LECTURE 15

Reductive groups schemes: an overview

39. Historical context at the beginning of the 1960 decade

Our source is a text given by Grothendieck to Demazure, certainly in the Fall of 1960. Grothendieck starts by saying that the language of schemes is arriving to a level of maturity where it can be used to formulate easily numerous questions of the theory of algebraic groups in the context of schemes and probably to tackle their solving. He asks, among others, the following question: what is the good definition of a semi-simple group scheme G over an arbitrary base scheme S? He says that to avoid pathologies one should require that the fibers of G (over points of S) be connected. He considers as basic models the semi-simple group schemes $\mathcal{G}(\mathcal{R})$ over $\mathbb{Z}$ constructed by Chevalley. (Here, $\mathcal{R}$ denotes a *semi-simple root datum*, which is a slightly enhanced version of a root system, see paragraph below). Then he says that the *safest* definition would be to require that for each $s \in S$ there exists an open neighborhood U of s and a faithfully flat morphism $S' \to U$ such that the pull-back $G \times_S S'$ would come from some Chevalley group scheme $\mathcal{G}(\mathcal{R})$ (here $\mathcal{R}$ is the root datum of the geometric fiber $G_{\overline{s}}$, see below).

To be precise, let K be an algebraically closed field, G a semi-simple algebraic K-group, $T \simeq \mathbb{G}_m^d$ a maximal torus of G and $X(T)$ its character group, which is isomorphic to $\mathbb{Z}^d$. Let $\mathfrak{g}$ and $\mathfrak{t}$ be the Lie algebras of G and T. One has $\mathfrak{g} = \mathfrak{t} \oplus \bigoplus_{\alpha \in R} \mathfrak{g}_\alpha$, where R is a root system in the real vector space $V = X(T) \otimes_{\mathbb{Z}} \mathbb{R}$ and $\dim \mathfrak{g}_\alpha = 1$ for each α. For each α, the coroot $\alpha^\vee$ is an element of the free $\mathbb{Z}$-module $X^\vee(T) = \mathrm{Hom}_{K\text{-}\mathrm{Gr}}(T, \mathbb{G}_m)$, which is dual to $X(T)$ via the pairing $\langle \lambda, \eta \rangle = n$, with n being the unique integer such that the morphism $\lambda \circ \eta : \mathbb{G}_m \to \mathbb{G}_m$ is given by $z \mapsto z^n$. The $\alpha^\vee$'s form a root system $R^\vee$ in $X^\vee(T) \otimes_{\mathbb{Z}} \mathbb{R} \simeq V^*$, called the *dual root system*. The root lattice $Q(R)$ is the sublattice of $X(T)$ generated by R and the coroot lattice $Q(R^\vee)$ is the sublattice of $X^\vee(T)$ generated by $R^\vee$. Then the weight lattice $P(R)$ is the dual of $Q(R^\vee)$, i.e. the set of $\lambda \in V$ such that $\langle \lambda, \alpha^\vee \rangle \in \mathbb{Z}$ for all $\alpha^\vee \in R^\vee$. Clearly, one has $Q(R) \subset X(T) \subset P(R)$ and the pair $\mathcal{R} = (R, X(T))$ is called the root datum (or *type*) of (G, T); it determines G up to isomorphism. Conversely, every pair (R, X) where X is a sublattice of $P(R)$ containing $Q(R)$ is obtained in this manner.[1] For each $\mathcal{R}$, Chevalley has constructed ([**Che61**]) an affine, flat group scheme $\mathcal{G}(\mathcal{R})$ over $\mathbb{Z}$ such that $\mathcal{G}(\mathcal{R}) \otimes_{\mathbb{Z}} K$ is a semi-simple algebraic K-group of type $\mathcal{R}$, for every algebraically closed field K.

