CMI Lectures on SGA3

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Patrick Polo

Contents

LECTURE 1

Diagonalisable groups and MT-groups.Representations of diagonalisable groups

1. Diagonalisable Groups and Groups of Multiplicative Type

In this section, M denotes a *finitely generated* abelian group.[1]

Definition 1.1. The group algebra $\mathbb{Z}[M]$ of M over $\mathbb{Z}$ is a Hopf algebra, with comultiplication, counit and antipode given, for all $m \in M$, by: [2]

$$\Delta(e_m) = e_m \otimes e_m, \qquad \varepsilon(e_m) = 1 \qquad \tau(e_m) = e_{-m}.$$

Hence the $\mathbb{Z}$-scheme $D(M) = \operatorname{Spec}\mathbb{Z}[M]$ is a commutative group scheme over $\mathbb{Z}$: for every scheme S, its set of S-points is:

$$D(M)(S) = \operatorname{Hom}_{\mathrm{Sch}}(S, \operatorname{Spec}\mathbb{Z}[M]) = \operatorname{Hom}_{\mathrm{Alg}}(\mathbb{Z}[M], \mathcal{O}_S(S)) = \operatorname{Hom}_{\mathrm{grp}}(M, \mathcal{O}_S(S)^\times), \tag{1.1}$$

endowed with the group structure $(\phi \cdot \psi)(m) = \phi(m)\psi(m)$.

The group scheme $D(M)$ is affine, finitely presented and faithfully flat over $\mathbb{Z}$, since $\mathbb{Z}[M]$ is a free $\mathbb{Z}$-module and a finitely presented $\mathbb{Z}$-algebra (see below).

Note first that $D(\mathbb{Z}) = \mathbb{G}_m$ and $D(\mathbb{Z}/n\mathbb{Z}) = \mu_n$ for each $n \geq 2$. Note also that if $M = N \oplus N'$ then $\mathbb{Z}[M] = \mathbb{Z}[N] \otimes_{\mathbb{Z}} \mathbb{Z}[N']$ and $D(M) = D(N) \times D(N')$. Thus, writing $M \simeq \mathbb{Z}^d \oplus \bigoplus_{i=1}^f \mathbb{Z}/n_i\mathbb{Z}$, one has

$$D(M) \simeq (\mathbb{G}_m)^d \times \mu_{n_1} \times \cdots \mu_{n_f}$$

and $\mathbb{Z}[M] \simeq \mathbb{Z}[T_1^{\pm 1}, \dots, T_d^{\pm 1}][X_1, \dots, X_f]/(X_1^{n_1} - 1, \dots, X_f^{n_f} - 1)$, which shows that it is a finitely presented $\mathbb{Z}$-algebra.

Next, for any base scheme S, one defines $D(M)_S = D(M) \times S$. For every scheme $S' \to S$, one still has: [3]

$$D(M)_S(S') = \operatorname{Hom}_{S\text{-}\mathrm{Sch}}(S', D(M)_S) = \operatorname{Hom}_{\mathrm{Sch}}(S', D(M)) = \operatorname{Hom}_{\mathrm{grp}}(M, \mathcal{O}_{S'}(S')^\times). \tag{1.2}$$

By base change, the group scheme $D(M)_S$ is affine, finitely presented and flat over S. It is smooth if and only if the order of the torsion part of M is invertible on S.

One says that a group scheme H over S is **diagonalisable** if it is isomorphic with $D(M)_S$ for some M. If M is free of rank d then $D(M)_S$ is isomorphic with $\mathbb{G}_{m,S}^d$ and is called a *split torus* of dimension d.

Remark 1.2. (added after the lecture) In view of (1.2) each $m \in M$ defines, for every S-scheme S', a group homomorphism $D(M)(S') \to \mathcal{O}_{S'}(S')^\times$, which is functorial in S'. One says that m is a **character** of $D(M)_S$ (see Def. 3.5 below). Thus, for every $h \in D(M)(S')$ one has an element $m(h)$ of $\mathcal{O}_{S'}(S')^\times$.

For the sake of simplicity, we adopt the definition below, more restrictive than SGA3 IX 1.1 as we fix the group M beforehand, but this entails no loss of generality (see remarks 7.7 and 7.9 later).

[0] version of August 16, 2023. Two minor typos corrected.

[1] In SGA3, arbitrary abelian groups are also considered, e.g. $M = \mathbb{Q}$, but the most interesting results are obtained when M is finitely generated.

[2] Here $(e_m)_{m \in M}$ is the canonical basis of $\mathbb{Z}[M]$. In the sequel we will often write simply m instead of e_m.

[3] That is, the functor $D(M)_S$ is the restriction of $D(M)$ to the category of S-schemes.

DEFINITION 1.3. A group scheme H over S is said to be **of multiplicative type** of type M if for each $s \in S$ there exists an affine open neighbourhood U of s and a surjective flat morphism $U' \to U$, with U' affine,[4] such that $H \times_S U' \simeq D(M)_{U'}$. Further, one says that H is :

- **quasi-isotrivial** if one may choose the maps $U' \to U$ to be étale;
- **isotrivial** if there exists a surjective **finite** étale map $S' \to S$ such that $H \times_S S' \simeq D(M)_{S'}$.
- locally isotrivial (resp. locally trivial) if each $s \in S$ admits an affine open neighbourhood U such that $H_U = H \times_S U$ is isotrivial (resp. diagonalisable).

If M is free of rank d, one says that H is a d-dimensional torus over S.

PROPOSITION 1.4. *Let H be a S-group scheme of multiplicative type of type M. Then H is affine, finitely presented and flat over S.*

PROOF. The assertion to prove is that the structural morphism $f : H \to S$ is affine, finitely presented and flat. This assertion is local on the base so we may assume that S is affine and there exists a surjective flat morphism $S' \to S$, with S' affine, such that $H_{S'} \simeq D(M)_{S'}$. Then the morphism $f_{S'} : H_{S'} \to S'$ is affine, finitely presented and flat. By [**EGA**] IV_2, Prop. 2.7.1, these properties are already true for f, since the morphism $S' \to S$ is flat, surjective and affine, hence faithfully flat and quasi-compact. ☐

It is easy to give examples of isotrivial groups of multiplicative type which are not diagonalisable.

EXAMPLE 1.5. Let R be a ring, $R \to R'$ an étale covering with Galois group Γ and $\Gamma \to \mathrm{Aut}(M)$ a morphism of groups. Then Γ acts by semi-linear automorphisms of Hopf algebra on $B = R'[M]$ via $\gamma(bm) = \gamma(b)\gamma(m)$,[5] and the invariants form a Hopf algebra A over R.

