LECTURE 1

Diagonalisable groups and MT-groups.Representations of diagonalisable groups

1. Diagonalisable Groups and Groups of Multiplicative Type

In this section, M denotes a *finitely generated* abelian group.[1]

DEFINITION 1.1. The group algebra $\mathbb{Z}[M]$ of M over $\mathbb{Z}$ is a Hopf algebra, with comultiplication, counit and antipode given, for all $m \in M$, by: [2]

$$\Delta(e_m) = e_m \otimes e_m, \qquad \varepsilon(e_m) = 1 \qquad \tau(e_m) = e_{-m}.$$

Hence the $\mathbb{Z}$-scheme $D(M) = \operatorname{Spec}\mathbb{Z}[M]$ is a commutative group scheme over $\mathbb{Z}$: for every scheme S, its set of S-points is:

$$D(M)(S) = \operatorname{Hom}_{\mathrm{Sch}}(S, \operatorname{Spec}\mathbb{Z}[M]) = \operatorname{Hom}_{\mathrm{Alg}}(\mathbb{Z}[M], \mathcal{O}_S(S)) = \operatorname{Hom}_{\mathrm{grp}}(M, \mathcal{O}_S(S)^\times), \tag{1.1}$$

endowed with the group structure $(\phi \cdot \psi)(m) = \phi(m)\psi(m)$.

The group scheme $D(M)$ is affine, finitely presented and faithfully flat over $\mathbb{Z}$, since $\mathbb{Z}[M]$ is a free $\mathbb{Z}$-module and a finitely presented $\mathbb{Z}$-algebra (see below).

Note first that $D(\mathbb{Z}) = \mathbb{G}_m$ and $D(\mathbb{Z}/n\mathbb{Z}) = \mu_n$ for each $n \geq 2$. Note also that if $M = N \oplus N'$ then $\mathbb{Z}[M] = \mathbb{Z}[N] \otimes_{\mathbb{Z}} \mathbb{Z}[N']$ and $D(M) = D(N) \times D(N')$. Thus, writing $M \simeq \mathbb{Z}^d \oplus \bigoplus_{i=1}^f \mathbb{Z}/n_i\mathbb{Z}$, one has

$$D(M) \simeq (\mathbb{G}_m)^d \times \mu_{n_1} \times \cdots \mu_{n_f}$$

and $\mathbb{Z}[M] \simeq \mathbb{Z}[T_1^{\pm 1}, \dots, T_d^{\pm 1}][X_1, \dots, X_f]/(X_1^{n_1} - 1, \dots, X_f^{n_f} - 1)$, which shows that it is a finitely presented $\mathbb{Z}$-algebra.

Next, for any base scheme S, one defines $D(M)_S = D(M) \times S$. For every scheme $S' \to S$, one still has: [3]

$$D(M)_S(S') = \operatorname{Hom}_{S\text{-}\mathrm{Sch}}(S', D(M)_S) = \operatorname{Hom}_{\mathrm{Sch}}(S', D(M)) = \operatorname{Hom}_{\mathrm{grp}}(M, \mathcal{O}_{S'}(S')^\times). \tag{1.2}$$

By base change, the group scheme $D(M)_S$ is affine, finitely presented and flat over S. It is smooth if and only if the order of the torsion part of M is invertible on S.

One says that a group scheme H over S is **diagonalisable** if it is isomorphic with $D(M)_S$ for some M. If M is free of rank d then $D(M)_S$ is isomorphic with $\mathbb{G}_{m,S}^d$ and is called a *split torus* of dimension d.

REMARK 1.2. (added after the lecture) In view of (1.2) each $m \in M$ defines, for every S-scheme S', a group homomorphism $D(M)(S') \to \mathcal{O}_{S'}(S')^\times$, which is functorial in S'. One says that m is a **character** of $D(M)_S$ (see Def. 3.5 below). Thus, for every $h \in D(M)(S')$ one has an element $m(h)$ of $\mathcal{O}_{S'}(S')^\times$.

For the sake of simplicity, we adopt the definition below, more restrictive than SGA3 IX 1.1 as we fix the group M beforehand, but this entails no loss of generality (see remarks 7.7 and 7.9 later).

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[0] version of August 16, 2023. Two minor typos corrected.

[1] In SGA3, arbitrary abelian groups are also considered, e.g. $M = \mathbb{Q}$, but the most interesting results are obtained when M is finitely generated.

[2] Here $(e_m)_{m \in M}$ is the canonical basis of $\mathbb{Z}[M]$. In the sequel we will often write simply m instead of e_m.

[3] That is, the functor $D(M)_S$ is the restriction of $D(M)$ to the category of S-schemes.