Then Grothendieck expresses the hope that it would suffice to require that G is affine and flat over S and that for each $s \in S$ the geometric fiber $G_{\overline{s}}$ is a connected semi-simple group over the algebraic closure $\overline{\kappa}(s)$ of the residue field $\kappa(s)$. Moreover, he conjectures that if G, G' are two such groups with isomorphic geometric fibers $G_{\overline{s}}, G'_{\overline{s}}$, there exists an open neighborhood U of s and a surjective étale morphism $S' \to U$ such that $G_{S'} \simeq G'_{S'}$. In particular, denoting by $\mathcal{R}$ the type of $G_{\overline{s}}$ and taking $G' = \mathcal{G}(\mathcal{R})_S$, this conjecture implies that the type of the geometric fibers should be a locally constant function of s.

Further, he indicates a strategy to prove that the weaker definition implies the stronger one: given $s \in S$, let $\mathcal{R}$ be the type of $G_{\overline{s}}$, set $G' = \mathcal{G}(\mathcal{R})_S$ and consider the functor F sending every S-scheme S' to the set of isomorphisms of S'-groups $G'_{S'} \xrightarrow{\sim} G_{S'}$; then one should prove that F is representable by a S-scheme X and then that there exists an open neighborhood U of s and a surjective étale morphism $S' \to U$ such that there exists a section of $X_{S'} \to S'$.

[0] version of Aug. 11, 2024.

[1] For *reductive* groups one has a more general notion of root datum, that we will see later.

Of course, further ideas and a lot of work were needed to achieve this project, and this was done by Demazure in his thesis, which is vol. 3 of SGA3 ([**SGA3**$_3$]).

For the sake of simplicity, we restricted ourselves to semi-simple group schemes in this section but, from now on, we return to the slightly more general setting of reductive group schemes.

40. Recollections from previous lectures

40.1. Existence of split maximal tori and system of roots. Let S denote an arbitrary base scheme. First, recall from Lect. 12, §28 the following definitions.

DEFINITION 40.1. One says that a S-group scheme G is **reductive** (resp. **semi-simple**) if:

(1) G is affine and smooth, with connected fibers.[2]
(2) Its geometric fibers are reductive (resp. semi-simple)

DEFINITION 40.2. Let G be a smooth affine S-group scheme. A **maximal torus** of G is a closed subgroup scheme T such that:

(1) T is a torus (in the sense of Def. 1.3).
(2) For every geometric point $\overline{s}$ of S, the subgroup $T_{\overline{s}}$ is a maximal torus of $G_{\overline{s}}$

From now on, we assume that G is a reductive S-group scheme. We proved in Lect. 13, Th. 35.1 that for each $s \in S$ there exists an étale neighbourhood U of s (that is, a surjective étale morphism $U \to V$, where V is a Zariski neighbourhood of s) such that G_U possesses a maximal torus, which is *split*, i.e. isomorphic to a diagonalisable group $D_U(M)$ for some finite free abelian group M. From now on, we take $S = U$. Observe then that, for each $s \in S$, we have a *natural identification* of M with the character group $X(T_{\overline{s}})$ of $T_{\overline{s}}$.

Let $\mathfrak{g} = \mathrm{Lie}(G)$; this is a locally free $\mathcal{O}_S$-module, equipped with the adjoint action of T. As T is diagonalisable and we saw that in Lect. 11 that the 0-weight space $\mathfrak{g}^T$ is just $\mathfrak{t} = \mathrm{Lie}(T)$, we can write as in Lect. 14:

$$\mathfrak{g} = \mathfrak{t} \oplus \bigoplus_{m \in M - \{0\}} \mathfrak{g}_m,$$

where each $\mathfrak{g}_m$ is a locally free $\mathcal{O}_S$-module. By the theory over an algebraically closed field, we know that the geometric fibers of a given $\mathfrak{g}_m$ are 0 or 1-dimensional. Hence each $\mathfrak{g}_m$ is a line bundle over some clopen (open and closed) subscheme of S, and 0 over the complement.

Now, recall from Lect. 14, §28 the following definitions and lemma.

