LECTURE 11

Representability of centralisers and transporters

Special Thanks. *I thank heartily Professor Madhavan Mukund, who arranged for a replacement laptop for me on the very day when my laptop broke down (Saturday, January 13). I also thank Rajarshi Gosh for giving me the tip that I could use Overleaf.*

25. Weil restriction

Let us fix a morphism of schemes $Z \to S$.

Definition 25.1. For any Z-scheme Y, its **Weil restriction of scalars** from Z to S, denoted by $\mathcal{R}_{Z/S}(Y)$ (or sometimes $\prod_{Z/S} Y$) is the contravariant functor from $(\mathrm{Sch}_{/S})$ to (Sets) such that, for every S-scheme T:

$$\mathrm{Hom}_S\big(T, \mathcal{R}_{Z/S}(Y)\big) = \mathrm{Hom}_Z(T \times_S Z, Y).$$

Remark 25.2. Note that $\mathcal{R}_{Z/S}(Y)$ is a sheaf for the fpqc topology. Indeed, let $T' \to T$ be a Zariski covering or a faithfully flat and quasi-compact morphism. Setting as usual $T'' = T' \times_T T'$, one has a commutative diagram:

$$\begin{array}{ccccc} \mathrm{Hom}_S(T, \mathcal{R}_{Z/S}(Y)) & \longrightarrow & \mathrm{Hom}_S(T', \mathcal{R}_{Z/S}(Y)) & \rightrightarrows & \mathrm{Hom}_S(T'', \mathcal{R}_{Z/S}(Y)) \\ \| & & \| & & \| \\ \mathrm{Hom}_Z(T \times_S Z, Y) & \longrightarrow & \mathrm{Hom}_Z(T' \times_S Z, Y) & \rightrightarrows & \mathrm{Hom}_Z(T'' \times_S Z, Y). \end{array} \tag{25.1}$$

Since the morphism $Z_{T'} \to Z_T$ obtained by base change is again a Zariski covering or faithfully flat and quasi-compact and since $Z_{T'} \times_{Z_T} Z_{T'} \simeq Z \times_S T''$, the second row is exact and hence so is the first row.

Remark 25.3. If $S = \mathrm{Spec}(A)$ and $Z = \mathrm{Spec}(B)$, where B is a finite free A-module of rank d, it is easy to see that the functor $\mathcal{R}_{Z/S}(\mathbb{A}^n_Z)$ is represented by the S-scheme $\mathbb{A}^{nd}_S$. Indeed, let $(e_1, \dots, e_d)$ be a basis of B as A-module. Since $\mathbb{A}^n_Z = \mathbb{A}^n_S$ one sees that a point of $\mathcal{R}_{Z/S}(\mathbb{A}^n_Z)$ with values in $\mathrm{Spec}(R)$, for an arbitrary A-algebra R, is the same thing as a n-tuple $(x_1, \dots, x_n)$ of elements of $R \otimes_A B \simeq \bigoplus_1^d Re_i$, hence a nd-tuple $(r_{1,1}, \dots, r_{1,d}, \dots, r_{n,d})$ of elements of R. [1].

Here, we will be interested in the case where Y is a *closed subscheme* of Z, with applications to the representability of centralisers and transporters, see below. Let us start with the following definition.

Definition 25.4. One says that the S-scheme Z is **essentially free** if there exists a covering (S_i) of S by affine open subsets S_i and for each i an affine and faithfully flat morphism $S'_i \to S_i$ such that $Z'_i = Z \times_S S'_i$ is covered by affine open subsets Z'_{ij} such that every $\mathcal{O}(Z'_{ij})$ is a projective module over $\mathcal{O}(S'_i)$.

[0]version of Jan. 22, 2024. Thanks to Manoj Kummimi for pointing out an inaccuracy in the previous version

[1]For further general results about Weil restriction, see e.g. [**BLR**], §7.6.

EXAMPLE 25.5. If H is a S-group of multiplicative type, it is essentially free over S. Indeed, by assumption there exists a covering $(S_i)_{i\in I}$ of S by affine open subsets S_i and for each i an affine and faithfully flat morphism $S'_i \to S_i$ such that $H'_i = H \times_S S'_i$ is a diagonalisable group over S'_i of type M_i, for some abelian group M_i. Then $\mathcal{O}(H'_i) = \mathcal{O}(S'_i)[M_i]$ is a free $\mathcal{O}(S'_i)$-module.