By Galois descent (see e.g. [**BLR**], §6.2, Example B), we know that $B \simeq A \otimes_R R'$ as Hopf algebras and as Γ-modules, where on the right-hand side Γ acts by $\gamma(a \otimes r') = a \otimes \gamma(r')$. Therefore, $H = \operatorname{Spec} A$ is an isotrivial group of multiplicative type of type M over $S = \operatorname{Spec} R$, which becomes diagonalisable over $S' = \operatorname{Spec} R'$. In general it is not diagonalisable; in fact, we will see later that H is diagonalisable if and only if the action of Γ on M is trivial.

(**Added after the lecture**) For every S-scheme T, one has natural identifications:

$$\begin{aligned}\mathrm{Hom}_S(T, H) = \mathrm{Hom}_S(T \times_S S', H)^\Gamma = \mathrm{Hom}_{S'}((T \times_S S', H_{S'})^\Gamma &= \mathrm{Hom}_{S'}((T \times_S S', D(M)_{S'})^\Gamma \\ &= \mathrm{Hom}_{\mathrm{grp}}(M, \mathbb{G}_m(T \times_S S'))^\Gamma.\end{aligned}$$

Thus, H represents the group functor $T \mapsto \mathrm{Hom}_{\mathrm{grp}}(M, \mathbb{G}_m(T \times_S S'))^\Gamma$.

A basic, and already instructive, example is:

EXAMPLE 1.6. Consider $\mathbb{R} \to \mathbb{C}$, with Galois group $\Gamma = \{\mathrm{id}, \tau\}$, and the morphism $\Gamma \to \mathrm{GL}(\mathbb{Z}) = \mathbb{Z}^\times$ which sends τ to -1. Then the $\mathbb{R}$-algebra A of Γ-invariants in $\mathbb{C}[\mathbb{Z}] = \mathbb{C}[T, T^{-1}]$ is generated by $X = (T + T^{-1})/2$ and $Y = (T - T^{-1})/2i$ and one has $A \simeq \mathbb{R}[X, Y]/(X^2 + Y^2 - 1)$. Note that $\mathbb{S}^1 = \operatorname{Spec} A$ represents the group functor which associates to every $\mathbb{R}$-algebra R the group of elements $z = a + ib$ in $R \otimes_{\mathbb{R}} \mathbb{C} = R \oplus iR$ such that $z^{-1} = \overline{z}$, that is, $(a + ib)(a - ib) = 1$.

A more elaborate example is:

EXAMPLE 1.7. Let k be a field of characteristic p, let x, y be indeterminates and consider the Artin-Schreier extension $k[y] \to k[x]$, given by $y \mapsto x^p - x$. It is a Galois covering with group $\Gamma = \mathbb{Z}/p\mathbb{Z}$. Let $\Gamma \to \mathrm{GL}(\mathbb{Z}^p)$ be the group morphism which sends the generator $\overline{1}$ of Γ to the automorphism which permutes cyclically the generators: $e_1 \mapsto e_2 \mapsto \cdots \mapsto e_p \mapsto e_1$, and let $A = (k[x][\mathbb{Z}^p])^\Gamma$. Then $H = \operatorname{Spec} A$ is an isotrivial torus of dimension p over

[4] If U is only required to be quasi-compact, it is covered by finitely many affine open subsets $V'_1, \dots, V'_n$, then their sum V' is affine and $V' \to U$ is flat and surjective. So there is no loss in assuming that U' is affine.

[5] The semi-linearity means in fact that the action of Γ on B is compatible with its action on R'.

the affine line $S = \operatorname{Spec} k[y]$, which becomes trivial over $S' = \operatorname{Spec} k[x]$. On can see that it represents the group functor which associates to every $k[y]$-algebra R the group of invertible elements in $R \otimes_{k[y]} k[x]$, see Remark 4.9 below.

REMARK 1.8. Over a connected base, it is not easy to give examples of groups of multiplicative type of type M which are not isotrivial, or which are locally trivial but not diagonalisable. In fact, as we shall see later, no such examples exist over a connected, *normal*, locally noetherian base S.

Two such examples over a nodal curve (irreducible or not) appear in [**SGA3**$_2$], Exp. X, 1.6 (see also [**Oes14**], end of §7.1)) and will be detailed in another lecture.

2. Representations of diagonalisable groups

Again, M denotes a finitely generated abelian group.

DEFINITION 2.1. Let G be an affine group scheme over S, given by a quasi-coherent $\mathcal{O}_S$-algebra $\mathcal{A}$. Let $\Delta : \mathcal{A} \to \mathcal{A} \otimes_{\mathcal{O}_S} \mathcal{A}$ (called the *comultiplication* map) and $\varepsilon : \mathcal{A} \to \mathcal{O}_S$ (called the *augmentation* map) be the morphisms of $\mathcal{O}_S$-algebras corresponding to the multiplication $G \times_S G \to G$ and to the unit section $S \to G$. They satisfy the following equalities:

$$(\Delta \otimes \mathrm{Id}_{\mathcal{A}}) \circ \Delta = (\mathrm{Id}_{\mathcal{A}} \otimes \Delta) \circ \Delta \qquad \text{and} \qquad (\varepsilon \otimes \mathrm{Id}_{\mathcal{A}}) \circ \Delta = \mathrm{Id}_{\mathcal{A}} = (\mathrm{Id}_{\mathcal{A}} \otimes \varepsilon) \circ \Delta \tag{2.1}$$

where in the second set of equalities we have used the identifications $\mathcal{O}_S \otimes_{\mathcal{O}_S} \mathcal{A} = \mathcal{A} = \mathcal{A} \otimes_{\mathcal{O}_S} \mathcal{O}_S$. The inversion map of G induces a morphism of $\mathcal{O}_S$-algebras $\tau : \mathcal{A} \to \mathcal{A}$ (the *antipodal map*), satisfying the equalities corresponding to the identities $gg^{-1} = e = g^{-1}g$ and $(gh)^{-1} = h^{-1}g^{-1}$.

