Quasi-isotriviality of MT-groups of finite type

22. The spreading theorem over a complete noetherian local ring

LEMMA 22.1. *Let A be a noetherian ring, with an ideal I such that A is separated and complete for the I-adic topology. Set $S = \operatorname{Spec}(A)$ and $S_0 = \operatorname{Spec}(A/I)$.*

(1) *Every maximal ideal of A contains I.*

(2) *Therefore, if U is an open subset of S containing S_0, then $U = S$.*

PROOF. (1) Let $x \in I$. For every $a \in A$ the element $1 - ax$ is invertible, its inverse being $1+\sum_{n\geq 1}(ax)^n$ (this sum converges since $(ax)^n \in I^n$ and A is I-adically complete). Now suppose that there exists a maximal ideal $\mathfrak{m}$ such that $x \notin \mathfrak{m}$, then there exists $y \in A$ and $z \in \mathfrak{m}$ such that $yx = 1 - z$, hence $1 - yx = z$ belongs to $\mathfrak{m}$, contradicting the fact that $1 - yx$ is invertible. This proves (1).

(2) The complement of U is a closed set $V(J) = \{P \in \operatorname{Spec}(A) \mid P \supset J\}$. If it is not empty (i.e. if J is a proper ideal), it contains a maximal ideal $\mathfrak{m}$, which is impossible since all maximal ideals belong to $V(I) = S_0$. Thus $U = S$. □

THEOREM 22.2. *Let A be a noetherian ring, with an ideal I such that A is separated and complete for the I-adic topology. Set $S = \operatorname{Spec}(A)$ and $S_0 = \operatorname{Spec}(A/I)$*

(1) *The functor $H \mapsto H_0 = H \times_S S_0$ is an* equivalence of categories*:*

$$\left\{ \begin{matrix} \textit{isotrivial } \mathrm{MT}\textit{-groups} \\ \textit{over } S \end{matrix} \right\} \xrightarrow{\sim} \left\{ \begin{matrix} \textit{isotrivial } \mathrm{MT}\textit{-groups} \\ \textit{over } S_0 \end{matrix} \right\}$$

Now, let G be a finite type affine S-group, flat over S at each point of G_0, and such that G_0 is an isotrivial MT*-group over S_0.*

(2) *There exists a finite type isotrivial* MT*-group H over S and a morphism of S-groups $u : H \to G$ such that $u_0 : H_0 \to G_0$ is an isomorphism.*

(3) *If one assumes further that G is a* MT*-group over S then u is an isomorphism; hence the hypothesis that G_0 be isotrivial implies that G is so.*

(4) *In general, u is an open and closed immersion.*

PROOF. (1)[1] By assertion (2) of Th. 19.2 we know already that this functor is fully faithful. Now, let H_0 be an isotrivial MT-group over S_0. The proof that there exists an isotrivial MT-group H over S such that $H \times_S S_0 \simeq H_0$ is similar to that of assertion (2) of Prop. 18.1. Namely, there exists a surjective *finite étale* morphism $S_0' \to S_0$ such that the pullback H_0' is a diagonalisable group $D(M)_{S_0'}$. By [**EGA**] IV$_4$, Prop. 18.3.2, the functor $X \longmapsto X_0 = X \times_S S_0$ is an equivalence between the category of schemes *finite and étale* over S and the corresponding one over S_0. Thus, there exists a surjective finite étale morphism $S' \to S$ such that $S_0' = S' \times_S S_0$. Then $H' = D(M)_{S'}$ is such that $H' \times_{S'} S_0' = H_0'$.

Next, one obtains as in the proof of assertion (2) of Th. 18.1 that the descent datum on H_0' relative to $S_0' \to S_0$ comes from a descent datum on H' relative to $S' \to S$. Since H' is

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[1]A shorter proof of assertion (1) is given in Lemma 23.10below.

affine over S', this descent datum is effective, hence there exists a S-group scheme H such that $H \times_S S' = H' = D_{S'}(M)$, and hence H is an isotrivial MT-group over S. Further, $H \times_S S_0$ and H_0 become isomorphic over S'_0, hence they are isomorphic because $S'_0 \to S_0$ is a morphism of descent. This completes the proof of assertion (1).

Now, let G be a finite type affine S-group, flat over S at each point of G_0, and such that G_0 is an isotrivial MT-group over S_0. Let M be the abelian group corresponding to G_0; it is finitely generated since G, hence G_0, is of finite type.

