LECTURE 13

Existence étale-locally of maximal tori in reductive groups

33. Proposition 6.1 of Exp. XIX in the strictly henselian local case

DEFINITION 33.1. Let A be a local ring. One says that A is **strictly henselian** if it is henselian (see Def. 23.1 in Lect. 10) and, further, its residue field is separably closed.

The goal of this lecture is to prove the following theorem, which is part of [**SGA3**$_3$], Exp. XIX, Prop. 6.1.

THEOREM 33.2. *Let A be a strictly henselian noetherian local ring, $S = \operatorname{Spec} A$, s_0 its closed point, and G a smooth, affine S-group scheme with connected fibers. Let us make the following assumption:* [1]

Each maximal torus of the geometric fiber $G_{\bar{s}_0}$ equals its own centraliser.

Then there exists a split maximal torus T in G.

PROOF. Let $\mathfrak{m}$ be the maximal ideal of A and k the residue field. Since k is infinite, we know that G_{s_0} has a maximal torus T_0, say of dimension d. Further, since k is separably closed, T_0 is a split torus $\mathbb{G}_{m,k}^d$, see Prop. 12.6 in Lect. 5. Hence we have a closed immersion $f_0 : \mathbb{G}_{m,k}^d \hookrightarrow G_0$.

In order to use the results of formal smoothness proved in Section 17 of Lect. 7, we first replace $\mathbb{G}_{m,k}$ by a finite MT-group. Let m be an integer > 1 invertible on S_0 hence on S. For each $h \in \mathbb{N}^*$, the centraliser $\underline{\mathrm{Cent}}_G({}_{m^h}T_0)$ is represented by a closed subgroup scheme of G_0. As the family of subgroups ${}_{m^h}T_0$ is schematically dense in T_0 and as G_0 is noetherian, there exists $h \in \mathbb{N}^*$ such that

$$\underline{\mathrm{Cent}}_{G_0}({}_{m^h}T_0) = \underline{\mathrm{Cent}}_{G_0}(T_0) = T_0.$$

Set $n = m^h$ and $E_0 = {}_nT_0 = \mu_{n,k}^d$. Then f_0 induces a closed immersion $u_0 : E_0 \to G_0$ such that $\underline{\mathrm{Cent}}_{G_0}(E_0) = T_0$. Set $E = \mu_{n,S}^d$ and consider the S-functor

$$P = \underline{\mathrm{Hom}}_{S\text{-}\mathrm{Gr}}(E, G).$$

Denote by ${}_nG$ the closed subscheme of G which is the "kernel" of the n-th power map $\pi_n : G \to G$, that is, ${}_nG$ is the fiber-product of π_n and the unit section $S \to G$. Since n is invertible on S, one has $\mu_{n,S} \simeq (\mathbb{Z}/n\mathbb{Z})_S$ and hence $\underline{\mathrm{Hom}}_{S\text{-}\mathrm{Gr}}(\mu_{n,S}, G)$ is represented by ${}_nG$. It follows that P is represented by a closed subscheme of $({}_nG)^d$, the fiber-product over S of d copies[2] of ${}_nG$; namely, it is the closed subscheme of d-tuples $(g_1, \dots, g_d)$ which pairwise commute. Therefore, since S is noetherian, P is of finite type over S.

On the other hand, $P \to S$ is formally smooth by Th. 17.1 in Lect. 7, hence P is a smooth S-scheme. Therefore, by Hensel's lemma, the section $u_0 \in P(k)$ lifts to a section $u \in P(S)$, that is, a morphism of S-group schemes $u : E \to G$. By Cor. 21.2 of Lect. 9, we know that $\mathrm{Ker}(u)$ is a MT-group over S and that u is a closed immersion if $\mathrm{Ker}(u)$ is the unit group. But the type of $\mathrm{Ker}(u)$ over the closed point s_0 is the abelian group $\{0\}$, and S is connected since A is local. Since the type of the fibers is a locally constant function, it follows that u is a closed immersion.

[0]Version of Jan. 23, 2024, after the lecture.

[1]Note that if G is a reductive group scheme over S, this assumption is satisfied for every point s of S.

[2]Unfortunately, the exponent d is missing in [**SGA3**$_3$], Exp. XIX (and also in the original edition).

