Reductive groups: roots and root subgroups

In this lecture, G is a reductive group scheme over a base scheme S (unless stated otherwise).

36. Roots

Let $s \in S$. By Th. 35.1 of Lect. 13, one can find an étale neighbourhood U of s such that G_U has a split maximal torus $T = D_U(M)$, where $M \simeq \mathbb{Z}^d$ for some $d > 0$. So, from now on, we take $S = U$. Recall in passing that $M = D_S(T)$ is called the *character group* of T and is often denoted by $X(T)$.

Since G is smooth over S, its Lie algebra $\mathfrak{g} = \mathrm{Lie}(G)$ is a vector bundle over S. It is endowed with the adjoint action of G, hence in particular of T. Set $\mathfrak{t} = \mathrm{Lie}(T)$ and recall from Prop. 27.6 of Lect. 11 that $\mathfrak{t} = \mathfrak{g}^T$. Since $T = D_S(M)$ we obtain (see Prop. 2.3 in Lect. 1) that $\mathfrak{g}$ splits as a direct sum of T-modules:

$$\mathfrak{g} = \mathfrak{t} \oplus \bigoplus_{m \in M - \{0\}} \mathfrak{g}_m,$$

where each $\mathfrak{g}_m$ is locally free sheaf of $\mathcal{O}_S$-modules[1]. By the theory over an algebraically closed field, we know that the geometric fibers of a given $\mathfrak{g}_m$ are 0 are 1-dimensional. Hence each $\mathfrak{g}_m$ is a line bundle over some clopen subset of S, and 0 over the complement.

Definition 36.1. Following [**SGA3**$_3$], Exp. XIX, Definitions 3.2 and 3.6, we say that a **root** of (G, T) is a section α of M_S, i.e. a locally constant function $\alpha : S \to M$, such that for each $s \in S$, one has:

(1) $\alpha(s) \neq 0$,
(2) Denoting simply by α this non-zero element of M, the fiber $\mathfrak{g}_\alpha(s)$ is non-zero, hence $\mathfrak{g}_\alpha$ is a line bundle over some open neighbourhood of s.

Then, we say that a set R of such roots is a **root system** of (G, T) if one has

$$(\star) \qquad \mathfrak{g} = \mathfrak{t} \oplus \bigoplus_{\alpha \in R} \mathfrak{g}_\alpha.$$

Or equivalently: for each $s \in S$, the map $\alpha \mapsto \alpha(\overline{s})$ is a bijection from R to the usual root system of $(G_{\overline{s}}, T_{\overline{s}})$.

In [**Co14**], Def. 4.1.1, Conrad takes the more restrictive definition that a root of (G, T) is a constant element α of M.

Lemma 36.2. *If α is a root of (G, T), then so is $-\alpha$.*

Proof. This is true fiberwise. ☐

Example 36.3. Let k be an algebraically closed field and $S = S_1 \sqcup S_2$ the direct sum of two copies of $\mathrm{Spec}(k)$. Let $G = \mathrm{SL}_{2,S}$ and T the maximal torus of diagonal matrices. Then $\mathfrak{g} = \mathfrak{t} \oplus \mathfrak{g}_\alpha \oplus \mathfrak{g}_{-\alpha}$, where α is the character of T sending the diagonal matrix $\mathrm{diag}(t, t^{-1})$ to t^2. But we may also consider the character β of T which equals α on S_1 and $-\alpha$ on S_2; then we also have $\mathfrak{g} = \mathfrak{t} \oplus \mathfrak{g}_\beta \oplus \mathfrak{g}_{-\beta}$.

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[1] As S is not required to be connected, the rank of this "vector bundle" may vary on different connected components of S.

For later use in connection with descent theory, let us record here the following lemma, of which the previous example is an illustration.

LEMMA 36.4. *Assume that R is a root system of (G,T). Then every root of (G,T) is* locally *on S equal to an element of R.*

PROOF. This follows from $(\star)$ above. $\square$

The next step is: given a root α, construct a closed smooth subgroup U_α of G which is normalized by T, with Lie algebra $\mathfrak{g}_\alpha$. The first reduction is the following.

DEFINITION 36.5. The connected component T_α of $\mathrm{Ker}(\alpha)$ is a codimension 1 subtorus of T. We know that its centraliser in G is represented by a closed smooth subgroup G_α of G. By the classical theory, its geometric fibers are reductive groups (of semi-simple rank one), hence G_α is a reductive group scheme over S.[2] Further one has

$$\mathfrak{g} = \mathfrak{t} \oplus \mathfrak{g}_\alpha \oplus \mathfrak{g}_{-\alpha}$$

and $\mathfrak{g}_\alpha$ and $\mathfrak{g}_{-\alpha}$ are line bundles over S.

Now, the goal is to prove the following theorem, which is [**SGA3**$_3$], Exp. XX, Th. 1.5 and Cor. 5.9 or, alternatively, [**Co14**], Th. 4.1.4. Recall that the adjoint action of T on $\mathfrak{g}_\alpha$ is given on arbitrary S'-points $t \in T(S')$ and $v \in \mathfrak{g}_\alpha \otimes_{\mathcal{O}(S)} \mathcal{O}(S')$ by $t \cdot v = \alpha(t)v$. On the other hand, T acts on G by inner automorphisms.