By assertion (1), there exists an isotrivial MT-group H over S and an isomorphism of S_0-groups $u_0 : H_0 \xrightarrow{\sim} G_0$. By assertion (2) of Th. 19.2 we know that u_0 lifts uniquely to a morphism of S-groups $u : H \longrightarrow G$.

On the other hand, since the type of the fibers H_s is a *locally constant function* of s, there exists an open subset U of S containing S_0 such that H_U is of type M. By Lemma 22.1, the only open subset of S containing S_0 is S itself. Thus, H is of type M over S, in particular it is of finite type. This proves (2).

(3) Assume further that G is a MT-group over S. The same reasoning as above, applied to G instead of H, shows that G is of type M over S, in particular it is of finite type. Then, by Prop. 7.10 of Lecture 3, $K = \mathrm{Ker}(u)$ and $C = \mathrm{Coker}(u)$ are MT-groups over S, hence the type of their fibers is again a locally constant function of s. Since u_0 is an isomorphism, the type of K and of C is the trivial abelian group $\{0\}$ over S_0 and hence over S. Thus K and C are trivial and hence u is an isomorphism. This proves (3)

Let us prove (4). Let $S' \to S$ be a finite étale map such that $H' \simeq D(M)_{S'}$. It suffices to prove that u' is an open and closed immersion, because then u will be so ([**EGA**] IV_2, Prop. 2.7.1). So, replacing S by S', we may assume that $H = D(M)_S$.

Let us denote by u_n the pullback over S_n of the morphism $u : H \to G$. By assumption, u_n is an isomorphism; in particular it is flat. By the local criterion of flatness (see e.g. [**EGA**], IV_3, Lemma 11.3.10.2 or [**Mat86**], Th. 22.3) it follows that u is flat at any point of H_0, in particular at any point of the unit section of H_0. Now, one knows that the locus V of points of H where u is flat is open ([**EGA**] IV_3, Th. 11.1.1), hence its inverse image by the unit section $\varepsilon : S \to H$ is an open subset U of S containing S_0. By Lemma 22.1 one has $U = S$, hence $u : H \to S$ is flat near every point of the unit section. For every $s \in S$ it follows that u_s is flat, because over a field one can show, going to an algebraic closure and using translations, that a morphism between finite type groups is flat as soon as it is flat near the identity (see [**SGA3**$_1$], VI_B, Prop. 1.3 and also VI_A, Lemma 2.5.3 for the stronger result without finiteness hypotheses). Thus, by the fibral criterion of flatness (see [**EGA**] IV_3, Cor. 11.3.11), $u : H \to G$ is *flat*.

Let $K = \mathrm{Ker}(u)$. As we have a cartesian diagram

$$\begin{array}{ccc} H & \xrightarrow{u} & G \\ \uparrow & & \uparrow{\scriptstyle \varepsilon} \\ K & \longrightarrow & S \end{array}$$

and the unit section $\varepsilon : S \to G$ is a closed immersion (G being affine hence separated over S), one obtains that K is a closed sugbgroup of H, flat over S, and such that K_0 is trivial. Let us prove that K is trivial.

For each $n \in \mathbb{N}^*$ the n-torsion subgroup ${}_nG = D(M/nM)_S$ is finite over S, hence so is its closed subgroup ${}_nK$. Its pullback ${}_nK_0$ over S_0 is trivial, hence by Nakayama's lemma ${}_nK$ is trivial. In particular, for each $s \in S$ we have that ${}_n(K_s) = ({}_nK)_s$ is trivial. One knows that over a field every closed subgroup of a diagonalisable group is diagonalisable (see [**SGA3**$_2$] IX, Prop. 8.1 or [**Oes14**], §5.4). Thus each fiber K_s is a diagonalisable group over $\kappa(s)$, and since ${}_n(K_s)$ is trivial for each n it follows from the density theorem 20.2 that K_s is trivial. Therefore, the unit section $\varepsilon : S \to K$ is an isomorphism on each fiber and hence, by Lemma 22.3 below, K is the trivial group. This proves that $u : H \to G$ is a monomorphism.