Next, $\underline{\mathrm{Cent}}_G(E)$ is represented by a smooth closed subgroup H of G, and its closed fiber is $\underline{\mathrm{Cent}}_{G_0}(E_0) = T_0$. Since S is local henselian, it follows from the remarkable Theorem 4.4 of [**SGA3**$_2$], Exp. X, whose proof we will give in the next section, that H contains an open and closed MT-subgroup T, whose closed fiber is T_0. As S is connected, the same reasoning as above gives that T is a d-dimensional torus, necessarily split since A is strictly henselian. It remains to prove that T is a maximal torus of G.

Again, we know that $\underline{\mathrm{Cent}}_G(T)$ is represented by a smooth closed subgroup C of G. By the classical result of Prop. 29.6 (1), we know that the fibers of C are connected, since those of G are. Then T is a closed subgroup of C and they have the same closed fiber. By Nakayama's lemma, it follows that $T = C$. This equality ensures that each geometric fiber $T_{\overline{s}}$ is a maximal torus of $G_{\overline{s}}$, hence T is a maximal torus of G. □

34. Theorem 4.4 of Exp. X in the henselian local case

Recall that, using the results about infinitesimal liftings of MT-groups, we proved in Lect. 10, Th. 22.2 the following result:

THEOREM 34.1. *Let A be a noetherian ring, with an ideal I such that A is separated and complete for the I-adic topology. Set $S = \mathrm{Spec}(A)$ and $S_0 = \mathrm{Spec}(A/I)$ Let G be an affine S-group scheme such that:*

(i) *$G \to S$ is of finite type and G is flat over S at each point of G_0,*
(ii) *G_0 is an isotrivial* MT-*group over S_0.*

Then there exists an open and closed subgroup H of G, which is an isotrivial MT-*group over S, such that $H_0 = G_0$.*

We are going to build on this result. We start with the following lemma.

LEMMA 34.2. *Let k be a field, G a* commutative *algebraic group over k. Assume that G has an open subgroup H of multiplicative type such that ${}_nH = {}_nG$ for all $n > 0$. Then $H = G$.*

PROOF. We may assume that k is algebraically closed. Then $H = D_k(M)$ for some finitely generated abelian group M. Let Q be the quotient of M by its torsion subgroup, then $T = D_k(Q)$ is the largest torus contained in H and H/T is finite, as well as G/T. Hence there exists an integer $d > 0$ such that $G/T = {}_d(G/T)$ and there exist a finite number of elements $g_i \in G(k)$ such that G is the disjoint union of the translates g_iH. One has $g_i^d \in T(k)$ and since k is algebraically closed, the d-th power map of $T(k)$ is surjective, there exist $t_i \in T(k)$ such that $t_i^d = g_i^d$. Hence, replacing each g_i by $g_it_i^{-1}$, we may assume that $g_i^d = 1$. Thus, $g_i \in {}_dG(k) = {}_dH(k)$ and it follows that $G = H$. □

Now, let us derive from Th. 34.1 the following result, which is interesting in its own right.

THEOREM 34.3. *Let A be a noetherian ring, with an ideal I such that A is separated and complete for the I-adic topology. Set $S = \mathrm{Spec}(A)$ and $S_0 = \mathrm{Spec}(A/I)$. Let G be an affine S-group such that:*

(i) *$G \to S$ is of finite type and G is flat over S,*[3]
(ii) *G_0 is an isotrivial* MT-*group over S_0.*

Then G is an isotrivial MT-*group over S if one of the following conditions is satisfied:*

(a) *G has connected fibers.*
(b) *G is abelian and the torsion subgroups ${}_nG$ are* finite *over S, for $n > 0$.*
(c) *The fibers of G are* MT-*groups, of constant type on each connected component of S.*

[3]Suffices to assume this at the points of G_0?

PROOF. Let H be the open and closed subgroup given by the previous theorem. If the fibers of G are connected, then $H = G$ and we are done. Now, denote by u the open immersion $H \hookrightarrow G$.