Then, a quasi-coherent $\mathcal{O}_S$-module $\mathcal{F}$ is called a G-**module** if it is endowed with a structure of *right* $\mathcal{A}$-*comodule*, that is, a map of $\mathcal{O}_S$-modules $\mu_{\mathcal{F}} : \mathcal{F} \to \mathcal{F} \otimes_{\mathcal{O}_S} \mathcal{A}$ satisfying:

$$(\mu_{\mathcal{F}} \otimes \mathrm{Id}_{\mathcal{A}}) \circ \mu_{\mathcal{F}} = (\mathrm{Id}_{\mathcal{F}} \otimes \Delta) \circ \mu_{\mathcal{F}} \qquad \text{and} \qquad (\mathrm{Id}_{\mathcal{F}} \otimes \varepsilon) \circ \mu_{\mathcal{F}} = \mathrm{Id}_{\mathcal{F}}. \tag{2.2}$$

REMARK 2.2. (added after the lecture) Let $U = \operatorname{Spec} R$ be an affine open subset of S. Then $\mathcal{F}(U)$ is a R-module, $G_U = \operatorname{Spec} A$ for some R-Hopf algebra A and we have the R-linear map $\mu_{\mathcal{F}(U)} : \mathcal{F}(U) \to \mathcal{F}(U) \otimes_R A$. Let $x \in \mathcal{F}(U)$. For every affine U-scheme $U' = \operatorname{Spec} R'$ and $g \in G(U') = \operatorname{Hom}_{R\text{-Alg}}(A, R')$, one has in $\mathcal{F}(U) \otimes_R R'$ the equality

$$g \cdot (x \otimes 1) = (\mathrm{Id}_{\mathcal{F}(U)} \otimes g)\mu_{\mathcal{F}(U)}(x). \tag{2.3}$$

Now, suppose that $G = D(M)_S$ and let $\mathcal{F}$ be a quasi-coherent G-$\mathcal{O}_S$-module. Thus, one has the comodule map $\mu_{\mathcal{F}} : \mathcal{F} \to \mathcal{F} \otimes_{\mathcal{O}_S} \mathcal{O}_S[M]$. This means that for any affine open subset U of S, one has an $\mathcal{O}_S(U)$-linear map

$$\mu_{\mathcal{F}}(U) : \mathcal{F}(U) \to \bigoplus_{m \in M} \mathcal{F}(U) \otimes_{\mathcal{O}_S(U)} \mathcal{O}_S(U)\, m,$$

sending each section $x \in \mathcal{F}(U)$ to a *finite* sum $\sum_{m \in M} p_m(x) \otimes m$, where the p_m's are $\mathcal{O}_S(U)$-linear endomorphisms of $\mathcal{F}(U)$, all zero on $\mathcal{F}(U)$ but a finite number of them. Then (2.2) gives that

$$\sum_{m', m \in M} p_{m'}(p_m(x)) \otimes m' \otimes m = \sum_{m \in M} p_m(x) \otimes m \otimes m \qquad \text{and} \qquad x = \sum_{m \in M} p_\mu(x).$$

In particular, the p_m's are orthogonal projectors whose sum is the identity of $\mathcal{F}(U)$. Moreover, the image of p_m is the direct summand

$$\mathcal{F}_m(U) = \{x \in \mathcal{F}(U) \mid \mu_{\mathcal{F}}(x) = x \otimes m\}.$$

Thus $\mathcal{F}(U) = \bigoplus_{m \in M} \mathcal{F}_m(U)$ and hence $\mathcal{F}$ coincides with the sheaf $\bigoplus_{m \in M} \mathcal{F}_m$, since it does so on each affine open subset of S. This proves the following proposition:

Proposition 2.3. *Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_S$-module. Then giving a structure of $D(M)_S$-module on $\mathcal{F}$ is the same thing as giving to $\mathcal{F}$ a M-grading.*

Remark 2.4. (added after the lecture) Note that $\mathcal{F}_m$ is the "*weight space corresponding to the character m*", that is, for any affine open subset $U = \operatorname{Spec} R$ of S, $\mathcal{F}_m(U)$ is the R-submodule of those $x \in \mathcal{F}_m(U)$ such that for every affine U-scheme $U' = \operatorname{Spec} R'$ and $h \in D(M)(U')$ one has in $\mathcal{F}(U) \otimes_R R'$ the equality

$$h \cdot (x \otimes 1) = (\mathrm{Id}_{\mathcal{F}(U)} \otimes h)(x \otimes m) = m(h)(x \otimes 1). \tag{2.4}$$

So the M-grading is in fact a decomposition into weight spaces.

Remark 2.5. Beware that if the base S is not quasi-compact, the (separated) presheaf $U \mapsto \bigoplus_{m \in M} \mathcal{F}_m(U)$ is not a sheaf, and needs to be sheafified to get the direct sum sheaf $\bigoplus_{m \in M} \mathcal{F}_m$. For example, take M infinite, for example $M = \mathbb{Z}$, let k be a field, say algebraically closed, and let S be the constant scheme $M_{\operatorname{Spec}(k)}$. For each $m \in M$, denote by f_m the function equal to 1 on the m-th copy and to 0 on the other copies. Define a $D(M)_S$-action on $\mathcal{F} = \mathcal{O}_S$ by $\mu(f_m) = f_m \otimes m$ for each m. Then any quasi-compact open subset U of S corresponds to a finite subset E of M, and $\mathcal{F}(U)$ is the direct sum of the $\mathcal{F}_m(U)$, all zero except for those $m \in E$. However, $\mathcal{F}(S) = \prod_{m \in M} k_m$ is not the direct sum of the $\mathcal{F}_m(S) = k_m$.

Notes for this Lecture

Diagonalisable groups are introduced in Exp. I, §4.7 (where Proposition 2.3 is proved), and again in Exp. VIII, Def. 1.1.

LECTURE 2

Constant and twisted constant groups, biduality

3. Constant groups and biduality. Character groups

In this section we fix a base scheme S. For the sake of brevity, we will sometimes write D-group (resp. MT-group) over S instead of diagonalisable group over S (resp. S-group of multiplicative type).

DEFINITION 3.1. To every non-empty set M one associates the S-scheme M_S which is the direct sum of a family $(S_m)_{m\in M}$ of copies of S indexed by M. It is étale over S, and is finite over S if and only M is finite. Such a scheme is called a **constant scheme** over S.

The sections $\Gamma(M_S/S)$ of the projection $M_S \to S$ are the locally constant functions from the topological space S to M, denoted by $\mathrm{Loc}(S, M)$. For any S-scheme S', one has $M_S \times_S S' \simeq M_{S'}$ hence:

$$\mathrm{Hom}_{S\text{-Sch}}(S', M_S) = \mathrm{Hom}_{S'\text{-Sch}}(S', M_{S'}) = \mathrm{Loc}(S', M). \tag{3.1}$$

Thus, M_S represents the functor which associates $\mathrm{Loc}(S', M)$ to every S-scheme S'.