Since H is diagonalisable, Cor. 21.2 tells us that u is in fact a closed immersion. On the other hand, as it is flat and of finite presentation (because H and G are of finite type over the noetherian base S) it is open, and is therefore an isomorphism from H to an open and closed subgroup of G. This completes the proof of assertion (4). ☐

In the proof of assertion (4), we have used the lemma below, which is [**EGA**] IV_4, Cor. 17.9.5.

LEMMA 22.3. *Let $f : X \to Y$ be a morphism of S-schemes, where X, Y are locally of finite presentation over S and X is flat over S. If for each $s \in S$ the morphism $f_s : X_s \to Y_s$ is an open immersion, resp. an isomorphism, so is f.*

23. MT-groups of finite type over a henselian local ring

In this section, $(A, \mathfrak{m})$ denotes a local ring, $S = \operatorname{Spec} A$, s the closed point of S and $S_0 = \{s\}$.

DEFINITION 23.1. One says that $(A, \mathfrak{m})$ is **henselian** if it satisfies the following equivalent conditions:

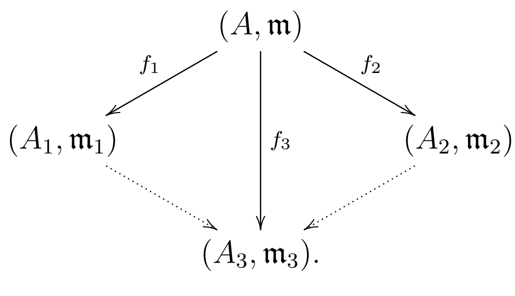
(1) Every finite A-algebra B decomposes as a product of local rings.
(2) For every morphisms $X \to S$ *finite* and $Y \to S$ *étale and separated*, the natural map $\operatorname{Hom}_S(X, Y) \to \operatorname{Hom}_{S_0}(X_0, Y_0)$ is bijective.
(3) For every smooth morphism $f : X \to S$ and every point $x \in X$ over s such that $\kappa(x) = \kappa(s)$, there exists a section $u : S \to X$ of f such that $u(s) = x$.

REMARK 23.2. Of course, the equivalence of the conditions is far from trivial. The first one is usually taken as the definition, see [$\mathbf{SGA3}_2$], Exp. X, §4 and [**EGA**] IV_4, Def. 18.5.8 and Prop. 18.5.9 (ii). The equivalence with (2) is proved in [**EGA**] IV_4, Cor. 18.5.12 and that with (3) in *loc. cit.* Th. 18.5.17.

NOTATION 23.3. A morphism of local rings $f : (A, \mathfrak{m}) \to (B, \mathfrak{n})$ is *local* if $f^{-1}(\mathfrak{n}) = \mathfrak{m}$. In this case[2] it induces an extension of residue fields $A/\mathfrak{m} \hookrightarrow B/\mathfrak{n}$; if further this extension is trivial, we will say for brevity that f is *tlocal*. Beware that this is not standard terminology!

REMARK 23.4. Recall that a *flat* local morphism $(A, \mathfrak{m}) \to (B, \mathfrak{n})$ is *faithfully flat.*

For what follows, we refer to [**EGA**] IV_4, Th. 18.6.6 or [**StaPr**], Algebra, §155 (Tag 0BSK). One can prove that the set of tlocal étale morphims $f_1 : (A, \mathfrak{m}) \to (A_1, \mathfrak{m}_1)$ is *filtered*, that is, if f_2 is another such morphism, there exists a third one f_3 which dominates f_1 and f_2, i.e. such that f_3 factors through f_1 and f_2:



DEFINITION 23.5. Using this, one can construct the filtered inductive limit of these morphisms. One obtains a flat tlocal morphism $A \to \widetilde{A}$, where $\widetilde{A}$ is a local henselian ring with maximal ideal $\widetilde{A}\mathfrak{m}$, which is determined up to unique isomorphism by the universal property that, for every local henselian ring B,

$$\operatorname{Loc.Hom}(\widetilde{A}, B) = \operatorname{Loc.Hom}(A, B). \qquad (23.1)$$

[2]Note that, for example, the inclusion $A \hookrightarrow K$ of a DVR in its field of fractions is **not** a local morphism.

One says that $\widetilde{A}$ is the **henselisation** of A.

From now on, suppose that A is **noetherian** and denote its $\mathfrak{m}$-adic completion by $\widehat{A}$. Then one knows that $A \longrightarrow \widehat{A}$ is faithfully flat (in particular, injective). Further, in this case one has the following proposition.

