LECTURE 12

Reductive center. Cartan subalgebras. Maximal tori over a field

28. Goals about maximal tori.

Let us recall three definitions from Lect. 6.

DEFINITION 28.1. Let G be a connected affine smooth group scheme over an algebraically closed field k. One knows that all maximal tori T of G are conjugate under $G(k)$; in particular they have the same dimension, which is called the *reductive rank* of G and denoted by $\mathrm{rk}_{\mathrm{red}}(G)$. Of course, if H is a smooth closed subgroup of G, then $\mathrm{rk}_{\mathrm{red}}(H) \leq \mathrm{rk}_{\mathrm{red}}(G)$. If equality holds, then a maximal torus of H is the same thing as a maximal torus of G contained in H.

One also knows that there exists a largest *normal* smooth connected solvable (resp. unipotent) subgroup of G, it is called the *radical* (resp. *unipotent radical*) of G and is denoted by $\mathrm{rad}(G)$ (resp. $\mathrm{rad}^u(G)$).

One says that G is *reductive* (resp. *semi-simple*) if $\mathrm{rad}^u(G) = \{e\}$ (resp. $\mathrm{rad}(G) = \{e\}$). If K is a larger algebraically closed field, G_K is reductive (resp. semi-simple) if and only if G is.

DEFINITION 28.2. Let S be a base scheme. One says that a S-group scheme G is **reductive** (resp. **semi-simple**) if:

(1) G is affine and smooth, with connected fibers.
(2) Its geometric fibers are reductive (resp. semi-simple), that is, for every $s \in S$, denoting by $\overline{s}$ the spectrum of an algebraic closure $\overline{\kappa}(s)$ of the residue field $\kappa(s)$, the $\overline{\kappa}(s)$-group $G_{\overline{s}}$ is reductive (resp. semi-simple).

DEFINITION 28.3. Let G be a smooth affine S-group scheme. A **maximal torus** of G is a closed subgroup scheme T such that:

(1) T is a torus (in the sense of Def. 1.3).
(2) For every geometric point $\overline{s}$ of S, the subgroup $T_{\overline{s}}$ is a maximal torus of $G_{\overline{s}}$

REMARK 28.4. Note that a maximal torus need not always exist. For example, if S is the spectrum of a discrete valuation ring R, with fraction field K and uniformizing parameter π, let G be the group functor such that for every S-scheme U,

$$G(U) = \{(x,t) \in \mathbb{G}_a(U) \times \mathbb{G}_m(U) \mid t = 1 + \pi x\},$$

with group law $(x_1,t_1)(x_2,t_2) = (x_1 + x_2 + \pi x_1 x_2, t_1 t_2)$. Then G is represented by the sub-Hopf algebra $A = R[X, T^{-1}]$ of $K[T, T^{-1}]$, where $X = (T-1)/pi$. The generic fiber of G is $\mathbb{G}_{m,K}$ whereas is special fiber is $\mathbb{G}_{a,k}$, where k is the residue field of A. Then the unique torus contained in G is the unit group T, which is not a maximal torus because in the generic fiber $T_K = \{e\}$ is not a maximal torus of $\mathbb{G}_{m,K}$.

REMARK 28.5. In [**SGA3**$_2$], XII, Th. 1.7, Grothendieck proves, using that the functor of maximal tori is represented by a smooth S-scheme, that the following conditions are equivalent:

(1) There exist étale-locally a maximal torus in G.
(2) The function $s \mapsto \mathrm{rk}_{\mathrm{red}}(G_{\overline{s}})$ is locally constant on S.

[0]version of Jan. 16, 2024. I thank Rajarshi Gosh for pushing me to include in this lecture statement (b) of Th. 31.2 and for pointing out 2 typos in a previous version.

Firstly, the goal of the present lecture is to prove the following theorem, which is proved in [**SGA3**$_2$], XIV, Th. 1.1 for an arbitrary field. For the sake of simplicity, we restrict to the case of an infinite field.

THEOREM 28.6. *Let G be a smooth affine group scheme over an* infinite *field k. Then there exists a maximal torus in G.*

Secondly, the goal for the next 2 or 3 lectures will be to prove, following [**SGA3**$_3$], Exp. XIX, that if G is reductive over S, then maximal tori exist étale-locally: for each $s \in S$, there exist an open neighborhood U of s and a surjective étale map $V \to U$ such that G_V possesses a maximal torus. Time permitting, we will present the results of [**SGA3**$_2$], XIV, proving that maximal tori exist even Zariski-locally.

To achieve these goals, several concepts and results, some over a general base and some over an algebraically closed field, will be needed.

29. Reductive center

Let S be a base scheme and G a smooth affine S-group.

DEFINITION 29.1 (**Reductive center**). Let Z be a closed subgroup scheme of G. One says that Z is a *reductive center* of G if:

(1) Z is central and of multiplicative type.
(2) For every base change $S' \to S$ and every central homomorphism $u : H \to G_{S'}$, where H is a MT-group over S' of finite type, u factors through $Z_{S'}$.

Note that in this case $Z_{S'}$ is a reductive center of $G_{S'}$ for every base change $S' \to S$.

REMARK 29.2. The group G considered in Rem. 28.4 does not have a reductive center.

Before proving a theorem about reductive centers, we need the following proposition and notation.