On the other hand, to give a morphism from M_S to a S-scheme H is the same as giving, for each $m \in M$, a morphism of S-schemes $S \to H$, i.e. an element of $H(S)$; thus one has:

$$\mathrm{Hom}_{S\text{-Sch}}(M_S, H) = \mathrm{Hom}_{\mathrm{Sets}}(M, H(S)). \tag{3.2}$$

If $u : M \to N$ is a map of sets, it induces a morphism of S-schemes $u_S : M_S \to N_S$. One therefore obtains a functor $M \mapsto M_S$ from the category of non-empty sets to that of S-schemes. It commutes with products, i.e. one has

$$M_S \times_S N_S \simeq (M \times N)_S.$$

Thus, if M is a group one obtains that M_S is a group scheme, called a **constant group scheme** over S. If $u : M \to N$ is a morphism of groups, then $u_S : M_S \to N_S$ is a morphism of S-group schemes. Thus, $M \to M_S$ is a functor from the category of groups to that of S-group schemes. Further, as in (3.2), for every S-group scheme H one has:

$$\mathrm{Hom}_{S\text{-Gr}}(M_S, H) = \mathrm{Hom}_{\mathrm{grps}}(M, H(S)). \tag{3.3}$$

DEFINITION 3.2. Let G, H be S-group schemes. **The functor** $\boxed{\underline{\mathrm{Hom}}_{S\text{-Gr}}(G, H)}$ is defined as follows: its value on a S-scheme T is the set $\mathrm{Hom}_{T\text{-Gr}}(G_T, H_T)$ of morphisms of T-group schemes from G_T to H_T. This is a *contravariant* functor from the category of S-group schemes to the category of sets. This functor is clearly "compatible with base change", that is, for any S-scheme T one has:

$$\underline{\mathrm{Hom}}_{S\text{-Gr}}(G, H) \times_S T = \underline{\mathrm{Hom}}_{T\text{-Gr}}(G_T, H_T) \tag{3.4}$$

as both sides send any T-scheme U to $\mathrm{Hom}_{U\text{-Gr}}(G_U, H_U)$.

Further, if the group law of H is commutative, it induces on $\underline{\mathrm{Hom}}_{S\text{-Gr}}(G, H)$ the structure of a **commutative group functor**. In particular, the group functor $\underline{\mathrm{Hom}}_{S\text{-Gr}}(G, \mathbb{G}_{m,S})$ is denoted by $D(G)$. For future use, let us record (3.4) in this case as:

$$D(G)_T = D(G_T) \tag{3.5}$$

[0]version of August 16, 2023, after the lecture.

In the rest of this section, M denotes a finitely generated abelian group.

REMARK 3.3. Suppose that $G = M_S$. Then, for every S-scheme T one has, by the previous definition and (1.2):

$$D(M_S)(T) = \mathrm{Hom}_{T\text{-Gr}}(M_T, \mathbb{G}_{m,T}) = \mathrm{Hom}_{\mathrm{grps}}(M, \mathcal{O}_T(T)^\times) = D(M)_S(T).$$

Thus the functor $D(M_S)$ is represented by the diagonalisable S-group scheme $D(M)_S$.

Next, let $G = D(M)_S$. For every S-scheme T, both $G_T = D(M)_T$ and $\mathbb{G}_{m,T}$ are affine over T and correspond to the quasi-coherent $\mathcal{O}_T$-Hopf algebras $\mathcal{O}_T[M]$ and $\mathcal{O}_T[X, X^{-1}]$ (the comultiplication of the latter being $\Delta(X) = X \otimes X$). Hence, one has:

$$D(D(M)_S)(T) = \mathrm{Hom}_{T\text{-Gr}}(D(M)_T, \mathbb{G}_{m,T}) = \mathrm{Hom}_{\mathcal{O}_T\text{-Hopf}}(\mathcal{O}_T[X, X^{-1}], \mathcal{O}_T[M]).$$

Note that any locally constant function $\phi : T \to M$ defines a partition of S into the open and closed subschemes S_m on which ϕ takes the value m, and on S_m this defines the Hopf algebra morphism given by $X \mapsto m$. This defines a monomorphism $M_S \to D(D(M)_S)$. Further, one has the following **biduality theorem**:

THEOREM 3.4. *The natural morphism $M_S \to D(D(M)_S)$ is an isomorphism.*

PROOF. We must prove that every morphim of $\mathcal{O}_T$-Hopf algebras $\psi : \mathcal{O}_T[X, X^{-1}] \to \mathcal{O}_T[M]$ is obtained as above. But $\mathcal{O}_T$ has a natural structure of $\mathcal{O}_T[X, X^{-1}]$-comodule, given by $\mu_{\mathbb{G}_m}(f) = f \otimes X$ for any local section of $\mathcal{O}_T$ (this corresponds to the natural action of $\mathcal{O}_{T'}(T')^\times$ on $\mathcal{O}_{T'}(T')$ for any T-scheme T').

Therefore, ψ makes $\mathcal{F} = \mathcal{O}_T$ into an $\mathcal{O}_T[M]$-comodule, with coaction μ_G given by

$$(*) \qquad \mu_G(f) = f \otimes \psi(X).$$

for any local section of $\mathcal{O}_T$. Hence, by Proposition 2.3, for each $t \in T$, the local ring $\mathcal{O}_{T,t}$ is the direct sum of the stalks $\mathcal{F}_{m,t}$, for $m \in M$, which are therefore projective $\mathcal{O}_{T,t}$-modules of rank 0 or 1. It follows that for each $t \in T$ there exists a unique $m \in M$ such that $\mathcal{F}_m \neq 0$ on some open neighbourhood of t, and one has $\mathcal{F}_m = \mathcal{O}_T$ on this neighbourhood.

This gives a partition of T into open and closed subschemes T_m, hence a locally constant function $\phi : T \to M$. Further, over each T_m one has $\mathcal{O}_T = \mathcal{F}_m$ hence for any local section f of $\mathcal{O}_{T_m}$ one has $\mu_G(f) = f \otimes m$.

Comparing with $(*)$ above, we see that on T_m the Hopf algebra morphism ψ is given by $X \mapsto m$, hence ψ is the Hopf algebra morphism corresponding to the locally constant function ϕ. This proves the theorem. □

DEFINITION 3.5. The constant group scheme M_S is called the *character group* of the diagonalisable group $D(M)_S$.

4. Twisted constant groups. Anti-equivalence with groups of multiplicative type

DEFINITION 4.1. A group scheme E over S is said to be a **twisted constant group** of type M if for each $s \in S$ there exists an affine open neighbourhood U of s and a surjective flat morphism $U' \to U$, with U' affine, such that $E \times_S U' \simeq M_{U'}$. Further, one says that E is :

- **quasi-isotrivial** if one may choose the maps $U' \to U$ to be étale;
- **isotrivial** if there exists a surjective **finite** étale map $S' \to S$ such that $E \times_S S' \simeq M_{S'}$.
- locally isotrivial (resp. locally trivial) if each $s \in S$ admits an affine open neighbourhood U such that $E \times_S U$ is isotrivial (resp. constant).