Proposition 23.6. *Let $(A, \mathfrak{m})$ be a noetherian local ring. Then:*

(1) *$\widetilde{A}$ is noetherian and has the same $\mathfrak{m}$-adic completion than A.*

(2) *Thus, one has tlocal, flat morphisms $A \longrightarrow \widetilde{A} \longrightarrow \widehat{A}$ of local noetherian rings.*

Remark 23.7. Under the assumption that A be noetherian, one can define informally $\widetilde{A}$ as follows. Set $k = A/\mathfrak{m}$ and $\widehat{S} = \operatorname{Spec}\widehat{A}$. We know already that $\widehat{A}$ is henselian. Let $(A, \mathfrak{m}) \to (A', \mathfrak{m}')$ be an étale tlocal morphism, corresponding to an étale map $f : S' \to S$ sending s' to s (with obvious notation). Since $\kappa(\widehat{s}) \otimes_k \kappa(s') = k$, there is a unique point x of $\widehat{S} \times_S S'$ mapping to $\widehat{s}$ and to s' and, by condition (2) of Def. 23.1, the morphism $\widehat{S} \times_S S' \to \widehat{S}$ admits a section sending s to x, which is necessarily étale (see [**EGA**] IV_4, Prop. 17.3.4). In other words, the morphism $\widehat{A} \to \widehat{A} \otimes_A A'$ admits a retraction τ which is a tlocal étale morphism and this gives a flat tlocal morphism $A' \to \widehat{A}$. Then $\widetilde{A}$ is the union of the images of these morphisms (since they form a filtered set, the union of their images is a subring).

Notation 23.8. For the rest of this section, we fix a noetherian local **henselian** ring $(A, \mathfrak{m})$, denote by A' its completion, by S, S' their spectra, and we set $S_0 = \operatorname{Spec}(k)$, where $k = A/\mathfrak{m}$.

Remark 23.9. Consider the following diagram of categories and base-change functors:

$$\begin{array}{ccc} \left\{\begin{array}{c}\text{Schemes finite and}\\ \text{étale over } S\end{array}\right\} & \xrightarrow{X \mapsto X \times_S S'} & \left\{\begin{array}{c}\text{Schemes finite and}\\ \text{étale over } S'\end{array}\right\} \\ & {\scriptstyle X \mapsto X \times_S S_0}\searrow^{\simeq} \qquad {}^{\simeq}\swarrow{\scriptstyle X' \mapsto X' \times_{S'} S_0} & \\ & \left\{\begin{array}{c}\text{Schemes finite and}\\ \text{étale over } S_0\end{array}\right\}. & \end{array}$$

The two oblique arrows are equivalence of categories, hence so is the horizontal one.

Note that S, S' and S_0 are connected. So, choosing a geometric point $\overline{s}$ over s, the observations above imply that fundamental (profinite) groups are isomorphic:

$$\pi_1(S, \overline{s}) \xleftarrow{\sim} \pi_1(S_0, \overline{s}) \xrightarrow{\sim} \pi_1(S', \overline{s}). \tag{23.2}$$

Lemma 23.10. *The functor $H \mapsto H_0 = H \times_S S_0$ is an equivalence between the category of* **isotrivial** MT*-groups over S, resp. S', and the corresponding one over S_0.*

Proof. Since S, S' and S_0 are connected, it follows from Th. 12.3 of Lecture 5 that the category of isotrivial MT-groups over S is anti-equivalent to the category of $\pi_1(S_0, \overline{s})$-modules M such that the kernel of $\pi_1(S, \overline{s}) \to \operatorname{Aut}(M)$ is an open subgroup, and similarly for S_0 and S'. Since the three fundamental groups are the same, the result follows.

Note that this argument also proves assertion (1) of Th. 22.2. ☐

Proposition 23.11. *In the following diagram of categories and base-change functors, all arrows are equivalence of categories:*

$$\begin{array}{ccc} \left\{\begin{array}{c}\textit{MT-groups}\\ \textit{finite over } S\end{array}\right\} & \xrightarrow{X \mapsto X \times_S S'} & \left\{\begin{array}{c}\textit{MT-groups}\\ \textit{finite over } S'\end{array}\right\} \\ & {\scriptstyle X \mapsto X \times_S S_0}\searrow^{\simeq} \qquad {}^{\simeq}\swarrow{\scriptstyle X' \mapsto X' \times_{S'} S_0} & \\ & \left\{\begin{array}{c}\textit{MT-groups}\\ \textit{finite over } S_0\end{array}\right\}. & \end{array}$$

Proof. Any MT-group H_0 finite over S_0 is, in particular, of finite type, hence isotrivial by Prop. 12.6 of Lecture 5, so it comes by change from a MT-group H over S, of the same type as H_0, hence finite over S.