DEFINITION 40.3. With assumptions as above, a **root** of (G, T) is a section α of the constant scheme $(M - \{0\})_S$, i.e. a locally constant function $\alpha : S \to M - \{0\}$, such that for each $s \in S$, the fiber $(\mathfrak{g}_{\alpha(s)})_s$ is non-zero, so that $\mathfrak{g}_{\alpha(s)}$ is a line bundle over some open neighbourhood of s. Then, we say that a set R of such roots is a **system of roots**[3] for (G, T) if one has

$$(\star) \qquad \mathfrak{g} = \mathfrak{t} \oplus \bigoplus_{\alpha \in R} \mathfrak{g}_\alpha.$$

Or equivalently: for each $s \in S$, the map $\alpha \mapsto \alpha(s)$ is a bijection from R to the usual root system of $(G_{\overline{s}}, T_{\overline{s}})$.

LEMMA 40.4. (i) *If α is a root of (G, T), then so is $-\alpha$.*

(ii) *If R is a system of roots for (G, T), every root of (G, T) is* locally on S *equal to an element of R.*

[2]Since a connected algebraic group over a field is geometrically connected (see [**SGA3**$_1$], VI$_A$, Prop. 2.1.1), the fibers are then geometrically connected.

[3]We suggest this terminology, in order to use “root system” only when we actually have a true root system in the usual sense.

PROOF. (i) is true since it is true fiberwise, and (ii) follows from ($\star$) above. $\square$

REMARKS 40.5. (a) The flexibility in the previous definition, taken from [**SGA3**$_3$], Exp. XIX, Def. 3.2 and 3.6, seems to be needed for purposes of descent theory. However, one should be aware that a system of roots for (G, T) need not be a root system in the usual sense (but becomes one if S is replaced by an appropriate clopen subscheme), below.

(b) In [**Co14**], Def. 4.1.1, Conrad takes the more restrictive definition that a root of (G, T) is a *constant* element α of M.

EXAMPLES 40.6. Let k be an algebraically closed field and $S = S_1 \sqcup S_2$ the direct sum of two copies of $\mathrm{Spec}(k)$.

(1) Let G be the S-group scheme which is PGL_2 over S_1 and SL_2 over S_2. The maximal torus T of diagonal matrices is isomorphic to $\mathbb{G}_{m,S}$ and its character group to $\mathbb{Z}$. Over S_1 (resp. S_2), its acts on the Lie algebra with the non-zero weights ± 1 (resp. ± 2), hence we have

$$\mathfrak{g} = \mathfrak{t} \oplus \mathfrak{g}_1 \oplus \mathfrak{g}_{-1} \oplus \mathfrak{g}_2 \oplus \mathfrak{g}_{-2}$$

with $\mathfrak{g}_{\pm 1}$ a line bundle over S_1 and zero over S_2, and conversely for $\mathfrak{g}_{\pm 2}$.

We could define the root $\alpha : S \to \mathbb{Z}$ by $\alpha = i$ on S_i for $i = 1, 2$. Then $\{\pm\alpha\}$ is a system of roots for (G, T). Or we could replace S by either of the clopen subschemes S_i, over which we have the constant root system $\{\pm i\}$.

(2) Perhaps more disturbing is the following example: Let $n \geq 3$ and let G be the S-group scheme which is the simply-connected group of type B_n (that is, $\mathrm{Spin}(2n + 1)$) over S_1 and of type C_n (that is, $\mathrm{Sp}(2n)$) over S_2. Then the maximal torus T of diagonal matrices is isomorphic to $(\mathbb{G}_{m,S})^n$ and its character group to $\mathbb{Z}^n$, with canonical basis $(\varepsilon_1, \dots, \varepsilon_n)$. The constant sections $\varepsilon_i - \varepsilon_j$, for $i \neq j$, are roots over S_1 and S_2 and we may complete these with the locally constant functions $\pm\eta_i$, for $i = 1, \dots, n$, equal to $\pm\varepsilon_i$ on S_1 and $\pm 2\varepsilon_i$ on S_2. These n^2 functions form a system of roots for (G, T). However, one may prefer to restrict to S_1 (resp. S_2) over which we have the honest, constant root system of type B_n (resp. C_n).

Let us now record from [**SGA3**$_3$], Exp. XXII, Prop. 2.1 the following result.