Assume condition (b). Then u induces for each $n > 0$ an open immersion ${}_n u : {}_n H \hookrightarrow {}_n G$ which induces an isomorphism $({}_n H)_0 \xrightarrow{\sim} ({}_n G)_0$. Then the complement of its image is finite over S, hence its projection to S is a closed subset of S which does not meet S_0, hence is empty (see Lemma 22.1 in Lect. 10). Therefore, ${}_n u$ is an isomorphism, and it follows from Lemma 34.2 that the open immersion u is an isomorphism on each fiber, hence an isomorphism. This proves (b).

Assume condition (c). As S is noetherian and complete for the I-adic topology, the connected components of S are open and are in bijection with those of S_0. So, we may assume that S and S_0 are connected (which is the case when A is local with maximal ideal I). Then for each geometric point $\overline{s}$, the fibers of $H_{\overline{s}}$ and $G_{\overline{s}}$ have isomorphic types N and M respectively (since it is so on S_0), and the closed immersion $u_{\overline{s}} : H_{\overline{s}} \to G_{\overline{s}}$ corresponds to a surjective morphism $\pi : M \to N$. Since M, N are isomorphic abelian groups of finite type, π must be an isomorphism, hence so is each u_s. As in (b), it follows that u is an isomorphism. □

Before we can state the main proposition and theorem, we need the following result about local henselian schemes.

PROPOSITION 34.4. *Let S be a local henselian scheme, s its closed point, X a S-scheme locally of finite type, x an isolated point in the fiber X_s.*

(i) *Then $\mathcal{O}_{X,x}$ is finite over $\mathcal{O}_{S,s}$.*
(ii) *Further, if $X \to S$ is separated, then $X' = \operatorname{Spec}(\mathcal{O}_{X,x})$ is an open and closed subscheme of X.*

PROOF. By the local form of Zariski's main theorem (see e.g. [**Ray70**], Ch. IV, Th. 1), x has an affine open neighbourhood $U = \operatorname{Spec}(B)$ which is of finite type and quai-finite over $A = \mathcal{O}_{S,s}$, and there exists an open immersion $U \hookrightarrow Y = \operatorname{Spec}(C)$, where C is a finite A-algebra.

Since A is henselian, Y is the disjoint sum of a finite number of local schemes $Y_i = \operatorname{Spec}(C_i)$, each finite over S, and the points of Y over s are the closed points y_i. Hence $x = y_i$ for some i and $\mathcal{O}_{X,x} = \mathcal{O}_{U,x} = C_i$ is finite over A. Further, $X' = Y_i$ is an open subscheme of U hence of X.

Assume further that $X \to S$ is separated. Then, by [**EGA**] II, 6.1.5 (v), since $X' \to S$ is finite, so is $X' \to X$, hence X' is also a closed subscheme of X. □

Now we can state and prove the following proposition, which generalizes Prop. 23.12 of Lect. 10.

PROPOSITION 34.5. *Let A be a noetherian local henselian ring, A' its completion, S and S' their spectra, s the closed point of S, and H, G two S-group schemes such that:*

- *H is a* MT*-group of finite type over S,*
- *$G \to S$ is separated and locally of finite type, G_s is a* MT*-group over $\kappa(s)$ and G is flat over S at the points of G_s.*

Let H', G' be the pull-backs of H, G to S'. Then the natural map below is bijective:

$$\operatorname{Hom}_{S\text{-Gr}}(H, G) \xrightarrow{\sim} \operatorname{Hom}_{S'\text{-Gr}}(H', G').$$

PROOF. Since $A \to A'$ is faithfully flat, we know already that this map is a bijection from the LHS to the subset of S'-group morphisms $u' : H' \to G'$ whose two pull-backs u''_1 and u''_2 to $S'' = S' \times_S S'$ coincide. So let $u' \in \operatorname{Hom}_{S'\text{-Gr}}(H', G')$, let us prove that $u''_1 = u''_2$.

By the density theorem of Lect. 9 (Th. 20.2) it suffices, as in the proof of Prop. 23.12 of Lect. 10, to prove that u''_1 and u''_2 coincide on the torsion subgroup ${}_n H''$, for each $n > 0$. So, let us fix $n > 0$ and replace H by its subgroup ${}_n H$, which is **finite** over S (if M is the finitely generated abelian group corresponding to H, then ${}_n H$ corresponds to the finite abelian group

M/nM). Let ϕ_n denote the n-th power map of G (which need not be a group homomorphism) and denote by ${}_nG$ its "kernel", i.e. the fiber-product of ϕ_n and the unit section of G. Note that for any base change $T \to S$, one has ${}_n(G_T) = ({}_nG)_T$.