DEFINITION 1.3. A group scheme H over S is said to be **of multiplicative type** of type M if for each $s \in S$ there exists an affine open neighbourhood U of s and a surjective flat morphism $U' \to U$, with U' affine,[4] such that $H \times_S U' \simeq D(M)_{U'}$. Further, one says that H is :

- **quasi-isotrivial** if one may choose the maps $U' \to U$ to be étale;
- **isotrivial** if there exists a surjective **finite** étale map $S' \to S$ such that $H \times_S S' \simeq D(M)_{S'}$.
- locally isotrivial (resp. locally trivial) if each $s \in S$ admits an affine open neighbourhood U such that $H_U = H \times_S U$ is isotrivial (resp. diagonalisable).

If M is free of rank d, one says that H is a d-dimensional torus over S.

PROPOSITION 1.4. *Let H be a S-group scheme of multiplicative type of type M. Then H is affine, finitely presented and flat over S.*

PROOF. The assertion to prove is that the structural morphism $f : H \to S$ is affine, finitely presented and flat. This assertion is local on the base so we may assume that S is affine and there exists a surjective flat morphism $S' \to S$, with S' affine, such that $H_{S'} \simeq D(M)_{S'}$. Then the morphism $f_{S'} : H_{S'} \to S'$ is affine, finitely presented and flat. By [**EGA**] IV_2, Prop. 2.7.1, these properties are already true for f, since the morphism $S' \to S$ is flat, surjective and affine, hence faithfully flat and quasi-compact. □

It is easy to give examples of isotrivial groups of multiplicative type which are not diagonalisable.

EXAMPLE 1.5. Let R be a ring, $R \to R'$ an étale covering with Galois group Γ and $\Gamma \to \text{Aut}(M)$ a morphism of groups. Then Γ acts by semi-linear automorphisms of Hopf algebra on $B = R'[M]$ via $\gamma(bm) = \gamma(b)\gamma(m)$,[5] and the invariants form a Hopf algebra A over R.

By Galois descent (see e.g. [**BLR**], §6.2, Example B), we know that $B \simeq A \otimes_R R'$ as Hopf algebras and as Γ-modules, where on the right-hand side Γ acts by $\gamma(a \otimes r') = a \otimes \gamma(r')$. Therefore, $H = \operatorname{Spec} A$ is an isotrivial group of multiplicative type of type M over $S = \operatorname{Spec} R$, which becomes diagonalisable over $S' = \operatorname{Spec} R'$. In general it is not diagonalisable; in fact, we will see later that H is diagonalisable if and only if the action of Γ on M is trivial.

(**Added after the lecture**) For every S-scheme T, one has natural identifications:

$$\begin{aligned}\operatorname{Hom}_S(T, H) = \operatorname{Hom}_S(T \times_S S', H)^\Gamma = \operatorname{Hom}_{S'}((T \times_S S', H_{S'})^\Gamma &= \operatorname{Hom}_{S'}((T \times_S S', D(M)_{S'})^\Gamma \\ &= \operatorname{Hom}_{\text{grp}}(M, \mathbb{G}_m(T \times_S S'))^\Gamma.\end{aligned}$$

Thus, H represents the group functor $T \mapsto \operatorname{Hom}_{\text{grp}}(M, \mathbb{G}_m(T \times_S S'))^\Gamma$.

A basic, and already instructive, example is:

EXAMPLE 1.6. Consider $\mathbb{R} \to \mathbb{C}$, with Galois group $\Gamma = \{\text{id}, \tau\}$, and the morphism $\Gamma \to \text{GL}(\mathbb{Z}) = \mathbb{Z}^\times$ which sends τ to -1. Then the $\mathbb{R}$-algebra A of Γ-invariants in $\mathbb{C}[\mathbb{Z}] = \mathbb{C}[T, T^{-1}]$ is generated by $X = (T + T^{-1})/2$ and $Y = (T - T^{-1})/2i$ and one has $A \simeq \mathbb{R}[X, Y]/(X^2 + Y^2 - 1)$. Note that $\mathbb{S}^1 = \operatorname{Spec} A$ represents the group functor which associates to every $\mathbb{R}$-algebra R the group of elements $z = a + ib$ in $R \otimes_\mathbb{R} \mathbb{C} = R \oplus iR$ such that $z^{-1} = \overline{z}$, that is, $(a + ib)(a - ib) = 1$.