LEMMA 40.7. *Keep notation as above. Then:*

(a) *S is the disjoint union of clopen subschemes U_R, such that (G, T) possesses over each U_R a constant system of roots R.*

(b) *Further, for each $s \in S$ belonging to some U_R, there exists a neighborhood V of s such that over V one has $\mathfrak{g} = \mathfrak{t} \oplus \bigoplus_{\alpha \in R} \mathfrak{g}_\alpha$ with each $\mathfrak{g}_\alpha$ a trivial line bundle.*

PROOF. (a) Let us write $\mathfrak{g} = \bigoplus_{m \in M} \mathfrak{g}_m$, with $\mathfrak{g}_0 = \mathfrak{t}$ and $\mathfrak{g}_m$ is locally free of rank 0 or 1 at each point. Let $s_0 \in S$ and let $R = \{m \in M - \{0\} \mid (\mathfrak{g}_m)_{s_0} \neq 0\}$. Then the set of $s \in S$ such that $\mathfrak{g}_s = \mathfrak{t}_s \oplus \bigoplus_{\alpha \in R} (\mathfrak{g}_\alpha)_s\}$ is an open subscheme U_R of S, and S is the disjoint union of these subsets as R runs through the subsets of M which occur in some fiber. Therefore, each U_R is also a closed subscheme.

Then (b) follows immediately: we have a finite number of line bundles $\mathfrak{g}_\alpha$, for $\alpha \in R$, and they can be all trivialised over some neighborhood V of s. $\square$

40.2. Existence of root subgroups. Further, using the dynamical method of Conrad-Gabber-Prasad[4] (see [**Co14**], §4.1) we proved in Lect. 14 that for each (constant) root α of (G, T), there exists a *unique* closed subgroup scheme U_α normalised by T and such that $\mathrm{Lie}(U_\alpha) = \mathfrak{g}_\alpha$ and a *unique* T-equivariant morphism of S-groups $\exp_\alpha : \mathfrak{g}_\alpha \to G$ whose differential along the unit section is the giving embedding $\mathfrak{g}_\alpha \hookrightarrow \mathfrak{g}$. Moreover, it induces a T-equivariant isomorphism of groups $\mathfrak{g}_\alpha \xrightarrow{\sim} U_\alpha$. The same is true for $-\alpha$.

Further, let Z_α be the centraliser in G of the subgroup $\mathrm{Ker}(\alpha) \subset T$; it is a smooth closed subgroup of G and $\mathrm{Lie}(Z_\alpha) = \mathfrak{t} \oplus \mathfrak{g}_\alpha \oplus \mathfrak{g}_{-\alpha}$; form this it follows that the multiplication morphism induces an *isomorphism*:

$$U_{-\alpha} \times T \times U_\alpha \xrightarrow{\sim} \Omega, \tag{40.1}$$

[4] It is shorter than Demazure's proof of Th. 1.5 in [**SGA3**$_3$], Exp. XIX.

where Ω is a dense open subscheme of Z_α, called the big cell.

41. Semi-simple Lie algebras and reductive groups over $\mathbb{C}$, root data

41.1. The Tits convention of signs. Let $\mathfrak{g}$ be a complex semi-simple Lie algebra, κ its Killing form and $\mathfrak{h}$ a Cartan subalgebra. Recall that one obtains first the set of roots $R \subset \mathfrak{h}^*$ as the set of non-zero weights of the adjoint action of $\mathfrak{h}$ on $\mathfrak{g}$, i.e. one has $\mathfrak{g}_0 = \mathfrak{h}$ and

$$\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha \in R} \mathfrak{g}_\alpha.$$