THEOREM 36.6. (1) *There exists a unique closed smooth subgroup U_α of G normalized by T and such that* $\mathrm{Lie}(U) = \mathfrak{g}_\alpha$*; further, one has $U_\alpha \subset G_\alpha$.*

(2) *There exists a unique T-equivariant homomorphism of S-groups* $\exp_\alpha : \mathfrak{g}_\alpha \to G$ *whose differential at the identity is the canonical inclusion $\mathfrak{g}_\alpha \hookrightarrow \mathfrak{g}$. It gives a T-equivariant group isomorphism $\mathfrak{g}_\alpha \xrightarrow{\sim} U_\alpha$.*

(3) *The same is true for $-\alpha$, and the multiplication map of G induces an isomorphism*

$$U_{-\alpha} \times T \times U_\alpha \xrightarrow{\sim} \Omega$$

where Ω is an open subscheme of G_α, called the big cell *of G_α.*

The proof will be given in the next sections. Firstly, replacing if necessary S by an open subset, we may assume that α is a constant section of M. Let $M^\vee = \mathrm{Hom}_{\mathbb{Z}}(M,\mathbb{Z})$. Since $\alpha \neq 0$, we can find an element $\lambda \in M^\vee$ such that $\alpha \circ \lambda$ is an integer $n > 0$. Then λ is an element of $\mathrm{Hom}_{S\text{-Gr}}(\mathbb{G}_{m,S}, T)$ (which is the group of locally constant functions $S \to M^\vee$), hence through $\lambda : \mathbb{G}_{m,S} \to T$ we obtain an action of $\mathbb{G}_{m,S}$ on G.

37. The dynamical method of Conrad-Gabber-Prasad

The following lemma is [**CGP**], Lemma 2.1.4.

LEMMA 37.1. *Let V be an affine S-scheme with an action of $\mathbb{G}_{m,S}$. Then:*

(1) *The subfunctor V^+ of V defined by: for any $S' \to S$,*

$$V^+(S') = \left\{ v \in V(S') \;\middle|\; \begin{array}{l} \text{the morphism } \rho_v : \mathbb{G}_{m,S'} \to V_{S'},\ t \mapsto t \cdot v \\ \text{extends to a morphism } \widetilde{\rho}_v : \mathbb{A}^1_{S'} \to V_{S'} \end{array} \right\}$$

is represented by a closed subscheme of V. For v as above, $\widetilde{\rho}_v(0)$ will be denoted by $\lim_{t\to 0} t \cdot v$.

(2) *The same is true for the subfunctor V^- defined by the condition "$\lim_{t\to 0} t^{-1} \cdot v$ exists", and for the subfunctor of invariants V^0, which is $V^+ \cap V^-$.*

(3) *For any $v \in V^+(S')$, the point $v_0 = \lim_{t\to 0} t \cdot v$ belongs to $V^0(S')$. Therefore, we have a morphism of S-schemes $\ell_0 : V^+ \to V^0$.*

[2]And in fact G_α equals the centraliser of $\mathrm{Ker}(\alpha)$.

(4) *The formation of V^+ commutes with base change. Further, if Z is a closed subscheme of V stable by the action of $\mathbb{G}_{m,S}$, then $Z^+ = Z \cap V^+$.*

PROOF. We may assume that $S = \mathrm{Spec}(\Lambda)$ is affine. Then $V = \mathrm{Spec}(A)$ for a Λ-algebra A. The $\mathbb{G}_m$-action makes A into a graded algebra:

$$A = \bigoplus_{n\in\mathbb{Z}} A_n.$$

Denote by $\Delta_A : A \to A \otimes_\Lambda \Lambda[T, T^{-1}]$ the morphism of algebras corresponding to the action of $\mathbb{G}_{m,S}$ on V. For any $S' = \mathrm{Spec}(R)$, where R is a Λ-algebra, any point $v \in V(S')$ corresponds to a morphism of Λ-algebras $A \to R$, denoted by $f \mapsto f(v)$, and the morphism ρ_v corresponds to the morphism of Λ-algebras $\varphi_v = (v \otimes \mathrm{id}) \circ \Delta_A : A \to R \otimes_\Lambda \Lambda[T, T^{-1}]$, that is,

$$f \mapsto \sum_{n\in\mathbb{Z}} f_n(v) \otimes T^n, \quad \text{if} \quad \Delta_A(f) = \sum_{n\in\mathbb{Z}} f_n \otimes T^n.$$

Thus, the condition that ρ_v extends to $\mathbb{A}^1$, i.e. that φ_v factors through $R \otimes_\Lambda \Lambda[T]$, is equivalent to $f_n(v) = 0$ for all f_n with $n < 0$ occuring in an expression as above. But for any $f = f_n \in A_n$, one has $\Delta_A(f_n) = f_n \otimes T^n$. It follows that V^+ is the closed subscheme of V defined by the ideal of A generated by $\bigoplus_{n<0} A_n$.