EXAMPLE 4.2. Let $S' \to S$ be a finite étale Galois covering with Galois group Γ and let $\Gamma \to \mathrm{Aut}(M)$ be a morphism of groups. Then recall (see e.g. [**SGA1**], Exp. V) the following facts:

(1) One has $S' \times_S S' \simeq S' \times \Gamma$ (a disjoint sum of copies of S').
(2) For each subgroup G of Γ, there exists a scheme Y, étale over S, which is the quotient S'/G, that is, for every S-scheme Z one has

$$\mathrm{Hom}_S(S'/G, Z) = \mathrm{Hom}_S(S', Z)^G$$

where the right-hand side denotes the G-equivariant S-morphims $f : S' \to Z$, that is, $f \circ \gamma = f$ for all $\gamma \in \Gamma$ (note that the action of G on Z is trivial).[1]
(3) For any S-scheme T, one has $(S'/G) \times_S T \simeq (S' \times_S T)/G$.
(4) In particular, one has $(S'/G) \times_S S' \simeq (S' \times \Gamma)/G \simeq S' \times (\Gamma/G)$.

Now Γ acts on $M_{S'} = \coprod_{m\in M} S'_m$ by sending the m-th copy of S' to the $\gamma(m)$-th copy via the automorphism γ of S'. Denote by M/Γ the set of Γ-orbits in M, choose a representative m in each orbit and denote by Γ_m its stabilizer. Consider the étale S-scheme

$$E = \coprod_{m\in M/\Gamma} S'/\Gamma_m$$

and denote it by $M_{S'}/\Gamma$. As we will see in a later lecture, this is indeed the quotient of $M_{S'}$ by Γ, in the sense that for any S-scheme Z, we have natural identifications

$$\mathrm{Hom}_S(E, Z) = \prod_{m\in M/\Gamma} \mathrm{Hom}_S(S'/\Gamma_m, Z) = \prod_{m\in M/\Gamma} \mathrm{Hom}_S(S', Z)^{\Gamma_m} = \prod_{m\in M} \mathrm{Hom}_S(M_{S'}, Z)^{\Gamma}.$$

Further, applying Fact (4) above to each S'/Γ_m, one obtains that $\boxed{E \times_S S' \simeq M_{S'}.}$

On the other hand, for any S-scheme T, one has natural identifications

$$\begin{aligned}\mathrm{Hom}_S(T, E) = \mathrm{Hom}_S(T \times_S S', E)^{\Gamma} &= \mathrm{Hom}_{S'}(T \times_S S', E_{S'})^{\Gamma} \\ &= \mathrm{Hom}_{S'}(T \times_S S', M_{S'})^{\Gamma} = \mathrm{Loc}(T \times_S S', M)^{\Gamma}.\end{aligned}$$

Therefore, E represents the group functor $T \to \mathrm{Loc}(T \times_S S', M)^{\Gamma}$. Since $E_{S'} \simeq M_{S'}$, it an isotrivial twisted constant group of type M, which splits over S'.

EXAMPLE 4.3. Let $S = \mathrm{Spec}\,\mathbb{R}$ and $S' = \mathrm{Spec}\,\mathbb{C}$, with Galois group $\Gamma = \{\mathrm{id}, \tau\}$ acting on $M = \mathbb{Z}$ by $\tau(n) = -n$. Then τ acts on $\mathbb{Z}_{S'} = \coprod_{n\in\mathbb{Z}}(\mathrm{Spec}\,\mathbb{C})_n$ by swapping $(\mathrm{Spec}\,\mathbb{C})_n$ and $\mathrm{Spec}(\mathbb{C}_{-n})$, the comorphism being $\tau : \mathbb{C} \to \mathbb{C}$. The quotient scheme $E = \mathbb{Z}_{S'}/\Gamma$ is the sum of $\mathrm{Spec}(\mathbb{R})_0$ and a sum of copies of $\mathrm{Spec}\,\mathbb{C}$ indexed by $(\mathbb{Z} - \{0\})/\Gamma$. This is an isotrivial twisted constant group of type $\mathbb{Z}$ over $\mathrm{Spec}\,\mathbb{R}$, which splits over $\mathrm{Spec}\,\mathbb{C}$.

Now, we have the following three results. The complete proofs rely on the powerful technique of faithfully flat descent, to be discussed in another lecture.

PROPOSITION 4.4. *Let E be a twisted constant S-group scheme of type M. Then:*
(1) *$D(E)$ is representable by a S-group scheme H of multiplicative type of type M.*
(2) *One has $E \simeq D(H)$. Thus E is reflexive.*
(3) *E is constant if and only if H is diagonalisable, and E is quasi-isotrivial (resp. isotrivial, resp. locally isotrivial, resp. locally trivial) if and only if H is so.*

PROPOSITION 4.5. *Let H be a S-group scheme of multiplicative type,* quasi-isotrivial *of type M. Then:*
(1) *$D(H)$ is representable by a quasi-isotrival twisted constant group E of type M.*
(2) *One has $H \simeq D(E)$. Thus H is reflexive.*
(3) *H is diagonalisable if and only if E is constant, and is isotrivial (resp. locally isotrivial, resp. locally trivial) if and only if E is so.*

[1] If $S = \mathrm{Spec}\,R$ and $S' = \mathrm{Spec}\,R'$, then $S'/G = \mathrm{Spec}\,R'^G$.

THEOREM 4.6. *Fix a base scheme S.*

(1) *The functors $E \mapsto D(E)$ and $H \mapsto D(H)$ are type-preserving anti-equivalences, quasi-inverse one to another, between the category of twisted constant finitely generated abelian groups E, quasi-isotrivial over S, and the category of finitely presented groups of multiplicative type, quasi-isotrivial over S.*
(2) *These functors induce anti-equivalences, quasi-inverse one to another, between the subcategories of groups which are isotrivial, resp. locally isotrivial, resp. locally trivial.*

We will begin the proof of these results in the next section, and complete it in the next lecture. Before that, let us describe everything explicitly in the Galoisian case: for the rest of this section, $S' \to S$ is a Galois covering with group Γ, and Γ acts by group automorphisms on M.