A more elaborate example is:

EXAMPLE 1.7. Let k be a field of characteristic p, let x, y be indeterminates and consider the Artin-Schreier extension $k[y] \to k[x]$, given by $y \mapsto x^p - x$. It is a Galois covering with group $\Gamma = \mathbb{Z}/p\mathbb{Z}$. Let $\Gamma \to \text{GL}(\mathbb{Z}^p)$ be the group morphism which sends the generator $\overline{1}$ of Γ to the automorphism which permutes cyclically the generators: $e_1 \mapsto e_2 \mapsto \cdots \mapsto e_p \mapsto e_1$, and let $A = (k[x][\mathbb{Z}^p])^\Gamma$. Then $H = \operatorname{Spec} A$ is an isotrivial torus of dimension p over

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[4] If U is only required to be quasi-compact, it is covered by finitely many affine open subsets $V_1', \dots, V_n'$, then their sum V' is affine and $V' \to U$ is flat and surjective. So there is no loss in assuming that U' is affine.

[5] The semi-linearity means in fact that the action of Γ on B is compatible with its action on R'.

the affine line $S = \operatorname{Spec} k[y]$, which becomes trivial over $S' = \operatorname{Spec} k[x]$. On can see that it represents the group functor which associates to every $k[y]$-algebra R the group of invertible elements in $R \otimes_{k[y]} k[x]$, see Remark 4.9 below.

REMARK 1.8. Over a connected base, it is not easy to give examples of groups of multiplicative type of type M which are not isotrivial, or which are locally trivial but not diagonalisable. In fact, as we shall see later, no such examples exist over a connected, *normal*, locally noetherian base S.

Two such examples over a nodal curve (irreducible or not) appear in [**SGA3**$_2$], Exp. X, 1.6 (see also [**Oes14**], end of §7.1)) and will be detailed in another lecture.

2. Representations of diagonalisable groups

Again, M denotes a finitely generated abelian group.

DEFINITION 2.1. Let G be an affine group scheme over S, given by a quasi-coherent $\mathcal{O}_S$-algebra $\mathcal{A}$. Let $\Delta : \mathcal{A} \to \mathcal{A} \otimes_{\mathcal{O}_S} \mathcal{A}$ (called the *comultiplication* map) and $\varepsilon : \mathcal{A} \to \mathcal{O}_S$ (called the *augmentation* map) be the morphisms of $\mathcal{O}_S$-algebras corresponding to the multiplication $G \times_S G \to G$ and to the unit section $S \to G$. They satisfy the following equalities:

$$(\Delta \otimes \mathrm{Id}_{\mathcal{A}}) \circ \Delta = (\mathrm{Id}_{\mathcal{A}} \otimes \Delta) \circ \Delta \qquad \text{and} \qquad (\varepsilon \otimes \mathrm{Id}_{\mathcal{A}}) \circ \Delta = \mathrm{Id}_{\mathcal{A}} = (\mathrm{Id}_{\mathcal{A}} \otimes \varepsilon) \circ \Delta \tag{2.1}$$

where in the second set of equalities we have used the identifications $\mathcal{O}_S \otimes_{\mathcal{O}_S} \mathcal{A} = \mathcal{A} = \mathcal{A} \otimes_{\mathcal{O}_S} \mathcal{O}_S$. The inversion map of G induces a morphism of $\mathcal{O}_S$-algebras $\tau : \mathcal{A} \to \mathcal{A}$ (the *antipodal map*), satisfying the equalities corresponding to the identities $gg^{-1} = e = g^{-1}g$ and $(gh)^{-1} = h^{-1}g^{-1}$.

Then, a quasi-coherent $\mathcal{O}_S$-module $\mathcal{F}$ is called a G-**module** if it is endowed with a structure of *right* $\mathcal{A}$-*comodule*, that is, a map of $\mathcal{O}_S$-modules $\mu_{\mathcal{F}} : \mathcal{F} \to \mathcal{F} \otimes_{\mathcal{O}_S} \mathcal{A}$ satisfying:

$$(\mu_{\mathcal{F}} \otimes \mathrm{Id}_{\mathcal{A}}) \circ \mu_{\mathcal{F}} = (\mathrm{Id}_{\mathcal{F}} \otimes \Delta) \circ \mu_{\mathcal{F}} \qquad \text{and} \qquad (\mathrm{Id}_{\mathcal{F}} \otimes \varepsilon) \circ \mu_{\mathcal{F}} = \mathrm{Id}_{\mathcal{F}}. \tag{2.2}$$