PROPOSITION 29.3. *Let H be a* commutative *affine group scheme over S. Suppose that there exist* MT-*groups K, Q over S such that K is a closed subgroup of H and the quotient H/K is isomorphic with Q. Then H is a* MT-*group over S.*

PROOF. See [**SGA3**$_2$], XVII, Prop. 7.1.1, or [**DG70**], §IV, §1.4, Prop. 4.5 when S is the spectrum of a field. □

NOTATION 29.4. Let T be a closed MT-subgroup of G. By Prop. 27.6 of Lect. 11, the centraliser $\underline{\mathrm{Cent}}_G(T)$ is represented by a smooth closed subgroup scheme of G, denoted by $C(T)$. Further, we recall that $\mathrm{Lie}(C(T)) = \mathrm{Lie}(G)^T$ (the T-invariants in $\mathrm{Lie}(G)$.

THEOREM 29.5. *Let G be a smooth affine S-group scheme.*

(a) *If G has a reductive center locally for the* fpqc *topology, it has a reductive center.*

(b) *A subgroup scheme Z of G is a reductive center if and only if it is so locally for the* fpqc *topology.*

(c) *Suppose that G has a reductive center Z. Then the quotient group scheme G/Z exists and has trivial reductive center.*

(d) *Suppose that G has connected fibers and admits a maximal torus T.*

(1) *Let H be a* MT-*group of finite type over S. Then any central homomorphism $u : H \to G$ factors through T.*

(2) *Let $\mathfrak{g} = \mathrm{Lie}(G)$ and let $\theta : T \to \mathrm{GL}(\mathfrak{g})$ be the restriction to T of the the adjoint representation of G. Then $\mathrm{Ker}(\theta)$ is a reductive center of G.*

(e) *Suppose that S is the spectrum of a field k. Then G admits a reductive center Z. Further, the maximal tori of G are in bijection with those of $G' = G/Z$.*

PROOF. Let us prove (a). Let $S' \to S$ be a covering morphism for the fpqc topology such that $G' = G_{S'}$ admits a reductive center Z'. Let $S'' = S' \times_S S'$ for $i = 1, 2$ let G''_i and Z''_i denote the pull-backs over S'' by the projection pr_i. Then Z''_i is the reductive center of G''_i hence the descent datum $c : G''_1 \xrightarrow{\sim} G''_2$ relative to S'/S induces a descent datum $c : Z''_1 \xrightarrow{\sim} Z''_2$. Since Z' is affine over S' (being a MT-group over S'), this descent datum is effective, i.e. Z' comes by base change from a closed subgroup scheme Z of G, of multiplicative type. Let us prove that Z is indeed a reductive center of G.

Let $T \to S$ and let $u : H \to G_T$ be a central homomorphism, where H is a MT-group over T of finite type. Set $T' = T \times_S S'$. By hypothesis, the pulback $u' : H_{T'} \to G_{T'}$ factors through $Z' \times_{S'} T = Z_T \times T'$, hence it belongs to the equaliser of the double arrow

$$\mathrm{Hom}_{T'\text{-Gr}}(H_{T'}, Z_{T'}) \rightrightarrows \mathrm{Hom}_{T''\text{-Gr}}(H_{T''}, Z_{T''}).$$

Since $S' \to S$ is a universal effective epimorphism, this equaliser is $\mathrm{Hom}_{T\text{-Gr}}(H_T, Z_T)$. This proves that Z is a reductive center for G. This proof also gives assertion (b).

Let us prove (c). By Cor. 21.2 of Lect. 9, the quotient G/Z is represented by an affine S-group scheme G'. Further, by [**SGA3**$_1$], VI_B, Prop. 9.2 (xii), G' is smooth over S, and its fibers are connected since those of G are.

Let Z' be a central MT-subgroup of G'; we want to prove that Z' is the trivial group. It suffices to prove this fiberwise, so we may and do assume that S is the spectrum of an algebraically closed field k. Let $Z_1 = \pi^{-1}(Z')$, where π is the projection $G \to G'$; let us prove that Z_1 is central in G. Consider the action of Z_1 on G by inner automorphisms; since Z is central this action factors through G'. Further, the action is trivial on Z (which is central) and the induced action on $G' = G/Z$ is trivial, since Z' is central in G'. Hence, for points $z' \in Z'(U)$ and $g \in G(U)$, where U is an arbitrary S-scheme, we may write:

$$z' \cdot g = z' g z'^{-1} = g \cdot b(z', \pi(g)),$$

with $b(z', \pi(g)) \in Z(U)$. As Z is central in G, it follows that the S-scheme morphism $b : Z' \times_S G' \to Z$ is bimultiplicative:

$$b(z', g'_1 g'_2) = b(z', g'_1) b(z', g'_2) \qquad \text{and} \qquad b(z'_1 z'_2, g') = b(z'_1, g) b(z'_2, g').$$

Thus, it corresponds to a morphism of S-group functors $u : G' \to \underline{\mathrm{Hom}}_{S\text{-Gr}}(Z', Z)$. But the RHS, call it E, is represented by an étale S-group scheme: if U is an open subset of S over which the MT-groups Z and Z' have type M and M', for some abelian groups M, M', then $E|_U$ RHS is a twisted constant group of type $\mathrm{Hom}_{\mathrm{grp}}(M, M')$, see Prop. 7.5 in Lect. 3.

Since E is étale, its unit section is an open immersion, hence $\mathrm{Ker}(u)$ is an open subgroup of G'. Since G' is connected it follows that u is trivial, hence b is the constant morphism with value the unit element e. This proves that the action of Z_1 on G is trivial, i.e. Z_1 is central in G.