EXAMPLE 4.7. Let $E = M_{S'}/\Gamma$ be as in example 4.2. For every S-scheme T one has

$$E \times_S T = (M_{S'} \times_S T)/\Gamma = M_{S' \times_S T}/\Gamma,$$

by Fact (3) of 4.2, and hence one has natural identifications

$$D(E)(T) = \mathrm{Hom}_{T\text{-Gr}}(E_T, \mathbb{G}_{m,T}) = \mathrm{Hom}_{T\text{-Gr}}(M_{S' \times_S T}, \mathbb{G}_{m,T})^{\Gamma} = \mathrm{Hom}_{\mathrm{grp}}\big(M, \mathbb{G}_m(S' \times_S T)\big)^{\Gamma}.$$

Combined with the discussion in Example 1.5, this shows that $D(E)$ is represented by the S-group of multiplicative type $H = \operatorname{Spec} R'[M]^{\Gamma}$, assuming for simplicity that $S = \operatorname{Spec} R$ and $S' = \operatorname{Spec} R'$.

EXAMPLE 4.8. Conversely, if $H = \operatorname{Spec} R'[M]^{\Gamma}$, it follows from the reflexivity part of Prop. 4.4 that $D(H) = M_{S'}/\Gamma$. This can also be seen directly, as follows. Let T be a S-scheme. Firstly, one has

$$H_T \times_S S' = H \times_S T \times_S S' = H_{S'} \times_S T = D(M)_{S'} \times_S T = D(M)_{S' \times_S T}$$

and hence

$$\begin{aligned}\mathrm{Hom}_{T\text{-Gr}}(H_T \times_S S', \mathbb{G}_{m,T}) &= \mathrm{Hom}_{T\text{-Gr}}(D(M)_{S' \times_S T}, \mathbb{G}_{m,T}) \\ &= \mathrm{Hom}_{T \times_S S'\text{-Gr}}(D(M)_{S' \times_S T}, \mathbb{G}_{m,T \times_S S'}) = \mathrm{Loc}(T \times_S S', M).\end{aligned}$$

Therefore, one has

$$D(H)(T) = \mathrm{Hom}_{T\text{-Gr}}(H_T, \mathbb{G}_{m,T}) = \mathrm{Hom}_{T\text{-Gr}}(H_T \times_S S', \mathbb{G}_{m,T})^{\Gamma} = \mathrm{Loc}(T \times_S S', M)^{\Gamma}$$

and it follows that $D(H)$ is represented by $M_{S'}/\Gamma$.

EXAMPLE 4.9. Now, consider the case where H corresponds to the permutation representation $M = \mathbb{Z}[\Gamma]$. Then, by the discussion in Example 1.5, one has for every S-scheme T:

$$H(T) = \mathrm{Hom}_{\mathrm{grp}}\big(\mathbb{Z}[\Gamma], \mathbb{G}_m(T \times_S S')\big)^{\Gamma} = \mathrm{Hom}_{\mathrm{grp}}\big(\mathbb{Z}, \mathbb{G}_m(T \times_S S')\big) = \mathbb{G}_m(T \times_S S').$$

Thus, H is the Weil restriction $\mathrm{Res}_S^{S'} \mathbb{G}_{m,S'}$. (This generalizes Example 1.7.)

REMARK 4.10. To answer a question of Prof. Balaji during the lecture, let us give immediately an example of a MT-group over S which is quasi-isotrivial but not locally isotrivial. Let k be a field, algebraically closed if one wants, and let S be the affine curve obtained by identifying the points 0 and 1 of $\mathbb{A}^1_k$, that is, its ring of functions is $R = \mathcal{O}(S) = \{P \in k[t] \mid P(0) = P(1)\}$. As k-algebra, R is generated by the elements $x = t(t-1)$ and $y = t^2(t-1)$, which satisfy the equation $x^3 = y(y-x)$ and one finds that S is the nodal cubic given by this equation.

Consider the auxiliary curve S obtained by glueing two copies of $\mathbb{A}^1_k$ by identifying 0 of each copy with 1 of the other copy. Then $\Gamma = \mathbb{Z}/22\mathbb{Z}$ acts freely on Q and the quotient is S; thus $Q \to S$ is an étale covering.[2] Then the open subscheme U of Q obtained by removing one of the singular points is still étale over S. Let us say that it is copy 0 of $\mathbb{A}^1_k$ with its point 1 glued to point 0 of the copy number 1 of $\mathbb{A}^1_k$. Then we can glue the point 0

[2]For another proof, see [**Tsi14**], Lect. 7, §5.3.

of that copy to the point 0 of a copy number 2 of $\mathbb{A}^1_k$, and then the point 1 of that copy to the point 0 of copy number 3, and so on. We can do the same in the negative direction, that is, glue the point 0 of copy 0 to the point 1 of copy -1, and so on. In this way, we obtain a curve P (not quasi-compact!) which is étale over S, is endowed with an obvious action of the constant group ZZ, and is in fact a principal $\mathbb{Z}$-bundle over S in the étale topology; that is, $P \times_S P \simeq P \times \mathbb{Z}$.

Using the group morphism $\mathbb{Z} \to \mathrm{GL}(\mathbb{Z}^2)$ given by $n \mapsto \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix}$, ones obtains an action of $\Gamma = \mathbb{Z}$ on the split 2-dimensional torus $D(\mathbb{Z}^2)_P$, extending the action of P. Using that $D(\mathbb{Z}^2)_P$ is affine over P, one can construct the quotient by Γ (more on this in a later lecture) and one obtains a S-group scheme H such that $H_P \simeq D(\mathbb{Z}^2)_P$, hence H is isotrivial. From the principal $\mathbb{Z}$-bundle P we can construct a $(\mathbb{Z}/n\mathbb{Z})$-bundle P over S, for each integer $n > 1$. (Note that P_2 is the previous auxiliary curve Q.) Clearly, the pull-back of H to P_n is *not* trivial because the given action of $\mathbb{Z}$ on $\mathbb{Z}^2$ does not factor trough any quotient $\mathbb{Z}/n\mathbb{Z}$.

Finally, one can prove that P is a universal cover of S, in the sense that any finite Galois covering $S' \to S$ is dominated by some P_n. (This is implicit in [**SGA1**], Exp. I, §11 a) together with [**SGA3**$_2$], Exp. X, 1.6, and a proof can be found, e.g. in [**Tsi14**], Lect. 7, §5.3.) It follows that H is not isotrivial on any neighbourhood of the singular point s. (However one sees that is is trivial on $S - \{s\}$.)

5. Notes for this Lecture

The biduality theorem (Th. 3.5) is proved in Exp. VIII, Th. 1.2 in a greater generality.

Groups of multiplicative type over S are defined in Exp. IX, Def. 1.1. Their duals, the twisted constant groups, are introduced in Exp. X, Def. 5.1.

The example of the previous remark is discussed in Exp. X, 1.6.