LEMMA 25.6. (a) *If Z is essentially free over S, it is flat over S.*

(b) *If $S = \operatorname{Spec}(k)$, where k is a field, every S-scheme is essentially free.*

(c) *If Z is essentially free over S, then for any base-change morphism $X \to S$, the morphism $Z \times_S X \to X$ is essentially free.*

PROOF. Left to the reader. □

REMARK 25.7. Suppose that $\tau : Y \hookrightarrow Z$ is a closed immersion. Then, for any S-scheme T one has:

$$\mathcal{R}_{Z/S}(Y)(T) = \begin{cases} \{\tau_T^{-1}\} & \text{if } \tau_T : Y_T \hookrightarrow Z_T \text{ is an isomorphism,} \\ \varnothing & \text{otherwise.} \end{cases}$$

Indeed, $\mathcal{R}_{Z/S}(Y)(T) = \operatorname{Hom}_Z(Z \times_S T, Y) = \operatorname{Hom}_{Z_T}(Z_T, Y_T)$ is the set of morphisms $f : Z_T \to Y_T$ such that $\tau_T \circ f = \operatorname{id}_{Z_T}$. Since τ_T is a closed immersion, such an f exists if and only if τ_T is an isomorphism, and then $f = \tau_T^{-1}$.

THEOREM 25.8. *Suppose that $Y \hookrightarrow Z$ is a closed immersion and that Z is essentially free over S.*

(i) *Then $\mathcal{R}_{Z/S}(Y)$ is represented by a closed subscheme C of S.*

(ii) *Further, if $Z \to S$ is quasi-compact and $Y \hookrightarrow Z$ is of finite presentation, then $C \hookrightarrow S$ is of finite presentation.*

PROOF. Set $F = \mathcal{R}_{Z/S}(Y)$. The proof is in four steps.

(1) Suppose firstly that $S = \operatorname{Spec}(A)$ and $Z = \operatorname{Spec}(B)$, where B is a projective A-module. Hence B is a direct summand of a free A-module L with basis $(e_\lambda)_{\lambda\in\Lambda}$. Let $\varphi_\lambda : L \to A$ be the coordinate forms with respect to this basis. Let E be a set of generators of the ideal J of B defining the closed subscheme $Y \subset Z$ and let I be the ideal of A generated by the $\varphi_\lambda(x)$, for $x \in E$ and $\lambda \in \Lambda$.

Now, let $T \to S$ be a morphism such that the closed immersion $Y_T \to Z_T$ is an isomorphism. Then, for any affine open subset $T' = \operatorname{Spec}(R)$ of T, one has a morphism of rings $f : A \to R$ and one obtains that the surjective morphism $B \otimes_A R \to (B/J) \otimes_A R$ is an isomorphism, which amounts to saying that for any $x \in E$ the image of $x \otimes 1$ in $B \otimes_A R$ or, equivalently, in $L \otimes_A R$ is zero. Since $x = \sum_\lambda \varphi_\lambda(x) e_\lambda$, the latter image is $\sum_\lambda e_\lambda \otimes f(\varphi_\lambda(x))$ and this is zero if and only if $f(\varphi_\lambda(x)) = 0$. Thus $\operatorname{Ker}(f)$ contains I and hence $T' \to S$ factors through the closed subscheme $C = \mathcal{V}(I)$. Since this is true for any open affine subset of T, one obtains thats $T \to S$ factors through C. Conversely, under this condition one has $Y_T = Z_T$. This proves the first assertion. Further, if J is finitely generated we may take E to be finite and as each $x \in E$ as only finitely many non-zero coordinates $\varphi_\lambda(x)$, it follows that I is finitely generated.

