$f(a)f(b) = f(ab) = f(cd) = f(c)f(d).$$

Hence the sought-for equality is true on the open subset $V \times_G V$ of $X \times_G X$. So, it suffices to prove that $V \times_G V$ is universally schematically dense in $X \times_G X$ relative to S. But one has:

$$V \times_G V = \Big(V \times_G (G \times_S G)\Big) \cap \Big((G \times_S G) \times_G V\Big)$$

hence, by symmetry and point (i) of Lemma 42.5, it suffices to prove that $V \times_G (G \times_S G)$ is universally schematically dense in $(G\times_S G)\times_G(G\times_S G)$ relative to S. But we have a commutative diagram

$$\begin{array}{ccc} V \times_G (G \times_S G) & \hookrightarrow & (G \times_S G) \times_G (G \times_S G) \\ \downarrow \simeq & & \downarrow \simeq \\ V \times_S G & \hookrightarrow & G \times_S G \times_S G \end{array}$$

where the vertical isomorphisms are given by $(a, b, c, d) \mapsto (a, b, c)$. And by base change we know that the open immersion on the bottom line is universally schematically dense relative to S, hence so is the one on the top line. This completes the proof of the proposition. □