LECTURE 3

Duality between twisted constant and MT-groups. Exactness of the functor D

6. More on the duality functor D: reflexive groups

In this section we fix a base scheme S. As in SGA3 VIII §1, we denote by I the commutative group scheme $\mathbb{G}_{m,S}$. Let G be a group scheme over S, and assume that the functor $D(G) = \underline{\mathrm{Hom}}_{S\text{-Gr}}(G, I)$ is representable.[1] For every S-scheme T, one has:

$$\mathrm{Hom}_{S\text{-Sch}}(T, D(G)) = D(G)(T) = \mathrm{Hom}_{T\text{-Gr}}(G_T, I_T) = \mathrm{Hom}_{S\text{-Gr}}(G \times_S T, I).$$

This is the subset of morphisms of S-schemes $G \times_S T \to I$ which are "*multiplicative with respect to the first argument*". If $T = G'$ is another S-group scheme, we may consider the subset $\mathrm{Hom}_{S\text{-Gr}}(G', D(G))$; it is the subset of morphims of S-schemes $G \times_S G' \to I$ which are *bimultiplicative*, that is, multiplicative with respect to both arguments. As here G and G' play symmetric roles, we obtain the first assertion of the following proposition:

Proposition 6.1. *Let G, G' be S-group schemes, and assume that $D(G)$ is representable. Then one has the first equality below, and also the second if $D(G')$ is representable:*

$$\mathrm{Hom}_{S\text{-Gr}}(G', D(G)) = D(G')(G) = \mathrm{Hom}_{S\text{-Gr}}(G, D(G')). \tag{6.1}$$

This is compatible with any base change $T \to S$, i.e. if $f : G' \to D(G)$ is a morphism of S-group schemes corresponding to $g : G \to D(G')$, then the morphism $f_T : G'_T \to D(G)_T = D(G_T)$ corresponds to $g_T : G_T \to D(G')_T = D(G'_T)$.

The second assertion follows since f and g correspond to a given bimultiplicative morphism $\phi : G \times_S G' \to I$; by base change it defines a bimultiplicative map $\phi_T : (G \times_S G')_T = G_T \times_T G'_T \to I_T$ which gives rise to f_T on the one hand and to g_T on the other hand.

Definition 6.2. Let G be an S-group scheme. We say that G is **reflexive** if $D(G)$ is representable and the canonical morphism $G \to D(D(G))$ is an isomorphism.[2] Note that this implies that G is commutative.

In this case, for any S-group scheme G' such that $D(G')$ is representable, (6.1) gives:

$$\mathrm{Hom}_{S\text{-Gr}}(G', G) = \mathrm{Hom}_{S\text{-Gr}}(D(G), D(G')). \tag{6.2}$$

Corollary 6.3. *The functor D induces an anti-equivalence of categories from the category of reflexive S-group schemes to itself.*

In view of this corollary, we see that Theorem 4.6 follows from Propositions 4.4 and4.5.

We will prove in the next lecture assertion (1) of both propositions. We take this for granted for the moment and we prove assertions (2) and (3), firstly in the case of 4.4.

[0] version of August 16, 2023, after the 2nd lecture.

[1] This is not really needed, see SGA3 VIII §1.

[2] This is more restrictive than Exp. VIII, Def. 1.0.1, which does not require that $D(G)$ be representable, but this suffices for our purposes.

Proof. Let us prove assertion (2). Since $H = D(E)$, Proposition 6.1 gives us a S-morphism $u : E \to D(H)$. The assertion that u is an S-isomorphism is local on the base so we may assume that S is affine and that there exists a surjective flat morphism $S' \to S$, with S' affine, such that $E_{S'} \simeq M_{S'}$. Then $H_{S'} = D(E)_{S'} = D(E_{S'}) \simeq D(M_{S'})$.

Further, the morphism $u_{S'} : E_{S'} \to D(H)_{S'} = D(H_{S'})$ obtained by base-change corresponds to the bimultiplicative map

$$E_{S'} \times_{S'} H_{S'} \simeq M_{S'} \times_{S'} D(M_{S'}) \to \mathbb{G}_{m,S'},$$

hence $u_{S'}$ is an isomorphism since $M_{S'}$ is reflexive. Then one can invoke again [**EGA**] IV_2, Prop. 2.7.1, which says that u is an isomorphim.[3] This proves assertion (2).

Now, over any $S' \to S$, if $E_{S'}$ is constant then $D(E)_{S'} = D(E_{S'})$ is diagonalisable, and the converse is true by the biduality theorem 3.4. This proves assertion (3). The proof is completely similar in the case of 4.5. □

7. Exactness of the functor D

In this section, we fix a base scheme S. Before we can speak of kernels and quotients in Proposition 7.4, we need to introduce some definitions. Consider a morphism of S-group schemes $\phi : G \to Y$.

Definition 7.1. Its **kernel** $K = \operatorname{Ker}\phi$ is the S-group scheme defined as the fiber product:

$$\begin{array}{ccc} K & \dashrightarrow & G \\ \vdots & & \downarrow \phi \\ S & \xrightarrow{e} & Y \end{array}$$

where $e : S \to Y$ denotes the unit section. For any S-scheme T, one has $K(T) = \operatorname{Ker}\phi(T)$, which is a normal subgroup of $G(T)$; thus K is a *normal* subgroup scheme of G.[4] If e is a closed immersion[5], so is $K \to G$.

Definition 7.2. Note that the morphism $G \times_Y G \to G \times_S G$ is an *immersion*[6], see [**EGA**], I 5.3.10 together with the correction III_2, Err_{10} to I 5.3.9. The multiplication of G induces a morphism of S-schemes $G \times_S K \to G \times_S G$, given on arbitrary T-points by $(g, k) \mapsto (g, gk)$, and this morphism induces an isomorphism of S-schemes:

$$G \times_S K \to G \times_Y G. \tag{7.1}$$

Set $R_\phi = G \times_Y G$ and denote by p_1, p_2 the two projections from R_ϕ to G.

Given a S-scheme Z, we say that a morphism of S-schemes $u : G \to Z$ is K-*invariant* if $p_1^*(u) = p_2^*(u)$; this is equivalent to saying that for an arbitrary S-scheme T and any $x \in G(T)$ and $h \in K(T)$, one has $u(T)(xh) = u(T)(x)$.

Next, we say that the morphism $\phi : G \to Y$ "is **the quotient scheme** G/K" if the map ϕ^* induces, for every S-scheme Z, a bijection:

$$\operatorname{Hom}_{S\text{-Sch}}(Y, Z) \xrightarrow[\sim]{\phi^*} \{K\text{-invariant } S\text{-morphisms } X \to Z\}. \tag{7.2}$$

[3]In *loc. cit.* this is buried as one among 17 cases of fpqc descent of properties of morphisms, but in fact this follows from first principles of descent theory for fpqc morphisms ([**SGA1**], VIII, Cor. 5.3 combined with [$\mathbf{SGA3}_1$], IV, Prop. 2.4 a)), as we shall explain in another lecture.