Let H, G be finite MT-groups over S. We have a commutative diagram

$$\begin{array}{ccc} \mathrm{Hom}_{S\text{-Gr}}(H,G) & \longrightarrow & \mathrm{Hom}_{S_0\text{-Gr}}(H_0,G_0) \\ \| & & \| \\ \mathrm{Hom}_{S\text{-Gr}}(D(G),D(H)) & \longrightarrow & \mathrm{Hom}_{S_0\text{-Gr}}(D(G_0),D(H_0)) \end{array}$$

and, since $D(G), D(H)$ are *finite étale* group schemes, the bottom horizontal map is bijective (the unique lift u of a group morphism $u_0 : D(G_0) \to D(H_0)$ is a group morphism, as one sees by considering the diagram involving the group laws of $D(G_0)$ and $D(H)$). Therefore, the top horizontal map is bijective too.

This proves that the base-change from S to S_0 is fully faithful. The same argument applies to the base-change from S' to S_0, since S' is also local henselian with closed point s. Thus the two oblique arrows are equivalence of categories, hence so is the horizontal one. □

Proposition 23.12. *Recall the hypotheses of* 23.8. *Let H, G be* MT*-groups of finite type over S and let H', G' be their pull-backs over S'. Then the natural map below is bijective :*

$$\mathrm{Hom}_{S\text{-Gr}}(H,G) \to \mathrm{Hom}_{S'\text{-Gr}}(H',G'). \tag{23.3}$$

Proof. Set $S'' = S' \times_S S'$ and let $\mathrm{pr}_1, \mathrm{pr}_2$ be its two projections to S'. As $S' \to S$ is faithfully flat and quasi-compact, one has an exact diagram

$$\mathrm{Hom}_{S\text{-Gr}}(H,G) \longrightarrow \mathrm{Hom}_{S'\text{-Gr}}(H',G') \overset{\mathrm{pr}_1^*}{\underset{\mathrm{pr}_2^*}{\rightrightarrows}} \mathrm{Hom}_{S''\text{-Gr}}(H'',G'')$$

hence we see that the assertion is that for **every** morphism of S'-groups $f' : H' \to G'$, the two morphisms of S''-groups $\mathrm{pr}_1^*(f'), \mathrm{pr}_2^*(f)' : H'' \to G''$ coincide.

Note that f' induces for each $n \in \mathbb{N}^*$ a morphism of S'-groups $f'_n : {}_nH' \to {}_nG'$. But ${}_nH'$ and ${}_nG'$ are *finite* MT-groups over S' hence, by the previous proposition, f'_n comes from a morphism of S-groups $f_n : {}_nH \to {}_nG$ and hence satisfies $\mathrm{pr}_1^*(f'_n) = \mathrm{pr}_2^*(f'_n)$.

Now, since G'' is affine hence separated over S'', the locus where $\mathrm{pr}_1^*(f')$ and $\mathrm{pr}_2^*(f')$ coincide is a closed subscheme of H'', and by the previous paragraph, it contains all the subgroups ${}_nH''$. Hence, by the density theorem 20.2, $\mathrm{pr}_1^*(f') = \mathrm{pr}_2^*(f')$. This proves the proposition. □

Corollary 23.13. *Recall the hypotheses of* 23.8. *Let G be a* MT*-group of finite type over S. Then G is isotrivial.*[3]

Proof. As G_0 is a MT-group of finite type over $S_0 = \mathrm{Spec}(k)$ it is isotrivial, say of type M for some finitely generated abelian group M (see Prop. 12.6 in Lecture 5). By Lemma 23.10, there exists an *isotrivial* MT-group H over S and an isomorphism $u_0 : H_0 \simeq G_0$. Since S is connected, both H and G are of constant type M over S.