Thus, one has an exact sequence $1 \longrightarrow Z \longrightarrow Z_1 \longrightarrow Z' \longrightarrow 1$, with Z_1 commutative. Hence, using Prop. 29.3, we obtain that Z_1 is a MT-group, central in G. It is therefore contained in (and in fact equal to) Z and hence Z' is the unit group. This completes the proof of (c).

Let us prove (d) (1). By Cor. 27.6 of Lect. 11, the centraliser $\underline{\mathrm{Cent}}_G(T)$ is represented by a closed subscheme C of G, which is affine and smooth over S. As T is central in C it is a fortiori invariant, and by Cor. 21.2 the quotient C/T is representable by an affine group scheme U. As u is central, it factors through C; thus, denoting by π the projection $C \to U$, we have to prove that the homomorphism $v = \pi \circ u$ is trivial. By Cor. 21.2 of Lect. 9, we know that $K = \mathrm{Ker}(v)$ is a MT-group over S and to prove that $K = H$ it suffices to prove that they have the same type on each geometric fiber, hence we are reduced to the case where S is the spectrum of an algebraically closed field. In this case, U is a smooth connected unipotent group and one knows that every homomorphism $H \to U$ is trivial. This proves (d) (1).

Let us prove (d) (2). Set $Z = \mathrm{Ker}(\theta)$; it is a MT-subgroup of T, by Prop. 7.10 of Lect. 3. If H is a MT-group over S and $u : H \to G$ a central homomorphism, then u factors through the kernel of the adjoint representation (since it is central) and through T by assertion (1), hence u factors through Z. This property is preserved by any base change $S' \to S$, since $Z_{S'} = \mathrm{Ker}(\theta_{S'})$. Thus, it remains to prove that Z is central in G, i.e. that $C = \underline{\mathrm{Cent}}_G(Z)$ equals G.

By Cor. 27.6 of Lect. 11, we know that C is a smooth closed subgroup of G and $\mathrm{Lie}(C) = \mathfrak{g}^Z$ equals $\mathfrak{g}$ since Z acts trivially on $\mathfrak{g}$. Then on each fiber the closed immersion $C \hookrightarrow G$ is étale at the origin, hence étale, hence is an open immersion (see [**SGA1**], Exp. I, Th. 5.1), hence is an isomorphism since the fibers of G are connected. Therefore, by the fibral criterion of isomorphism (see [**EGA**], IV_4, Cor. 17.9.5 or Lemma 22.3 in Lect. 10), one has $C = G$. This completes the proof of (d) (2).

Let us prove (e). We know that $\underline{\mathrm{Cent}}(G)$ is represented by a closed subgroup scheme of G (not necessarily smooth, even if G is). Thus, replacing G by its center, we may assume that G is *commutative*. Then, one knows that there exists a largest MT-subgroup scheme Z of G, the quotient $U = G/Z$ is unipotent, and Z_K is the largest MT-subgroup scheme of G_K for every field extension K/k (and $U_K = G_K/Z_K$ is unipotent), see [$\mathbf{SGA3}_3$], Exp. XVII, Th. 7.2.1 or [**DG70**], §IV.3, Th. 1.1 and Prop. 1.3 a).

Let us prove that Z is a reductive center of G. Let S be a k-scheme and $u : H \to G_S$ a morphism of S-groups, where H is a MT-group over S, say of type M. Composing with the morphism $G_S \to U_S$, whose kernel is Z_S, we want to prove that the morphism $v : H \to U_S$ is the trivial morphism. As $\mathrm{Ker}(v)$ is a MT-group over S, it suffices to prove that its type at each geometric point $\overline{s}$ of S is M. But as $H_{\overline{s}}$ is of multiplicative type and $U_{\overline{s}}$ unipotent, the morphism $v_{\overline{s}}$ is trivial. This proves that Z is a reductive center of G, whence the first assertion of (d).

Finally, denote by π the projection $G \to G' = G/Z$ and let $\overline{k}$ be an algebraic closure of k. Suppose first that T is a maximal torus of G, then $T_{\overline{k}}$ is a maximal torus of $G_{\overline{k}}$ and one knows that $\pi(T_{\overline{k}})$ is a maximal torus of $G'_{\overline{k}}$, see e.g. [**Che04**], §7.3, Th. 3 (a). Since $\pi(T_{\overline{k}}) = \pi(T)_{\overline{k}}$, this proves that $\pi(T)$ is a maximal torus of G'.

Conversely, suppose that T' is a maximal torus of G' and let $T = \pi^{-1}(T')$. Let T_0 be a maximal torus of $G_{\overline{k}}$, then $\pi(T_0)$ is a maximal torus of $G'_{\overline{k}}$, by *loc. cit.* Since all maximal tori of $G'_{\overline{k}}$ are conjugate under $G'(\overline{k}) = \pi(G(\overline{k}))$, there exists $g \in G(\overline{k})$ such that $\pi(gT_0g^{-1}) = T'_{\overline{k}}$. Since T_0 contains Z, this implies that $gT_0g^{-1} = \pi^{-1}(T'_{\overline{k}}) = T_{\overline{k}}$, and this proves that T is a maximal torus of G. ☐

Before deriving a corollary of the previous theorem, let us recall the following known proposition, see [**Bo91**], 11.5, 11.7, 11.12–13, or [**Che04**], §6.5, Cor. 2 and Cor. 4, and §6.7 Th. 6.7.