(2) Still with $S = \operatorname{Spec}(A)$, suppose now that Z is covered by affine open subsets Z_j such that each $B_j = \mathcal{O}(Z_j)$ is a projective A-module. For each j, set $Y_j = Y \cap Z_j$ and let the ideals $J_j \subset B_j$ and $I_j \subset A$ be defined as above. Then, for any S-scheme T, the base change of $Y \to Z$ is an isomorphism if and only if the same is true for each $Y_j \to Z_j$. It follows that F is represented by the intersection C of the closed subschemes $C_j = \mathcal{V}(I_j)$, defined by the ideal $I = \sum_j I_j$. Assume further that $Z \to S$ is quasi-compact, then Z is quasi-compact hence can be covered by finitely many open subsets Z_j. Therefore, if the closed immersion $Y \hookrightarrow Z$ is of finite presentation, so is the closed immersion $C \hookrightarrow S$.

(3) Suppose now that $S = \operatorname{Spec}(A)$ and there exists an affine and faithfully flat morphism $S' \to S$ such that $Z' = Z \times_S S'$ is covered by affine open subsets Z'_j such that every $\mathcal{O}(Z'_j)$

is a projective module over $\mathcal{O}(S')$. Then, by the previous step, $F_{S'}$ is represented by a closed subscheme C' of S'. It is endowed with a descent datum relative to S'/S (because $F_{S'}$ is) and, by [**SGA1**], Exp. VIII, Cor. 1.9, C' descends to a closed subscheme C of S; moreover, since F is a fpqc sheaf, C represents F (see the proof of Prop. 10.7 in Lecture 5). Further, if $Z \to S$ is quasi-compact and $Y \hookrightarrow Z$ is of finite presentation, then $C' \hookrightarrow S'$ is of finite presentation and hence so is $C \hookrightarrow S$, by [**EGA**] IV_2, Prop. 2.7.1.

(4) Finally, in the general case, with the notation of Def. 25.4, each functor $F_i = F \times_S S_i$ is represented by a closed subscheme C_i of S_i. Since F is a local functor, the C_i glue together to give a closed subscheme C of S, which represents F (see the proof of Lemma 10.2 in Lect. 5). Further, if $Z \to S$ is quasi-compact and $Y \hookrightarrow Z$ is of finite presentation, then each $C_i \hookrightarrow S_i$ is of finite presentation, hence $C \hookrightarrow S$ is locally of presentation, and being a closed immersion (hence quasi-compact and separated), it is of finite presentation. ☐

26. Transporters and normalisers

26.1. A first consequence of theorem 25.8. Let S be a base scheme and let G, U, X denote S-schemes, with U *essentially free* over S.

PROPOSITION 26.1. *Let $P = G \times_S U$, let $a : P \to X$ be a S-morphism, let V be a closed subscheme of X and let $P' = P \times_X V$. Then $\mathcal{R}_{P/G}(P')$ is represented by a closed subscheme G' of G and for any S-scheme T one has:*

$$G'(T) = \{g \in G(T) \mid \text{ the morphism } a \circ (g \times \mathrm{id}_U) : T \times_S U \to X \text{ factors through } V\}.$$

Further, if $U \to S$ is quasi-compact and $V \to X$ is of finite presentation, then $G' \to G$ is of finite presentation.

PROOF. Since P' is a closed subscheme of P and $P = G \times_S U$ is essentially free over G, the first assertion follows from Theorem 25.8 applied to $S = G$, $Z = P$ and $Y = P'$. For any S-scheme T one has, by Remark 25.7,

$$G'(T) = \{g \in G(T) \mid T \times_G P' = T \times_G P, \quad \text{where } T \text{ is over } G \text{ via } g : T \to G\}.$$

Since $T \times_G P = T \times_S U$ and $T \times_G P' = T \times_S U \times_X V$, one obtains that $G'(T)$ is the set of those $g \in G(T)$ such that $a \circ (g \times \mathrm{id}_U)(T \times_S U) \subset V$.

Further, the second assertion follows from assertion (ii) of Th. 25.8. ☐

26.2. Transporters and normalisers. Suppose now that G is a S-group scheme acting on a S-scheme X. Let U, V be subschemes of X.

DEFINITION 26.2. (1) The *transporter* of U in V, denoted $\underline{\mathrm{Tran}}_G(U, V)$, is the subfunctor of G whose T-points are those $g \in G(T)$ such that $g(U_T) \subset V_T$.