Let $\mathfrak{m}$ and k denote the maximal ideal and residue field of A; for each $m \in \mathbb{N}$ denote by a subscript m on the right the reduction modulo $\mathfrak{m}^{m+1}$. Then each G_m is flat over S_m (since G is flat over S at the points of G_0) and since G_0 is a finite type MT-group over $S_0 = \mathrm{Spec}(k)$, necessarily isotrivial (see Lect. 5, Prop. 12.6), it follows from Th. 18.3 of Lect. 8 that:

($*$) G_m is a finite type MT-group over S_m.

($**$) Hence $({}_nG)_m = {}_n(G_m)$ is a MT-group, **finite** and flat over S_m.

This is true, in particular, for $m = 0$. Let us assume for the moment the following claim:

Claim 34.6. (1) *There exists an open and closed subscheme Z of ${}_nG$, which is finite over S and such that every morphism of S-schemes $Y \to {}_nG$, with Y finite over S, factors through Z.*

(2) *The subscheme $Z' = Z \times_S S'$ of ${}_nG'$ has the analogous universal property.*

(3) *Z' is an isotrivial* MT-*group over S', and Z an isotrivial* MT-*group over S.*

Taking this claim for granted for the moment, let us complete the proof of Prop. 34.5. Recall that $H = {}_nH$ is finite over S and let $u' : H' \to G'$ be a morphism of S'-groups. It factors obviously through ${}_nG'$ and hence, by assertion (2) of the Claim, through Z'. As H and Z are MT-groups **finite** over S, it follows from Prop. 23.11 of Lect. 10 that $u' : H' \to Z'$ comes from a morphism of S-groups $u : H \to Z$, and therefore the two pull-backs u_1'', u_2'' coincide. This completes the proof of Prop. 34.5, up to the proof of the previous claim.

Let us now prove the claim. Since ${}_nG$ is, like G, separated and locally of finite type over S and since $({}_nG)_0$ is finite over S_0, it follows from Lemma 34.4 that the local rings of ${}_nG$ at the points over s are finite over S and that one has a decomposition into two open and closed subschemes:

$$(\star) \qquad {}_nG = Z \coprod ({}_nG)^*$$

where Z is finite over S and $({}_nG)^*$ lies above $S - \{s\}$. Further, every morphism of S-schemes $Y \to {}_nG$, with Y finite over S, factors through Z, and one sees that the pull-back of $(\star)$ to S' is the analogous decomposition of ${}_nG'$. This proves assertions (1) and (2).

Let us prove that Z' is a commutative subgroup scheme of G'. As Z' is finite over S', so is $P' = Z' \times_{S'} Z'$. Denote by μ the restriction to P' of the multiplication map of H'. Note that it is not clear that μ factors through ${}_nG'$, so we cannot invoke the claim to assert that $\mu(P') \subset Z'$. Instead, we use the following argument.

By [**EGA**], II 5.4.3 and IV$_1$ 1.1.3, since $P' \to S'$ is finite and $H' \to S'$ separated and locally of finite type, $Y = \mu(P')$ is a closed subscheme of H', which is **proper** over S' (it is universally closed and quasi-compact, hence of finite type). Further, $Y \to S'$ has finite fibers (since this is true for $P' \to S'$) and hence, since S' is noetherian, it follows from [**EGA**] III, 4.4.2 that $Y \to S'$ is **finite**. Hence $Y = \mathrm{Spec}(B)$ for some finite A'-algebra B.

As Z' contains the unit section, one has $Z' \subset Y$. Over each S_m, we saw in ($**$) above that $({}_nG)_m = {}_n(G_m)$ is a MT-group, finite over S_m, hence $Z'_m = ({}_nG)_m = Y_m$. Thus, if I is the ideal of B defining Z', one has $I \subset \mathfrak{m}^n B$ for all n. Since B is finite over A', it is separated for the $\mathfrak{m}$-adic topology, hence $I = 0$ and $Z' = Y$. Therefore Z' is a subgroup scheme of H'. Let us prove that it is commutative.