PROPOSITION 29.6. *Let K be an algebraically closed field and G a connected smooth affine K-group.*

(1) *If S is a torus of G, the centraliser $C(S)$ is connected.*

(2) *For a connected smooth affine K-group H, the following conditions are equivalent:*

(a) *H has a maximal torus T which is central.*

(b) *H has a unique maximal torus T.*

(c) *H is nilpotent.*

In this case, H is the direct product of T and a unipotent group H_u and T is the largest K-subgroup of H of multiplicative type.

(3) *If T is a maximal torus of G, then $C(T)$ is connected and nilpotent.*

In the setting of (2) above note that, by the proof of the first part of assertion (e) in Th. 29.5, T is then the reductive center of H.

COROLLARY 29.7. *Let G be a smooth affine algebraic group over a field k. Let K be an algebraically closed field containing k and suppose that G_K is nilpotent. Then G admits a maximal torus T.*

PROOF. By hypothesis, $G' = G_K$ has a unique maximal torus T' and T' is a reductive center of G'. By assertion (a) of Th. 29.5, T' comes from a MT-subgroup T of G, and T is a maximal torus of G. $\square$

30. Cartan subalgebras over an infinite field

In this section, let k be an *infinite* field and K an algebraically closed field containing k. Let $\mathfrak{g}$ be a finite-dimensional Lie algebra over k.

DEFINITION 30.1 (**Cartan subalgebras**). One says that a Lie subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ is a *Cartan subalgebra* if $\mathfrak{h}$ is nilpotent and equals its own normaliser, that is, $\mathfrak{h} = \{x \in \mathfrak{g} \mid \mathrm{ad}(x)(\mathfrak{h}) \subset \mathfrak{h}\}$.

DEFINITION 30.2. (1) Let $x \in \mathfrak{g}$. Its **nilspace**, denoted $\mathrm{Nil}(x, \mathfrak{g})$ is $\bigcup_{i \in \mathbb{N}} \mathrm{Ker}\, \mathrm{ad}(x)^i$. One says that x is **regular** if $\dim_k \mathrm{Nil}(x, \mathfrak{g})$ is minimal.

(2) Let r be this minimal value, it is called the *nilpotent rank* of $\mathfrak{g}$ and denoted by $\mathrm{rk}_{\mathrm{nil}}(\mathfrak{g})$.

(3) Let T be an indeterminate and $d = \dim_k \mathfrak{g}$. For each $x \in \mathfrak{g}$, one may write

$$\det\big(T - \mathrm{ad}(x)\big) = T^d + c_1(x)T^{n-1} + \cdots + c_d(x)$$

for some polynomial functions $c_1, \ldots, c_d$ on $\mathfrak{g}$, and since k is infinite, these define actual polynomials $c_1, \ldots, c_d \in S(\mathfrak{g}^*)$ and a polynomial $P_{\mathfrak{g}}(T) \in S(\mathfrak{g}^*)[T]$. Then $r = \mathrm{rk}_{\mathrm{nil}}(\mathfrak{g})$ is the largest integer such that $P_{\mathfrak{g}}(T)$ is divisible by T^r, i.e. such that the polynomials c_i are identically 0 for $i > d - r$. In particular, $r = \mathrm{rk}_{\mathrm{nil}}(\mathfrak{g})$ is invariant under any field extension k'/k. Further, the set of regular elements is the dense open set $\mathrm{Reg}(\mathfrak{g})$ defined by the non-vanishing of c_{n-r}, and since k is infinite, we know that regular elements do exist.

PROPOSITION 30.3. *Let $x \in \mathfrak{g}$. Set $\mathfrak{h} = \mathrm{Nil}(x, \mathfrak{g})$.*

(1) *Then $\mathfrak{h}$ is a Lie subalgebra of $\mathfrak{g}$ and equals its own normalizer. Moreover, $\mathrm{ad}(x)_{\mathfrak{g}/\mathfrak{h}}$ is injective.*

(2) *Further, if x is a* regular *element, then $\mathfrak{h}$ is* nilpotent.

PROOF. (1) Set $D = \mathrm{ad}(x)$. Then $\mathfrak{h}$ is the set of elements $y \in \mathfrak{g}$ such that $D^n(y) = 0$ for n large enough (and in fact, for $n \geq d = \dim_k \mathfrak{g}$). Since D is a derivation of $\mathfrak{g}$, that is, $D([y,z]) = [D(y), z] + [y, D(z)]$ for all $y, z \in \mathfrak{g}$, the Leibniz formula tells us that, for every $n \in \mathbb{N}^*$, one has:

$$D^n([y,z]) = \sum_{i=0}^{n} \binom{n}{i} \big[D^i(y), D^{n-i}(z)\big].$$

It follows that if $y, z \in \mathfrak{h}$ then $[y, z] \in \mathfrak{h}$, that is, $\mathfrak{h}$ is a Lie subalgebra of $\mathfrak{g}$.

Next, let $y \in \mathfrak{g}$ such that $[y, \mathfrak{h}] \subset \mathfrak{h}$. Since $x \in \mathfrak{h}$, this gives $D(y) \in \mathfrak{h}$ and hence there exists $n \in \mathbb{N}$ such that $D^{n+1}(y) = 0$. Thus $y \in \mathfrak{h}$. This proves that $\mathfrak{h}$ equals its own normalizer, and also that $\mathrm{ad}(x)_{\mathfrak{g}/\mathfrak{h}}$ is injective (since $D(y) \in \mathfrak{h}$ implies that $y \in \mathfrak{h}$).