(2) The *strict transporter* of U to V, denoted $\underline{\mathrm{Transt}}_G(U, V)$, is the subfunctor of G whose T-points are those $g \in G(T)$ such that $g(U_T) = V_T$.

(3) Denoting by $\phi : G \to G \times_S G$ the morphism sending an arbitrary T-point g to (g, g^{-1}), one sees that $\underline{\mathrm{Transt}}_G(U, V)$ is the inverse image by ϕ of $\underline{\mathrm{Tran}}_G(U, V) \times_S \underline{\mathrm{Tran}}_G(V, U)$. Therefore, if both $\underline{\mathrm{Tran}}_G(U, V)$ and $\underline{\mathrm{Tran}}_G(V, U)$ are represented by closed subschemes of G, so is $\underline{\mathrm{Transt}}_G(U, V)$.

Then one deduces from Prop. 26.1 the following corollary.

COROLLARY 26.3. *Let G be a S-group scheme acting on a S-scheme X and let U, V be subschemes of X.*

(1) *If V is a closed subscheme and U is essentially free over S, then $\underline{\mathrm{Tran}}_G(U, V)$ is represented by a closed subscheme of G.*

(2) *If U, V are closed subschemes and are essentially free over S, then $\underline{\mathrm{Transt}}_G(U,V)$ is represented by a closed subscheme of G.*

For $X = G$, on which G acts by conjugation, one obtains:

(3) *If H is a closed subscheme of G, essentially free over S, then $\underline{\mathrm{Norm}}_G(H)$ is represented by a closed subgroup scheme of G.*

(4) *In particular, if H is a closed S-subgroup scheme of multiplicative type, then $\underline{\mathrm{Norm}}_G(H)$ is represented by a closed subgroup scheme of G.*

Proof. Assertion (4) follows from assertion (3) since any S-group of multiplicative type is essentially free over S (Example 25.5). □

27. Kernels, fixed points and centralisers

27.1. A second consequence of theorem 25.8. Let S be a base scheme and let G, X, Y be S-schemes.

Proposition 27.1. *Suppose that $X \to S$ is* essentially free *and $Y \to S$* separated. *Let $P = G \times_S X$, let a_1, a_2 be two S-morphisms $P \to Y$, let $a = a_1 \times a_2$ be the corresponding morphism $P \to W = Y \times_S Y$ and let P' be the pull-back of the diagonal of W by a. Then $\mathcal{R}_{P/G}(P')$ is represented by a closed subscheme G' of G and for any S-scheme T one has:*

$$G'(T) = \{g \in G(T) \mid \text{ the morphisms } a_i \circ (g \times \mathrm{id}_X) : T \times_S X \to Y \text{ coincide, for } i = 1, 2\}.$$

Further, if $X \to S$ is quasi-compact and $Y \to S$ is locally of finite type, the closed immersion $G' \hookrightarrow G$ is of finite presentation.

Proof. As before, $P = G \times_S X$ is essentially free over G. Since $Y \to S$ is separated, the diagonal $\Delta_{Y/S}$ of $W = Y \times_S Y$ is a closed subscheme and hence P' is a closed subscheme of P. Thus, the first assertion follows from Theorem 25.8 applied to $S = G$, $Z = P$ and the closed subscheme P' of Z. For any S-scheme T one has, by Remark 25.7,

$$G'(T) = \{g \in G(T) \mid T \times_G P' = T \times_G P, \quad \text{where } T \text{ is over } G \text{ via } g : T \to G\}.$$

Since $T \times_G P = T \times_S X$ and $T \times_G P' = T \times_S X \times_W \Delta_{Y/S}$, one obtains that $G'(T)$ is the set of those $g \in G(T)$ such that $a \circ (g \times \mathrm{id}_X)(T \times_S X) \subset \Delta_{Y/S}$, i.e. such that $a_i \circ (g \times \mathrm{id}_X)$ coincide on $T \times_S X$, for $i = 1, 2$.