Then, one proves that $\kappa(\mathfrak{g}_\alpha, \mathfrak{g}_\beta) = 0$ if $\alpha + \beta \neq 0$; it follows that R is stable under $\alpha \mapsto -\alpha$ and that the restriction of κ to $\mathfrak{h}$ and to each subspace $\mathfrak{g}_\alpha \oplus \mathfrak{g}_{-\alpha}$ is non-degenerate. If we denote by h_α the unique element of $\mathfrak{h}$ such that $\kappa(h_\alpha, h) = \alpha(h)$ for all $h \in \mathfrak{h}$, then for all $x \in \mathfrak{g}_\alpha$, $y \in \mathfrak{g}_{-\alpha}$ and $h \in \mathfrak{h}$, one has

$$\kappa([x,y],h) = \kappa(x,[y,h]) = \alpha(h)\kappa(x,y) = \kappa(x,y)\kappa(h_\alpha,h) = \kappa\big(\kappa(x,y)h_\alpha, h\big)$$

and since $\kappa|_{\mathfrak{h}\times\mathfrak{h}}$ is non degenerate, it follows that $[x,y] = \kappa(x,y)h_\alpha$. In particular, $\mathfrak{h}_\alpha = [\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}]$ is 1-dimensional. Thus, there exists a unique $H_\alpha \in \mathfrak{h}_\alpha$ such that $\alpha(H_\alpha) = 2$, namely $H_\alpha = 2h_\alpha/\kappa(h_\alpha,h_\alpha)$ (it is called the *coroot* associated with α). Further, one may choose $X_{\pm\alpha} \in \mathfrak{g}_{\pm\alpha}$ such that $\mathfrak{sl}_\alpha = \mathrm{Vect}(X_\alpha, H_\alpha, X_{-\alpha})$ is a Lie subalgebra isomorphic to $\mathfrak{sl}_2$ and then, using the representation theory of $\mathfrak{sl}_2$, one obtains $\dim \mathfrak{g}_{\pm\alpha} = 1$ and the integrality property $\beta(H_\alpha) \in \mathbb{Z}$ for all $\alpha, \beta \in R$, see e.g. [**Hu72**], Prop. 8.3 and 8.4. The usual choice for the isomorphism $\mathfrak{sl}_2 \xrightarrow{\sim} \mathfrak{sl}_\alpha$ is:

$$E = E_{12} \mapsto X_\alpha, \qquad H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \mapsto H_\alpha, \qquad F = E_{21} \mapsto X_{-\alpha},$$

one then has $[X_\alpha, X_{-\alpha}] = H_\alpha$. In fact, as discovered by Tits ([**Ti66**], it is better to use the isomorphism

$$E_{12} \mapsto X_\alpha, \qquad H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \mapsto H_\alpha, \qquad -E_{21} \mapsto X_{-\alpha},$$

in which case one has $[X_{-\alpha}, X_\alpha] = H_\alpha$. This is the convention used in [**BLie75**].

One advantage of this convention, explained in [**Ti66**], bottom of p. 21 (see also [**BLie75**], §VIII.2.4, Prop. 7), is the following. If one wants to construct a Chevalley basis over $\mathbb{Z}$ of $\mathfrak{g}$ or, what amounts to the same, and admissible $\mathbb{Z}$-lattice in $\mathfrak{g}$ using the Kostant $\mathbb{Z}$-form (see [**Hu72**], §26 or [**BLie75**], §VIII.12.4), then with this convention the Chevalley constants $N_{\alpha,\beta}$ satisfy $N_{-\alpha,-\beta} = N_{\alpha,\beta}$.

In hindsight, one can also see that this is the "right" convention as follows: if we start with the highest weight vector X_α in the adjoint representation of $\mathfrak{sl}_\alpha$, then applying the divided powers $X_{-\alpha}^{(n)} = X_{-\alpha}^n/n!$ we get:

$$X_{-\alpha} \cdot X_\alpha = [X_{-\alpha}, X_\alpha] = H_\alpha, \qquad \text{and then} \qquad X_{-\alpha}^{(2)} \cdot X_\alpha = \frac{1}{2}[X_{-\alpha}, H_\alpha] = X_{-\alpha}$$

(with no minus signs) and, starting with the lowest weight vector $X_{-\alpha}$ and applying the divided powers $X_\alpha^{(n)}$ we have:

$$X_\alpha \cdot X_{-\alpha} = [X_\alpha, X_{-\alpha}] = -H_\alpha = H_{-\alpha}, \qquad \text{and then} \qquad X_\alpha^{(2)} \cdot X_{-\alpha} = \frac{1}{2}[X_\alpha, -H_\alpha] = X_\alpha.$$