(2) By Engel's theorem, it suffices to prove that $\mathrm{ad}(y)_{\mathfrak{h}}$ is nilpotent, for every $y \in \mathfrak{h}$. Set $\mathfrak{h}' = \mathfrak{h}' \otimes_k \overline{k}$ and $\mathfrak{g}' = \mathfrak{g} \otimes_k \overline{k}$. It suffices to prove that for every $y \in \mathfrak{h}'$ the adjoint action of y on $\mathfrak{h}'$ is nilpotent. Since $\mathfrak{h}' = \mathrm{Nil}(x, \mathfrak{g}')$ and $r = \dim_k \mathrm{Nil}(x, \mathfrak{g}) = \mathrm{rk}_{\mathrm{nil}}(\mathfrak{g})$ is invariant under the extension $\overline{k}/k$, we are reduced to the case where $k = \overline{k}$.

Since $\mathrm{ad}(x)_{\mathfrak{g}/\mathfrak{h}}$ is injective, there exists a non-empty open, hence dense, subset U of $\mathfrak{h}$ such that $\mathrm{ad}(y)_{\mathfrak{g}/\mathfrak{h}}$ is injective for $y \in U(k)$. Therefore, for $y \in U(k)$ one has $\mathrm{Nil}(y, \mathfrak{g}) \subset \mathfrak{h}$. Then $V = U \cap \mathrm{Reg}(\mathfrak{g})$ is a dense open subset of $\mathfrak{h}$ and for every $y \in V(k)$ the inclusion $\mathrm{Nil}(y, \mathfrak{g}) \subset \mathfrak{h} = \mathrm{Nil}(x, \mathfrak{g})$ is an equality, since both have dimension r. It follows that $\mathrm{ad}(y)_{\mathfrak{h}}$ is nilpotent, for every $y \in V(k)$, thus the map $\mathfrak{h} \times \mathfrak{h} \to \mathfrak{h}$, $(y, z) \mapsto \mathrm{ad}(y)^r(z)$ is zero on $V(k) \times \mathfrak{h}$ (recall that $r = \dim_k \mathfrak{h}$). Since $V(k)$ is dense in $\mathfrak{h}$, this map is identically zero. Hence, by Engel's theorem, $\mathfrak{h}$ is nilpotent. □

DEFINITION 30.4 (**G-Cartan subalgebras**). We will say that a Lie subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ is a *G-Cartan subalgebra* if $\mathfrak{h} = \mathrm{Nil}(x, \mathfrak{g})$ for a regular element x of $\mathfrak{g}$.

By the previous proposition, $\mathfrak{h}$ is then a nilpotent subalgebra equal to its own normaliser, hence is a Cartan subalgebra in the usual sense.

REMARK 30.5. Note that if $\mathfrak{g}$ itself is nilpotent, then $\mathrm{Nil}(x, \mathfrak{g}) = \mathfrak{g}$ for every $x \in \mathfrak{g}$, hence every $x \in \mathfrak{g}$ is regular and $\mathfrak{g}$ is a G-Cartan subalgebra of itself.

For future use, let us prove here the following proposition.

PROPOSITION 30.6. *Let x be a regular element and $\mathfrak{h}$ a Lie subalgebra of $\mathfrak{g}$ containing x.*

(1) *Then $\mathfrak{h}$ is nilpotent if and only if $\mathfrak{h} \subset \mathrm{Nil}(x, \mathfrak{g})$.*

(2) $\mathrm{Nil}(x, \mathfrak{g})$ *is the unique G-Cartan subalgebra of $\mathfrak{g}$ containing x.*

PROOF. (1) By Prop. 30.3, we know that $\mathrm{Nil}(x, \mathfrak{g})$ is nilpotent, hence the implication "if" is obvious. Conversely, if $\mathfrak{h}$ is nilpotent and $x \in \mathfrak{h}$, then $\mathrm{ad}(x)$ acts nilpotently on $\mathfrak{h}$ and hence $\mathfrak{h} \subset \mathrm{Nil}(x, \mathfrak{g})$.

(2) Let $\mathfrak{h}'$ be another G-Cartan subalgebra of $\mathfrak{g}$ containing x. Then $\mathfrak{h}'$ is nilpotent hence $\mathfrak{h}' \subset \mathrm{Nil}(x, \mathfrak{g})$. But $\dim_k \mathfrak{h}' = \mathrm{rk}_{\mathrm{nil}}(\mathfrak{g}) = \dim_k \mathrm{Nil}(x, \mathfrak{g})$, hence $\mathfrak{h}' = \mathrm{Nil}(x, \mathfrak{g})$. □

Now, a key result is the following theorem, whose proof we postpone to the next lecture.

THEOREM 30.7. *Let $\mathfrak{h} = \mathrm{Nil}(x, \mathfrak{g})$, where x is a regular element of $\mathfrak{g}$. Then:*

(1) *There exists a unique smooth connected closed subgroup H of G such that* $\mathrm{Lie}(H) = \mathfrak{h}$.