Let τ be the automorphism of P' which swaps the two factors. The equalizer K of μ and $\mu \circ \tau$ is a closed subscheme of P', hence is finite over S', and for each m one has $K_m = P'_m$ since $Z'_m = {}_n(G_m)$ is commutative. By the same reasoning as above, it follows that $K = P'$, hence Z' is a commutative S'-group scheme.

Moreover, as each $Z'_m = {}_n(G_m)$ is flat over S_m, it follows from the local criterion of flatness (see [**EGA**], 0_{III}, 10.2.2 or [**Mat86**]) that Z' is **flat** over S'. Further, $Z'_0 = {}_n(G_0)$ is a MT-group over $S_0 = \mathrm{Spec}(k)$, necessarily isotrivial.

Hence, assertion (b) of Th. 34.3 tells us that Z' is an isotrivial MT-group over S'. This proves the first assertion of (3). Finally, as $S' \to S$ is faithfully flat (and affine), it follows that the multiplication map $Z \times_S Z \to G$ factors through Z, hence Z is a S-group scheme, and of multiplicative type since Z' is. This completes the proof of Claim 34.6 and of Prop. 34.5. ☐

We then obtain the following theorem.

Theorem 34.7. *Let S be the spectrum of a noetherian local henselian ring A, and s its closed point. Let G be an affine S-group scheme of finite type, flat over S at the points of H_s and such that H_s is a* MT-*group.*

Then there exists an open and closed immersion $H \to G$, where H is an isotrivial MT-*group over S such that $H_s = G_s$.*

Remark 34.8. If G_s is a torus, then so is H, and H is the connected component of G. Therefore G normalizes T. One such example over $S = \mathrm{Spec}(k[[t]])$, with k a field, is given in [**SGA3**$_3$], Exp. XIX, §5.

Proof. The proof is analogous to that of Cor. 23.10 of Lect. 10, once that we have Prop. 34.5 at our disposal. Namely, let A' be the completion of A. Set $S' = \mathrm{Spec}(A')$ and $G' = G_{S'}$. Recall (Lect. 5, Prop. 12.6) that G_s is isotrivial over $\mathrm{Spec}\,\kappa(s)$. Hence, by Th. 34.1, there exists an open and closed immersion $u' : H' \to G'$, where H' is an isotrivial MT-group over S' such that $H'_s = G'_s$. Recall that, by Lemma 23.10 of Lect. 10, the categories of *isotrivial* MT-groups over S, $S_0 = \{s\}$ and S' are equivalent; thus H' comes from an isotrivial MT-group H over S.

Then, by Prop. 34.5, u' comes from a homomorphism of S-groups $u : H \to G$, and u is an open and closed immersion inducing an isomorphism $H_s \xrightarrow{\sim} G_s$ because this is so after the faithfully flat affine base change $S' \to S$. ☐

35. Existence étale-locally of maximal tori in reductive groups

Using Grothendieck's technique of reducing the general case to the case of a noetherian local henselian base (see [**EGA**] IV_3, §8, with additional results for group schemes in [**SGA3**$_1$], Exp. VI_B, §10), which we sketched in Section 24 of Lect. 10, one obtains the following:

Theorem 35.1. *Let S be a scheme, G a reductive S-group scheme and $s \in S$. There exists an étale neighbourhood U of s such that G_U possesses a split maximal torus T.*

Notes for this Lecture

Theorems 33.2 and 35.1 are contained in Exp. XIX, Prop. 6.1; we have arranged the proof slightly differently, using Exp. XIV, Th. 1.1 (Th. 30.2 in Lect. 12) in order not to rely on the results of Exp. XI (representability of the functor of maximal tori by a smooth S-scheme), and at the end of the proof of Th. 33.2 we derived the existence of T directly from Th. 4.4, instead of the more elaborate Th. 8.1 of Exp. X.

Lemma 34.2 is contained in Exp. X, Lemma 3.6, whereas Cor. 34.3 is Exp. X, Cor. 3.8.

Prop. 34.4 is X, Prop. 4.3.0, which is an addition in the new edition of [**SGA3**$_2$].

Prop. 34.5 is X, Lemma 4.3, whereas Th. 34.7 is contained in X, Th. 4.4. The detailed proof of X, Lemma 4.3, as it appears in the new edition of [**SGA3**$_2$], was communicated to the author by M. Raynaud.