Another advantage of this convention, emphasized by Demazure in [**De15**] (but already present in [**Ti66**]) is the following. In the group SL_2, consider the elements

$$u_\alpha = \exp(X_\alpha) = I_2 + X_\alpha = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \qquad \text{and} \qquad u_{-\alpha} = \exp(X_{-\alpha}) = I_2 + X_{-\alpha} = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}.$$

Then an easy computation shows that

$$u_\alpha u_{-\alpha} u_\alpha = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = u_{-\alpha} u_\alpha u_{-\alpha}\,; \tag{41.1}$$

this element m_α belongs to the normalizer of the torus, its image in the Weyl group is the reflection s_α, and it is *unchanged* if we exchange α and $-\alpha$.

Moreover, setting $U_\alpha(x) = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}$ and $U_{-\alpha}(y) = \begin{pmatrix} 1 & 0 \\ -y & 1 \end{pmatrix}$ an easy computation gives:

$$U_\alpha(x) U_{-\alpha}(y) = \begin{pmatrix} 1-xy & x \\ -y & 1 \end{pmatrix}$$

and this element belongs to the big celle $\Omega = U_{-\alpha} \times T \times U_\alpha$ if and only if $1 - xy$ is invertible, and in this case one has the formula:

$$U_\alpha(x) U_{-\alpha}(y) = U_{-\alpha}\Big(\frac{y}{1-xy}\Big) \alpha^\vee(1-xy) U_\alpha\Big(\frac{x}{1-xy}\Big), \tag{41.2}$$

where $\alpha^\vee$ denotes the morphism $\mathbb{G}_m \to \mathrm{SL}_2$, $z \mapsto \mathrm{diag}(z, z^{-1})$. Now, in the general setting of paragraph 40.2 above, Demazure's idea was to perform a similar computation **to define** the sought-for morphism $\alpha^\vee : \mathbb{G}_{m,S} \to Z_\alpha$. We will explain his computations and results in the next lecture.

41.2. Root data. Let us start with the following definition (see [**SGA3**$_3$], XXI, Def. 1.1.1 or [**Sp98**], §7.4.1).

Definition 41.1. Let M and $M^\vee$ be finite free $\mathbb{Z}$-modules in duality by a pairing $M \times M^\vee \to \mathbb{Z}$ denoted by $\langle\ ,\ \rangle$. Let be given a finite subset R of M and a map $\alpha \mapsto \alpha^\vee$ from R to $M^\vee$ and set $R^\vee = \{\alpha^\vee \mid \alpha \in R\}$. For $\alpha \in R$ we define endomorphisms s_α and $s_\alpha^\vee$ of M and $M^\vee$ by:

$$s_\alpha(\lambda) = \lambda - \langle \lambda, \alpha^\vee \rangle \alpha, \qquad \forall \lambda \in M, \tag{41.3}$$

$$s_\alpha^\vee(\eta) = \eta - \langle \alpha, \eta \rangle \alpha^\vee, \qquad \forall \eta \in M^\vee. \tag{41.4}$$

One says that the quadruple $\mathcal{R} = (M, R, M^\vee, R^\vee)$ is a *root datum* if the following axioms are satisfied:

(RD 1) For each $\alpha \in R$, one has $\langle \alpha, \alpha^\vee \rangle = 2$

(RD 2) For each $\alpha \in R$, one has $s_\alpha(R) \subset R$ and $s_\alpha^\vee(R^\vee) \subset R^\vee$.

It follows from (RD 1) that $s_\alpha^2 = \mathrm{id}_M$ and $s_\alpha(\alpha) = -\alpha$. One denotes by W the group of automorphisms of M generated by the s_α, for $\alpha \in R$.