(2) *Further, H_K contains a maximal torus of G_K.*

REMARK 30.8. Suppose that G is reductive. One could expect that a G-Cartan subalgebra of $\mathfrak{g}$ is the same thing as the Lie algebra of a maximal torus of G. Alas, this is not true in general: if $\mathrm{char}(k) = 2$ and $G = \mathrm{SL}_2$, then $\mathfrak{g} = \mathrm{Lie}(G)$ is isomorphic to a Heisenberg algebra ($[X, Y] = H$, with H central), hence nilpotent!

However, we will see in the next section that the expected result is true if G is replaced by its adjoint quotient $G' = G/Z$, where Z is the reductive center of G.

31. Existence of maximal tori over an infinite field

In this section, let k be an *infinite* field, G a connected smooth affine k-group and $\mathfrak{g} = \mathrm{Lie}(G)$. Let K be an algebraically closed field containing k.

PROPOSITION 31.1. *Suppose that the reductive center of G is the trivial group. If $\mathfrak{g}$ is nilpotent, then the trivial group is a maximal torus of G.*

PROOF. Let T be a maximal torus of G_K and $\mathfrak{t}$ its Lie algebra. It suffices to prove that T is the trivial group or, equivalently, that $\mathfrak{t} = 0$. Consider the adjoint action of T on $\mathfrak{g}$. By assertion (d) (2) of Th. 29.5, the kernel of $T \to \mathrm{GL}(\mathfrak{g})$ is the reductive center of G_K, which is the trivial group by assumption. Hence $T \to \mathrm{GL}(\mathfrak{g})$ is a closed immersion and hence the Lie algebra map $\mathrm{ad} : \mathfrak{t} \to \mathrm{End}(\mathfrak{g})$ is injective. On the other hand, the adjoint action of T on $\mathfrak{g}$ is semi-simple. Thus, for each $x \in \mathfrak{t}$, $\mathrm{ad}(x)$ is semi-simple, and also nilpotent, as $\mathfrak{g}_K$ is nilpotent, hence $\mathrm{ad}(x) = 0$. It follows that $\mathfrak{t} = 0$. □

THEOREM 31.2. *Let k be an* infinite *field and G a connected smooth affine k-group.*

(a) *G has a maximal torus T.*

(b) *If G is semi-simple of adjoint type, then a G-Cartan subalgebra of $\mathfrak{g}$ is the same thing as the Lie algebra of a maximal torus of G.*

PROOF. (a) We proceed by induction on $n = \dim(G)$. If $n = 0$, then G is the trivial group and there is nothing to prove. So we may assume $n > 0$ and the assertion proved in dimensions $< n$. Using assertions (e) and (c) of Th. 29.5, we may and do assume that the reductive center Z of G is trivial. Let $\mathfrak{h}$ be a G-Cartan subalgebra of $\mathfrak{g}$. By Th. 30.7, there exists a smooth connected closed subgroup H of G such that $\mathrm{Lie}(H) = \mathfrak{h}$ and, further, H_K contains a maximal torus of G_K. There are two cases.

(1) If $\dim(H) = \dim(G)$, then $H = G$ and hence $\mathfrak{g} = \mathfrak{h}$ is nilpotent. Since $Z = \{e\}$, this implies, by the previous proposition, that $\{e\}$ is a maximal torus of G.

(2) If $\dim(H) < \dim(G)$ then, by the inductive hypothesis, there exists in H a maximal torus T. Since H_K contains a maximal torus of G_K, one has $\dim(H) = \mathrm{rk}_{\mathrm{red}}(G)$ and hence T is a maximal torus of G. This completes the proof of assertion (a).

(b) Assume now that G is semi-simple of adjoint type and let $H \supset T$ be as in the proof of assertion (a). We want to prove that $H = T$ and for this it suffices to prove that they have the same dimension. For this, we may replace k by K and we drop the subscript K in the following.

One knows that $\dim(H) = \mathrm{rk}_{\mathrm{nil}}(\mathfrak{g})$ is the minimal possible dimension for a nilspace $\mathrm{Nil}(x, \mathfrak{g}_K)$. On the other hand, for the adjoint action of T on $\mathfrak{g}$ (we are now over K, which is algebraically closed), one has

$$(*) \qquad \mathfrak{g} = \mathfrak{g}^T \oplus \bigoplus_{\alpha \in R} \mathfrak{g}_\alpha,$$

where R is the root system of (G, T). One has $\mathfrak{g}^T = \mathrm{Lie}(T) = \mathfrak{t}$ and every $t \in \mathfrak{t}$ acts on $frakg_\alpha$ by $\mathrm{ad}(t)(x) = \alpha(t)x$, where by abuse of notation we still denote by α the linear form on $\mathfrak{t}$ which is the differential of the character $\alpha : T \to \mathbb{G}_m$. Let Δ be a set of simple roots of R; since G is of adjoint type, Δ is a basis of $\mathfrak{t}^* = X(T) \otimes_{\mathbb{Z}} K$, and since every root is conjugate under the Weyl group to a simple root, it follows α is a non-zero linear form on $\mathfrak{t}$, for every $\alpha \in R$. Since K is infinite, there exists $x \in \mathfrak{t}$ not belonging to any of the hyperplanes $\mathrm{Ker}(\alpha)$, for $\alpha \in R$. Then $(*)$ shows that $\mathrm{Nil}(x, \mathfrak{g}) = \mathfrak{t}$. Hence $\dim(T) = \dim \mathfrak{t}$ is $\geq \mathrm{rk}_{\mathrm{nil}}(\mathfrak{g}) = \dim(H)$. It follows that $H = T$, as desired. ☐

Let us conclude this section with some terminology and remarks.