By assertion (2) of Theorem Th. 19.2, that is, by the algebrisation theorem Th. 19.2 (1) and by the lifting results of Th. 18.3 and Prop. 18.1 (which use duality to extend the cohomological result of Th. 17.1 to MT-groups, not necessarily smooth), we know that u_0 lifts uniquely to a morphism of S'-groups $u' : H' \longrightarrow G'$. By the previous proposition, u comes by base-change from a morphism of S-groups $u : H \longrightarrow G$.

Now, both H and G are MT-groups of constant type M over S, with M finitely generated. Hence, by Prop. 7.10 of Lecture 3, $K = \mathrm{Ker}(u)$ and $C = \mathrm{Coker}(u)$ are MT-groups over S, and

[3] This extends assertion (3) of Th. 22.2 from the complete to the henselian case. There is also an extension of assertion (4), but the proof is more difficult, see Exp. X, Lemma 4.3 and Th. 4.4.

since S is connected they also have a constant type over S. Since u_0 is an isomorphism, the type of K and of C is the trivial abelian group $\{0\}$. Thus K and C are trivial and hence u is an isomorphism. ☐

24. Quasi-isotriviality of finitely generated MT-groups

From the previous corollary, one deduces the following theorem, valid over an arbitrary base scheme S.

THEOREM 24.1. *Let G be a* MT*-group of finite type over a scheme S. Then G is quasi-isotrivial, i.e. for every $s \in S$ there exist an open neighbourhood and a surjective étale morphism $U' \to U$ such that $G_{U'}$ is diagonalisable.*

SKETCH OF PROOF. This uses the principle, detailed in [**EGA**] IV_3, §8, that is $f : X \to Y$ is a morphism of schemes of finite presentation over the spectrum S of a ring Λ, and if Λ is the filtered inductive limit of a family of subrings $(\Lambda_i)_{i \in I}$, then, denoting by a subscript i the pull-backs over $S_i = \operatorname{Spec} \Lambda_i$, one has:

a) There exists an index $i \in I$ and a morphism $f_i : X_i \to Y_i$ such that f comes from f_i by base change.
b) If one considers a property (P) like being: an isomorphism, an open or closed immersion, an affine, quasi-affine, finite, quasi-finite, or proper morphism, then f has property (P) if and only if there exists an index $i_1 \geq i$ such that for every $j \geq i_1$ the morphism f_j obtained from f_i by base-change has property (P).

Applying this firstly to the local ring $\mathcal{O}_{S,s}$, it suffices to prove the theorem over $\operatorname{Spec} \mathcal{O}_{S,s}$. Next, $\mathcal{O}_{S,s}$ is the inductive limit of local subrings which are localizations of finitely generated $\mathbb{Z}$-algebras, so we are reduced to the case where $A = \mathcal{O}_{S,s}$ is *noetherian*. Let $\widetilde{A}$ be its henselisation.

Then, by the previous corollary, there exists a finite étale morphism $\widetilde{A} \to A'$ such that the pull-back G' of G over $\operatorname{Spec} A'$ is diagonalisable. Since $\widetilde{A}$ is the filtered inductive limit of "étale neighbourhoods of s", one obtains from the previous principle an open neighbourhood U of s in S and a surjective étale morphism $U' \to U$ such that $G_{U'}$ is diagonalisable. This proves the theorem. ☐

Notes for this Lecture

In Theorem 22.2, assertions (1) and (3) are in Th. 3.2 of Exp. X, whereas assertion (4) is Th. 3.7 of Exp. X.

The proof of assertion (4) is taken from the proof of [**Co14**], Th. B.3.2; it is easier to understand than the proof of X, Th. 3.7, which relies on the results 6.1 to 6.6 of Exp. IX.

Remark 23.9, Lemma 23.10 and Prop. 23.11 are Exp. X, 4.0, Rem. 4.0.1 and Lemma 4.1.

Prop. 23.12 and Cor. 23.13 are taken from Exp. X, Lemma 4.3 and Th. 4.4, which prove a more general result, namely that the spreading theorem (assertion (4) of Th. 22.2) holds true over a local henselian ring. We have followed the more comprehensible proof of [**Co14**], Prop. B.3.4.

Theorem 24.1 is Exp. X, Cor. 4.5. The reductions to the local ring $\mathcal{O}_{S,s}$, then to a noetherian local ring and then to a noetherian local henselian ring are detailed in the proof of X, Th. 4.4 in the new edition of [$\mathbf{SGA3}_2$], whose preliminary version is available on the lecturer's web page.