Set $V = \mathbb{Z}R \otimes_\mathbb{Z} \mathbb{R}$ and denote by π the natural map from $M^\vee$ to the dual vector space V^*. The axioms imply that R is a root system in V in the sense of [**BLie68**], the coroot of each root α being $\pi(\alpha^\vee)$. It follows (see *loc. cit.*) that the map $\alpha \mapsto \alpha^\vee$ is injective, that $\mathbb{Z}R^\vee \otimes_\mathbb{Z} \mathbb{R}$ identifies with V^* and that $R^\vee$ is a root system in V^*, called the dual root system of R. Further, W is called the Weyl group of $\mathcal{R}$.

Remark 41.2. Note that $s_\alpha^\vee$ is simply the *transpose* of s_α: for any $\eta \in M^\vee$ and $\lambda \in M$, one has:

$$\langle \lambda, {}^t s_\alpha(\eta) \rangle = \langle s_\alpha(\lambda), \eta \rangle = \langle \lambda - \langle \lambda, \alpha^\vee \rangle \alpha, \eta \rangle = \langle \lambda, \eta \rangle - \langle \lambda, \alpha^\vee \rangle \langle \alpha, \eta \rangle = \langle \lambda, \eta - \langle \alpha, \eta \rangle \alpha^\vee \rangle = \langle \lambda, s_\alpha^\vee(\eta) \rangle$$

and since $\langle\ ,\ \rangle$ is non-degenerate it follows that ${}^t s_\alpha(\eta) = s_\alpha^\vee(\eta)$. For this reason, one sometimes denote $s_\alpha^\vee$ simply by s_α.

Now let G be a reductive algebraic group over $\mathbb{C}$ (or, more generally, over an algebraically closed field K), let T be a maximal torus. To (G, T) one associates its root datum $\mathcal{R}$ as follows. Let $M = X(T)$ and $M^\vee = X^\vee(T)$ the groups of characters and cocharacters of T, which are in duality by the pairing $(\lambda \circ \eta)(z) = z^{\langle \lambda, \eta \rangle}$ for all $z \in \mathbb{G}_m(K)$. Let $\mathfrak{g} = \mathrm{Lie}(G)$ and $\mathfrak{t} = \mathrm{Lie}(T)$. One can prove that, w.r.t. the adjoint action of T, one has

$$(*) \qquad \mathfrak{g} = \mathfrak{t} \oplus \bigoplus_{\beta \in R} \mathfrak{g}_\beta$$

for some finite subset R of M, and that each $\mathfrak{g}_\alpha$ is 1-dimensional. This is the first, easy, step. Some more work is needed to get the coroots $\alpha^\vee$ and verify the axioms (RD 1) and (RD 2).

Here is an outline, see e.g. [**Sp98**] §7.4.3 for the details. Let $\alpha \in R$ and let Z_α be the centraliser in G of the subgroup $\mathrm{Ker}(\alpha) \subset T$; then Z_α is a reductive group and $\mathrm{Lie}(Z_\alpha) = \mathfrak{t} \oplus \mathfrak{g}_\alpha \oplus \mathfrak{g}_{-\alpha}$. Let Z'_α be the derived subgroup of Z_α; one can prove that Z'_α is isomorphic to SL_2 or PGL_2 and that there is a unique cocharacter $\alpha^\vee : \mathbb{G}_m \to T \cap Z'_\alpha$ such that $\langle \alpha, \alpha^\vee \rangle = 2$.

Next, note that the normaliser $N_G(T)$ acts naturally on $M = X(T)$ and $M^\vee = X^\vee(T)$. One deduces from the SL_2-computation (41.1) that Z'_α contains an element m_α which normalises T and acts on M and $M^\vee$ as s_α. Since the decomposition $(*)$ is preserved by the adjoint action of m_α, it follows that for each $\beta \in R$ one has that $m_\alpha(\beta) = s_\alpha(\beta)$ is still a root. Further, the conjugate $m_\alpha Z'_\beta m_\alpha^{-1}$ equals Z'_γ, where $\gamma = s_\alpha(\beta)$, and it follows that the cocharacter $m_\alpha(\beta^\vee) = s_\alpha(\beta^\vee)$ equals $\gamma^\vee$. This proves that axiom (RD 2) is satisfied.