DEFINITION 31.3 (**Cartan subgroups I**). Let K be an algebraically closed field, G a connected smooth affine K-group. Let T be a maximal torus of G. By Prop. 27.6 of Lect. 11, the centraliser $\underline{\mathrm{Cent}}_G(T)$ is represented by a smooth closed subgroup scheme $C(T)$ of G. One says that $C(T)$ is the *Cartan subgroup* corresponding to T. By Prop. 29.6, $C(T)$ is connected and nilpotent.

DEFINITION 31.4. Let k be a field, G a connected smooth affine k-group and $\mathfrak{g} = \mathrm{Lie}(G)$. A smooth connected closed subgroup H of G such that $\mathrm{Lie}(H)$ is a G-Cartan subalgebra of $\mathfrak{g}$ is a called a *subgroup of type* (C) in [**SGA3**$_2$], XIII, Def. 6.2. We suggest the more informative name *almost-Cartan subgroup.*

REMARKS 31.5. (a) Beware that an almost-Cartan subgroup is not necessarily a Cartan subgroup, as noted in Remark 30.8 (SL_2 over a field of characteristic 2).

(b) With notation as in the previous definition, let H be an almost Cartan subgroup of G and let H_K be an algebraically closed field containing k. We stated in Th. 30.7 and used in Th. 31.2 that H_K contains a maximal torus of G_K. In [**SGA3**$_2$], Grothendieck proves the stronger result that H_K contains a Cartan subgroup: see XIII, Rem. 6.6 (b), which builds on the proof of the implications (vii) $\implies$ (vi) $\implies$ (i) in XIII, Th. 2.1.

32. Proof of the key theorem 30.7.

Let us recall the standing hypotheses: k is an infinite field, G a connected smooth affine k-group, $\mathfrak{g} = \mathrm{Lie}(G)$, and K is an algebraically closed field containing k, and we now assume that the transcendence degree of K over its prime subfield is $\geq d = \mathrm{rk}_{\mathrm{red}}(G)$.

THEOREM 32.1. *Let* $\mathfrak{h} = \mathrm{Nil}(x, \mathfrak{g})$, *where* x *is a regular element of* $\mathfrak{g}$. *Then:*

(1) $\underline{\mathrm{Norm}}_G(\mathfrak{h})$ *is represented by a closed subgroup scheme* N *of* G. *Further,* N *is smooth and* $\mathrm{Lie}(N) = \mathfrak{h}$.

(2) $H = N^0$ *is the unique smooth connected closed subgroup* H *of* G *such that* $\mathrm{Lie}(H) = \mathfrak{h}$. *Moreover,* $\underline{\mathrm{Norm}}_G(H)$ *is represented by* N.

(3) *Further,* H_K *contains a maximal torus of* G_K.

PROOF. By assertion (1) of Cor. 26.3 in Lect. 11, applied to the vector bundles $X = W(\mathfrak{g})$ and $U = V = W(\mathfrak{h})$, we know that $\underline{\mathrm{Norm}}_G(\mathfrak{h})$ is represented by a closed subgroup scheme N of G. One has $\mathrm{Lie}(N) \subset \mathrm{Norm}_{\mathfrak{g}}(\mathfrak{h}) = \mathfrak{h}$.

By abuse of notation, let us write in the sequel $\mathfrak{h}$ and $\mathfrak{g}$ instead of $W(\mathfrak{h})$ and $W(\mathfrak{g})$. Consider the morphism $\varphi : G \times \mathfrak{h} \to \mathfrak{g}$, $(g, y) \mapsto \mathrm{Ad}(g)(y)$. The k-group N acts freely on the right on $G \times \mathfrak{h}$ by $(g, y) \cdot n = (gn, \mathrm{Ad}(n^{-1})(y))$ and the quotient is represented by a scheme $X = G \times^N \mathfrak{h}$ and φ factors a morphism $\psi : X \to \mathfrak{g}$. Denote by q the projection $G \times \mathfrak{h} \to X$.

Since $\mathfrak{h}$ is the unique G-Cartan subalgebra containing x, the fiber of ψ over the rational point $x \in \mathfrak{g}$ is the single point $q(e, x)$. Therefore, one has $\dim X \leq \dim \mathfrak{g}$. On the other hand, the tangent map of φ at the point (e, x) is the map:

$$\mathfrak{g} \times \mathfrak{h} \to \mathfrak{g}, \qquad (y, t) \mapsto t + [y, x].$$

It will be surjective if the map $\mathrm{ad}(x)_{\mathfrak{g}/\mathfrak{h}}$ is surjective, which is the case because this map is injective by Prop. 30.3. Thus, φ is smooth at (e, x) and hence its image, which is that of ψ, contains a non-empty open subset of $\mathfrak{g}$. Hence, $\dim X = \dim \mathfrak{g}$. Since $\dim X = \dim G - \dim N + \dim \mathfrak{h}$ and $\dim G = \dim \mathfrak{g}$ since G is smooth, we obtain $\dim N = \dim \mathfrak{h}$. Combined with the previously seen inclusion $\mathrm{Lie}(N) \subset \mathfrak{h}$, this gives that N is smooth and $\mathrm{Lie}(N) = \mathfrak{h}$. This completes the proof of assertion (1).