REMARK 2.2. (added after the lecture) Let $U = \operatorname{Spec} R$ be an affine open subset of S. Then $\mathcal{F}(U)$ is a R-module, $G_U = \operatorname{Spec} A$ for some R-Hopf algebra A and we have the R-linear map $\mu_{\mathcal{F}(U)} : \mathcal{F}(U) \to \mathcal{F}(U) \otimes_R A$. Let $x \in \mathcal{F}(U)$. For every affine U-scheme $U' = \operatorname{Spec} R'$ and $g \in G(U') = \mathrm{Hom}_{R\text{-Alg}}(A, R')$, one has in $\mathcal{F}(U) \otimes_R R'$ the equality

$$g \cdot (x \otimes 1) = (\mathrm{Id}_{\mathcal{F}(U)} \otimes g)\mu_{\mathcal{F}(U)}(x). \tag{2.3}$$

Now, suppose that $G = D(M)_S$ and let $\mathcal{F}$ be a quasi-coherent G-$\mathcal{O}_S$-module. Thus, one has the comodule map $\mu_{\mathcal{F}} : \mathcal{F} \to \mathcal{F} \otimes_{\mathcal{O}_S} \mathcal{O}_S[M]$. This means that for any affine open subset U of S, one has an $\mathcal{O}_S(U)$-linear map

$$\mu_{\mathcal{F}}(U) : \mathcal{F}(U) \to \bigoplus_{m \in M} \mathcal{F}(U) \otimes_{\mathcal{O}_S(U)} \mathcal{O}_S(U)\, m,$$

sending each section $x \in \mathcal{F}(U)$ to a *finite* sum $\sum_{m \in M} p_m(x) \otimes m$, where the p_m's are $\mathcal{O}_S(U)$-linear endomorphisms of $\mathcal{F}(U)$, all zero on $\mathcal{F}(U)$ but a finite number of them. Then (2.2) gives that

$$\sum_{m', m \in M} p_{m'}(p_m(x)) \otimes m' \otimes m = \sum_{m \in M} p_m(x) \otimes m \otimes m \qquad \text{and} \qquad x = \sum_{m \in M} p_\mu(x).$$

In particular, the p_m's are orthogonal projectors whose sum is the identity of $\mathcal{F}(U)$. Moreover, the image of p_m is the direct summand

$$\mathcal{F}_m(U) = \{x \in \mathcal{F}(U) \mid \mu_{\mathcal{F}}(x) = x \otimes m\}.$$

Thus $\mathcal{F}(U) = \bigoplus_{m \in M} \mathcal{F}_m(U)$ and hence $\mathcal{F}$ coincides with the sheaf $\bigoplus_{m \in M} \mathcal{F}_m$, since it does so on each affine open subset of S. This proves the following proposition:

PROPOSITION 2.3. *Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_S$-module. Then giving a structure of $D(M)_S$-module on $\mathcal{F}$ is the same thing as giving to $\mathcal{F}$ a M-grading.*

REMARK 2.4. (added after the lecture) Note that $\mathcal{F}_m$ is the "*weight space corresponding to the character m*", that is, for any affine open subset $U = \operatorname{Spec} R$ of S, $\mathcal{F}_m(U)$ is the R-submodule of those $x \in \mathcal{F}_m(U)$ such that for every affine U-scheme $U' = \operatorname{Spec} R'$ and $h \in D(M)(U')$ one has in $\mathcal{F}(U) \otimes_R R'$ the equality

$$h \cdot (x \otimes 1) = (\mathrm{Id}_{\mathcal{F}(U)} \otimes h)(x \otimes m) = m(h)(x \otimes 1). \tag{2.4}$$

So the M-grading is in fact a decomposition into weight spaces.

REMARK 2.5. Beware that if the base S is not quasi-compact, the (separated) presheaf $U \mapsto \bigoplus_{m \in M} \mathcal{F}_m(U)$ is not a sheaf, and needs to be sheafified to get the direct sum sheaf $\bigoplus_{m \in M} \mathcal{F}_m$. For example, take M infinite, for example $M = \mathbb{Z}$, let k be a field, say algebraically closed, and let S be the constant scheme $M_{\mathrm{Spec}(k)}$. For each $m \in M$, denote by f_m the function equal to 1 on the m-th copy and to 0 on the other copies. Define a $D(M)_S$-action on $\mathcal{F} = \mathcal{O}_S$ by $\mu(f_m) = f_m \otimes m$ for each m. Then any quasi-compact open subset U of S corresponds to a finite subset E of M, and $\mathcal{F}(U)$ is the direct sum of the $\mathcal{F}_m(U)$, all zero except for those $m \in E$. However, $\mathcal{F}(S) = \prod_{m \in M} k_m$ is not the direct sum of the $\mathcal{F}_m(S) = k_m$.

Notes for this Lecture

Diagonalisable groups are introduced in Exp. I, §4.7 (where Proposition 2.3 is proved), and again in Exp. VIII, Def. 1.1.