Finally, if $Y \to S$ is separated and locally of finite type, the closed immersion $\Delta_{Y/S} \hookrightarrow Y \times_S Y$ is of finite presentation (see [**EGA**] IV_1, Cor. 1.4.3.1), hence the second assertion follows from assertion (ii) of Th. 25.8. □

27.2. Kernels, fixed points. Suppose now that G is a S-group scheme acting on the S-scheme X. Let us now derive from Prop. 27.1 the following results about the kernel and fixed points of an action.

Corollary 27.2. *If X is essentially free and separated over S, then the kernel of the action of G on X is represented by a closed S-subgroup scheme K of G.*

If further $X \to S$ is of finite type, the closed immersion $K \hookrightarrow G$ is of finite presentation.

Proof. One applies Prop. 27.1 to the given action $a_1 : G \times_S X \to X$ of G on X and to the trivial action $a_2 : (g, x) \mapsto x$. (For the second assertion, recall that finite type = quasi-compact and locally of finite type.) □

Next, replacing (G, X, Y) in Prop. 27.1 by (X, G, X) in that order, that is, applying Prop. 27.1 to the maps $a_1 : X \times_S G \to X$, $(x, g) \mapsto gx$ and $a_2 : (x, g) \mapsto x$, one obtains the:

COROLLARY 27.3. *If G is essentially free over S and X separated over S, the subfunctor of invariants X^G is represented by a closed subscheme of X.*

If further $G \to S$ is quasi-compact and $X \to S$ locally of finite type, the closed immersion $X^G \hookrightarrow X$ is of finite presentation.

PROOF. Since G is essentially free over S then $P = X \times_S G$ is essentially free over X. And since $X \to S$ is separated, the diagonal $\Delta_{X/S}$ of $W = X \times_S X$ is a closed subscheme and hence P' is a closed subscheme of P. Thus, the first assertion follows from Theorem 25.8 applied to $S = X$, $Z = P$ and $Y = P'$. For any S-scheme T one has, by Remark 25.7,

$$X^G(T) = \{x \in X(T) \mid T \times_X P' = T \times_X P, \quad \text{where } T \text{ is over } X \text{ via } x : T \to X\}.$$

Since $T \times_X P = T \times_S G$ and $T \times_X P' = T \times_S G \times_W \Delta_{X/S}$, one obtains that $X^G(T)$ is the set of those $x \in X(T)$ such that $a_i \circ (x \times \mathrm{id}_G)$ coincide on $T \times_S G$, for $i = 1, 2$, which amounts to saying that for every $T' \to T$ and $g \in G(T')$, one has $gx_{T'} = x_{T'}$.

Further, the last assertion follows from the last assertion of Prop. 27.1. □

27.3. Centralisers. Let G be a S-group scheme and X a S-scheme. Suppose given two morphisms of S-schemes $u, v : X \to G$.

DEFINITION 27.4. The *transporter* of u into v, denoted by $\underline{\mathrm{Tran}}_G(u, v)$, is the S-subfunctor of G defined as follows. Note first that for any S-scheme T and $g \in G(T)$, conjugation by g defines an automorphism $\mathrm{int}(g)$ of $G_T = G \times_S T$. Then, for every S-scheme T, one has:

$$\underline{\mathrm{Tran}}_G(u,v)(T) = \left\{ g \in G(T) \;\middle|\; \begin{array}{l} \text{the morphisms } \mathrm{int}(g) \circ u_T,\ v_T : X_T \to G_T \text{ coincide,} \\ \text{i.e. , for every } T' \to T \text{ and } x \in X(T') \text{ one has } gu(x)g^{-1} = v(x). \end{array} \right\}$$

When $v = u$, it is a S-subgroup functor, denoted by $\underline{\mathrm{Cent}}_G(u)$ and called the *centraliser* of u.

Further, if X is a subscheme of G one sets $\underline{\mathrm{Cent}}_G(X) = \underline{\mathrm{Cent}}_G(u)$, where u denotes the immersion $X \hookrightarrow G$.