EXAMPLES 41.3. (1) For $G = \mathrm{GL}_2$, let T be the maximal torus of diagonal matrices. Then $X(T) = \mathbb{Z}\varepsilon_1 \oplus \mathbb{Z}\varepsilon_2$, where $\varepsilon_i(\mathrm{diag}(z_1, z_2)) = z_i$ for $i = 1, 2$, while $X^\vee(T) = \mathbb{Z}\eta_1 \oplus \mathbb{Z}\eta_2$, where $\eta_1(z) = \mathrm{diag}(z, 1)$ and $\eta_2(z) = \mathrm{diag}(1, z)$. Then T acts (by conjugation) on the matrix E_{12} (resp. E_{21}) with the weight $\alpha = \varepsilon_1 - \varepsilon_2$ (resp. $-\alpha$). The corresponding cocharacters are $\alpha^\vee = \eta_1 - \eta_2$ and $-\alpha^\vee$. Thus, the root datum of GL_2 is:

$$\mathcal{R}(\mathrm{GL}_2) = \Big(\mathbb{Z}\varepsilon_1 \oplus \mathbb{Z}\varepsilon_2, \{\pm(\varepsilon_1 - \varepsilon_2)\}, \mathbb{Z}\eta_1 \oplus \mathbb{Z}\eta_2, \{\pm(\eta_1 - \eta_2)\}\Big).$$

(2) Now, $T' = T \cap \mathrm{SL}_2$ is the kernel of $\varepsilon_1 + \varepsilon_2$ and one has $X(T') = X(T)/\mathbb{Z}(\varepsilon_1 + \varepsilon_2)$, whereas $X^\vee(T') = \mathbb{Z}(\eta_1 - \eta_2)$. Denoting by $\overline{\varepsilon}_i$ the image of ε_i in $X(T')$, one obtains that the root datum of SL_2 is:

$$\mathcal{R}(\mathrm{SL}_2) = \Big(\frac{\mathbb{Z}\varepsilon_1 \oplus \mathbb{Z}\varepsilon_2}{\mathbb{Z}(\varepsilon_1 + \varepsilon_2)}, \{\pm(\overline{\varepsilon}_1 - \overline{\varepsilon}_2)\}, \mathbb{Z}(\eta_1 - \eta_2), \{\pm(\eta_1 - \eta_2)\}\Big).$$

(3) Next, the image $\overline{T}$ of T in PGL_2 is the quotient of T by the subtorus $(\eta_1 + \eta_2)(\mathbb{G}_m)$. One has $X^\vee(\overline{T}) = X^\vee(T)/\mathbb{Z}(\eta_1 + \eta_2)$ and $X(\overline{T})$ is the orthogonal of $\eta_1 + \eta_2$ in $X(T)$, which is spanned by $\alpha = \varepsilon_1 - \varepsilon_2$. Thus, denoting by $\overline{\eta}_i$ the image of η_i in $X^\vee(\overline{T})$, one obtains that the root datum of PGL_2 is:

$$\mathcal{R}(\mathrm{PGL}_2) = \Big(\mathbb{Z}(\varepsilon_1 - \varepsilon_2), \{\pm(\varepsilon_1 - \varepsilon_2)\}, \frac{\mathbb{Z}\eta_1 \oplus \mathbb{Z}\eta_2}{\mathbb{Z}(\eta_1 + \eta_2)}, \{\pm(\overline{\eta}_1 - \overline{\eta}_2)\}, \Big).$$

(4) If M is a free $\mathbb{Z}$-module of rank d and $M^\vee$ its dual, the root datum of the torus $T = D(M)$ is $\mathcal{R}(T) = (M, \varnothing, M^\vee, \varnothing)$ (there are no roots!).

(5) There is an obvious notion of "direct sum" of root data. Note that $\mathcal{R}(\mathrm{GL}_2)$ is not the direct sum of $\mathcal{R}(\mathbb{G}_m)$ and the root datum of either SL_2 or PGL_2. (For more on this, see [**SGA3**$_3$], Exp. XIX, §6).