(2) Set $H = N^0$, then H is smooth and $\mathrm{Lie}(H) = \mathfrak{h}$. Suppose that H' is another connected smooth subgroup such that $\mathrm{Lie}(H') = \mathfrak{h}$. Since the adjoint action of H' stabilizes its Lie algebra $\mathfrak{h}$, we have $H' \subset \mathrm{Norm}_G(\mathfrak{h}) = N$ whence $H' \subset N^0 = H$ since H' is connected, hence $H' = H$ since both are smooth and connected of the same dimension. Moreover one has $\underline{\mathrm{Norm}}_G(H) \subset \underline{\mathrm{Norm}}_G(\mathfrak{h}) = N$, and N normalizes $N^0 = H$, hence $\underline{\mathrm{Norm}}_G(H) = N$. This proves assertion (2).

(3) We now extend scalars to K and omit the subscripts. We know that $\mathrm{ad}(x)_{\mathfrak{g}/\mathfrak{h}}$ is injective (and hence the set of $y \in \mathfrak{g}$ having this property is open and non-empty, hence dense). It follows that there exists $h_1 \in H(K)$ such that $\mathrm{Ad}(h_1)$ has no fixed point $\neq 0$ in $\mathfrak{g}/\mathfrak{h}$. Indeed, otherwise the regular function $H \to \mathbb{G}_a$ given by $h \mapsto \det_{\mathfrak{g}/\mathfrak{h}}(\mathrm{Id} - \mathrm{Ad}(h))$ is identically zero hence vanishes on every infinitesimal point $e + \varepsilon y$, which would give $\det_{\mathfrak{g}/\mathfrak{h}}(\mathrm{ad}(y)) = 0$ for all $y \in \mathfrak{h}$.

Consider now the morphism $\varphi : G \times H \to G$, $(g, h) \mapsto ghg^{-1}$, as before, it factors through a morphism $\psi : X \to G$, where $X = G \times^N H$. The tangent map to φ at the point (e, h_1) is the map

$$\mathfrak{g} \times \mathfrak{h} \to \mathfrak{g}, \qquad (y, t) \mapsto t + (\mathrm{Id} - \mathrm{Ad}(h_1))(y).$$

It will be surjective if the map $(\mathrm{Id} - \mathrm{Ad}(h_1))_{\mathfrak{g}/\mathfrak{h}}$ is surjective, which is the case because this map is injective by the foregoing discussion. Thus, φ is smooth at (e, h_1) and hence its image, which is that of ψ, contains a non-empty open subset of G. Since $\dim X = \dim G - \dim N + \dim H = \dim G$, one obtains that ψ is dominant and generically finite.

It follows that G contains a dense open set U such that each $g \in U(K)$ is contained in a conjugate of H (and in fact in finitely many of them). Let T be a maximal torus of G, necessarily split since K is algebraically closed. By the classical theory (Borel's density theorem), the union of the $G(K)$-conjugates of T is also dense in G. Hence it meets U hence, replacing T by a conjugate, we may assume that $T \cap U$ is not empty, hence it is a dense open set V of T such that each $t \in V(K)$ is contained in some $G(K)$-conjugate of H. Denote by Y the reduced closed subscheme $T - V$.

We assumed that the transcendence degree of K over its prime field K_0 is $\geq d = \dim T$, hence $T(K)$ contains elements $t = (t_1, \ldots, t_d)$ with the t_i algebraically independent over K_0. The set of all such elements is clearly dense in T, hence they cannot all lie in $Y(K)$. Thus, we can find such an element $t = (t_1, \ldots, t_d)$ in $V(K)$. It is not contained in any proper closed subgroup of T since it satisfies no equation $t_1^{n_1} \cdots t_d^{n_d} = 1$ with the $n_i \in \mathbb{Z}$ not all zero. On the other hand, since $t \in V(K)$ there exists $g \in G(K)$ such that t is contained in gHg^{-1}, hence in $T \cap gHg^{-1}$, which is a closed subgroup of G. It follows that $T \subset gHg^{-1}$ and hence H contains the maximal torus $g^{-1}Tg$. This completes the proof of assertion (3). □

Notes for this Lecture

The results about the reductive center are taken from [**SGA3**$_2$], XII, §4.

Cartan subalgebras (nilpotent Lie subalgebras which equal their own normaliser) go back to the works of Élie Cartan and W. Killing. We have chosen to use *G-Cartan subalgebra* for the notion introduced by Grothendieck in [**SGA3**$_2$], Exp. XIII, Def. 4.6.0, after he had mentioned in Exp. XIII, Prop. 4.4 that if $\mathfrak{g}$ is the Lie algebra of a smooth affine k-group, the two definitions coincide, a fact proved in Exp. XIII, Cor. 5.7 (a).

Assertion (1) of Theorem 30.7 is proved in XIII, Th. 6.1 and Cor. 6.3, whereas assertion (2) is mentioned in Rem. 6.6 (b).

Prop. 31.1 is proved in XIV, Lemma 1.3 together with XII, Cor. 4.9. Finally, assertion (a) of Theorem 31.2 is proved in XIV, Th. 1.1, whereas assertion (b) is contained in XIV, Th. 3.18.

The idea to use an algebraically field K with sufficiently large transcendence degree occurs already in Grothendieck's presentation of Borel's density theorem in [**Che04**], Exp. 6, §6.6, proof of Th. 6.