Then, setting $P = G \times_S X$ and considering the S-morphisms $a_1 : P \to G$, $(g, x) \mapsto gu(x)g^{-1}$ and $a_2 : (g, x) \mapsto v(x)$, we can use Prop. 27.1 where (G, X, Y) in that proposition is replaced by (G, X, G). Thus, we obtain:

COROLLARY 27.5. *Suppose that $G \to S$ is separated and that X is essentially free over S.*

(1) *$\underline{\mathrm{Tran}}_G(u, v)$ is represented by a closed subscheme $T_G(u, v)$ of G. If further $X \to S$ is quasi-compact and $G \to S$ locally of finite type, the closed immersion $T_G(u, v) \hookrightarrow G$ is of finite presentation.*

(2) *When $u = v$, $\underline{\mathrm{Cent}}_G(u)$ is represented by a closed* subgroup *scheme $C_G(u)$ of G.*

(3) *If $G \to S$ is locally of finite type and if H is a subgroup scheme of multiplicative type, the closed immersion $C_G(H) \hookrightarrow G$ is of finite presentation.*

PROOF. (1) and (2) follow from Prop. 27.1. If H is a subgroup scheme of multiplicative type, then $H \to S$ is essentially free (Example 25.5) and affine, hence quasi-compact. Thus (3) follows from (1) and (2). □

PROPOSITION 27.6. *Suppose that G is a smooth affine S-group and H is a subgroup scheme of multiplicative type.*

(1) *Then the closed subgroup scheme $C = C_G(H)$ is smooth and affine.*

(2) *Further,* $\mathrm{Lie}(C)$ *is the submodule* $\mathrm{Lie}(G)^H$ *of H-invariants in* $\mathrm{Lie}(G)$.

PROOF. (1) By the previous corollary, we know already that $C \to S$ is of finite presentation and affine over S (being closed in G), so it suffices to see that the functor $\underline{\mathrm{Cent}}_G(H)$ is formally smooth. Denoting by u the immersion $H \hookrightarrow G$, one has $\underline{\mathrm{Cent}}_G(H) = \underline{\mathrm{Cent}}_G(u)$. Let $S' =$

$\operatorname{Spec}(A)$ be an affine scheme over S, let I be a nilpotent ideal of A and $S'_0 = \operatorname{Spec}(A/I)$. Let u_0 denote the pull-back of u to S'_0 and let $z \in C(S'_0)$. That is, z is an element of $G(S'_0)$ such that $\operatorname{int}(z) \circ u_0 = u_0$. Since G is smooth, z lifts to an element $x \in G(S')$. Set $v = \operatorname{int}(x) \circ u$, then $v_0 = u_0$. By Th. 17.1 in Lect. 7, there exists $g \in \operatorname{Ker}(G(S') \to G(S'_0))$ such that $\operatorname{int}(g) \circ v = u$. Set $y = gx$, then $\operatorname{int}(y) \circ u = u$ hence $y \in C(S')$, and the image of y in $G(S'_0)$ is z. This proves that C is smooth.

Proof of (2) to be added. This is a general result, see [**SGA3**$_1$], Exp. II, Th. 5.2.3 (ii). □

Notes for this Lecture

The content of this lecture appears in Exp. VIII, §6, n^{os} 6.1 to 6.5 of [**SGA3**$_2$] and has also been reproduced in the new edition of [**SGA3**$_1$], Exp. VI$_B$, n^{os} 6.2.1 to 6.2.5, following a footnote by Grothendieck at the beginning of Exp. VIII, §6: "*The natural place for this paragraph would be in Exp. VI$_B$*".

The assertion (ii) of Th. 25.8 (and the similar assertions in all subsequent results) was not in [**SGA3**$_2$] and was added by the lecturer in [**SGA3**$_1$], Exp. VI$_B$, where the hypothesis that $Z \to S$ be quasi-compact has been overlooked, unfortunately.

The representability of Weil restrictions is also discussed in [**BLR**], §7.6, where a result similar to Th. 25.8 is proved under the more restrictive hypothesis that $Z \to S$ be finite and locally free.

Proposition 27.6 is proved several times in SGA3. In Exp. XI, Cor. 5.3 (a), it is derived from the (hard) result that the functor of homomorphisms of S-groups $H \to G$ is representable. A simpler proof is given in Exp. XI, Th. 6.2 (iii), see also Cor. 9.8 in the additional section XII.9 in the new edition. Finally, the direct proof given above is taken from Exp. XIX, proof of Prop. 6.1.