The proof for V^- is similar. Further, denoting by V^0 the subfunctor of fixed points, one sees that $v \in V(S')$ belongs to $V^0(S')$ if and only if $f_n(v) = 0$ for all $n \neq 0$ and $f_n \in A_n$. In particular, one has $V^0 = V^+ \cap V^-$.

Further, for $v \in V^+(S')$ and with notation as above, the point $v_0 = \lim_{t\to 0} t \cdot v$ sends any $f \in A$ to $f(v_0) = f_0(v)$; in other words, for $n \in \mathbb{Z}$ and $f \in A_n$, one has

$$f(v_0) = \begin{cases} f(v) & \text{if } n = 0, \\ 0 & \text{otherwise.} \end{cases}$$

Thus $\varphi_{v_0}(f) = f_0(v) \otimes 1$, and this shows that $v_0 \in V^0(S')$.

Finally, assertion (4) is not difficult and is left to the reader. □

DEFINITION 37.2. More generally, if T is a torus over S acting on a S-scheme V and if $\lambda : \mathbb{G}_{m,S} \to T$ is a morphism of S-group schemes (one then says that λ is a 1-psg of T), one denotes by $V^+(\lambda)$ the corresponding closed subscheme of V.

The following proposition is contained in [**CGP**], Lemma 2.1.5 and Prop. 2.1.8.

PROPOSITION 37.3. *Let G be an affine S-group scheme, $\lambda : \mathbb{G}_{m,S} \to G$ a 1-psg; consider the resulting action of $\mathbb{G}_{m,S}$ on G via inner automorphims, set $P(\lambda) = G^+(\lambda)$ and let $Z(\lambda)$ be the centraliser of λ. Then:*

1. *$P(\lambda)$ and $Z(\lambda)$ are closed subgroup schemes. Further, the S-morphism $\ell_0 : P(\lambda) \to Z(\lambda)$ is a group morphism, which is the identity on $Z(\lambda)$.*
2. *Let $U(\lambda) = \mathrm{Ker}(\ell_0)$. Then $U(\lambda)$ is a closed subgroup of G, and $P(\lambda)$ is the semi-direct product $Z(\lambda) \ltimes U(\lambda)$.*
3. *The formation of $P(\lambda), Z(\lambda)$ and $U(\lambda)$ commutes with base change.*
4. *If G is of finite presentation over S, so are $P(\lambda), Z(\lambda)$ and $U(\lambda)$.*
5. *If G is of finite type over S, the fibers of $U(\lambda)$ are unipotent groups.*
6. *Set $\mathfrak{g} = \mathrm{Lie}(G)$. Then, with obvious notation, one has $\mathrm{Lie}\, P(\lambda) = \mathfrak{g}^+(\lambda) = \bigoplus_{n\geq 0} \mathfrak{g}_n$ and $\mathrm{Lie}\, Z(\lambda) = \mathfrak{g}_0$.*

PROOF. Again, we may assume that $S = \mathrm{Spec}(\Lambda)$ is affine.

(1) is easy since $\mathbb{G}_{m,S}$ acts by group automorphisms. As for (2), recall that, being a fiber-product, the kernel is always a subscheme, which is closed if $G \to S$ is separated, which is the

case here since $G \to S$ is affine. Then the last assertion of (2) follows from the second sentence of (1).

(3) is left to the reader. As for (4), if $G \to S$ is of finite presentation, then G and λ can be descended to a noetherian subring Λ_1 of Λ, then the closed subschemes $P(\lambda)_1$, etc. are of finite presentation over $S_1 = \mathrm{Spec}(\lambda_1)$, hence by base change so are $P(\lambda)$, etc. over S.

For (5), we may assume that $S = \mathrm{Spec}(k)$ for a field k. Since G is affine of finite type, there is a closed immersion $\tau : G \hookrightarrow H = \mathrm{GL}_{n,k}$ for some n. Then $U(\lambda) = G \cap U_H(\tau \circ \lambda)$ and one can check by a direct calculation that $U_H(\mu)$ is unipotent for any 1-psg $\mu : \mathbb{G}_{m,k} \to H = \mathrm{GL}_{n,k}$, see [**CGP**], Example 2.1.1.

Finally, (6) follows from the functorial description of $G^+(\lambda)$ and $G^0(\lambda)$ by considering points with values in $\mathrm{Spec}(\Lambda[\varepsilon])$, see the proof of [**CGP**], Prop. 2.1.8 (1). $\square$

Remark 37.4. In [**CGP**], Prop. 2.1.8 it is proved under the hypotheses of the previous proposition that the multiplication map $U(-\lambda) \times Z(\lambda) \times U(\lambda) \to G$ is an open immersion if S is the spectrum of a field or if G is smooth over S. As the proof in this generality is not easy, we will content ourselves below with the result easily obtained for reductive groups by looking at the fibers.

38. Proof of theorem 36.6: the isomorphism $\exp_\alpha : \mathfrak{g}_\alpha \to U_\alpha$