[4]This can be expressed in terms of morphisms by saying that the morphism $G \times_S K \to G$ "given on arbitrary T-points by $(g, k) \mapsto gkg^{-1}$" factors through K.

[5]This is the case if Y is separated (in particular, affine) over S, because e is the pull-back of the diagonal map $G \to G \times_S G$ by the map $\mathrm{id}_G \times e$.

[6]That is, a closed immersion followed by an open immersion.

Using the language of the next lecture, (7.2) means that $\phi : G \to Y$ is an **effective epimorphism**.

REMARK 7.3. In the classical theory, if k is an algebraically closed field and $H \subset G$ are algebraic groups over k (that is, reduced k-group schemes of finite type), one shows that the set $G(k)/H(k)$ is the set of k-points of some algebraic variety W, namely the orbit $G(k) \cdot [v]$ in $\mathbb{P}(V)$, where V is a representation of G and $[v]$ is a line for which the isotropy group is $H(k)$ *and the isotropy Lie algebra* inside $\mathrm{Lie}(G)$ is $\mathrm{Lie}(H)$ (which ensures that H is actually the schematic stabiliser of $[v]$), and one defines G/H as W. Then it is proved, somehow as a side result, that W satisfies the universal property (7.2) above, with K replaced by H.

Over a general base scheme S, the situation is completely different: quotients G/H *do not always exist*! For this reason, one has to characterize the quotients G/H (if they exist) by the universal property (7.2) above (with K replaced by H), and then look for some S-scheme Y satisfying this property. By Yoneda lemma, such a Y is unique up to *unique isomorphism*; due to this strong uniqueness property, Y may be constructed *locally*, that is, if we find an open cover (U_i) of S and a scheme Y_i over U_i, then the Y_i glue to a scheme Y over S which is the sought-for quotient G/H. More on this in the next lecture.

In the rest of this section, given an abelian group M we write $D_S(M)$ instead of $D(M)_S$.

PROPOSITION 7.4. *Let* $0 \longrightarrow P \xrightarrow{u} M \xrightarrow{v} N \longrightarrow 0$ *be an exact sequence of abelian groups. Set* $G = D_S(M)$ *and* $Y = D_S(P)$*. Then:*

(1) $D_S(v)$ *is an isomorphism from* $K = D_S(N)$ *to* $\operatorname{Ker} D_S(u)$ *and is a closed immersion.*
(2) $D_S(u) : G \to Y$ *is affine and faithfully flat.*
(3) Y *is the quotient* G/K.
(4) The formation of this quotient commutes with base change, *i.e. for any S-scheme T, Y_T is the quotient G_T/K_T.*

PROOF. (1) Let us prove that the morphism $D_S(v) : D_S(N) \to K$ is an isomorphism. It suffices to prove that, for any S-scheme T, the map $D_S(N)(T) \to K(T)$ is bijective. But $K(T)$ is the set of group morphisms $f : M \to \mathcal{O}_T(T)^\times$ such that $f \circ u$ is the trivial morphism, which is the same as $\mathrm{Hom}_{\mathrm{grp}}(M/P, \mathcal{O}_T(T)^\times) = D_S(N)(T)$. This proves the first assertion. Further, the map $\mathcal{O}_S[M] \to \mathcal{O}_S[N]$ is surjective, hence $D_S(v)$ is a closed immersion.[7]

(2) Let (U_i) be a covering of S by affine open subschemes $U_i = \operatorname{Spec} A_i$. Then $Y = D_S(P)$ is covered by the affine open subschemes $Y_{U_i} = \operatorname{Spec} A_i[P]$ and $G_{U_i} = \operatorname{Spec} A_i[M]$ is affine over Y_{U_i}. Further, denoting by $\tau : N \to M$ a set-theoretic section of the projection $M \to N$, one sees that $A_i[M]$ is free over $A_i[P]$ with basis $(\tau(n))_{n\in N}$. It follows that G is affine and faithfully flat.

Assertion (3) follows since any faithfully flat quasi-compact morphism is an **effective epimorphism** (see Def. 7.2), as we shall see in the next lecture.

As for (4), we could invoke the general fact that a faithfully flat quasi-compact morphism remains so after base change. But here (4) follows directly since $G_T = D_T(M)$ and $K_T = D_T(N)$, hence by (3) applied to T instead of S one has $G_T/K_T = D_T(P) = D_S(P) \times_S T = Y_T$. $\square$

PROPOSITION 7.5. *Let M, N be abelian groups. Set* $E = \mathrm{Hom}_{\mathrm{grp}}(M, N)$.

(1) *There is a natural monomorphism* $E_S \to \underline{\mathrm{Hom}}_{S\text{-}\mathrm{Gr}}(M_S, N_S)$.
(2) *If M is finitely generated, this monomorphism is an isomorphism.*

PROOF. Set $F = \underline{\mathrm{Hom}}_{S\text{-}\mathrm{Gr}}(M_S, N_S)$. Let T be a S-scheme. Then $E_S(T) = \mathrm{Loc}(T, E)$ identifies with the set of maps $f : M \times T \to N$ which are additive in the first variable and "uniformly locally constant" in the second variable, i.e. each $t \in T$ admits a neighbourhood U such that $f(m,t) = f(m,u)$ for all $u \in U$ and $m \in M$, whereas $F(T)$ is the larger set of all maps $g : M \times T \to N$ which are additive in the first variable and such that for each $m \in M$ and $t \in T$, there exists a neighbourhood U_m of t such that the function $g_m : u \mapsto f(m,u)$ is constant on U_m.

[7]One could also say that $D_S(M) \to S$ is affine, hence separated, and apply a general result.

Note that, by additivity, $g_{m'}$ is constant on U_m for all m' in the subgroup generated by m. Therefore, if M is generated by elements $m_1, \dots, m_r$, then all g_m are constant on the open neighbourhood $\bigcap_{i=1}^r U_{m_i}$ of t. This proves that g belongs to the subset $\mathrm{Loc}(T, E)$.[8] □

From now on, we assume again that all abelian groups (resp. S-group schemes of multiplicative type) under consideration are **finitely generated**. Recall that a S-group scheme G is called *locally diagonalisable* if each $s \in S$ admits an open neighbourhood U such that $G_U \simeq D(M)_U$ for some abelian group M (uniquely defined by s since $M = \mathrm{Loc}(\mathrm{Spec}\,\mathcal{O}_{S,s}, M_U)$). Thus, one obtains a partition of S into open and closed subsets over which G is diagonalisable.

PROPOSITION 7.6. *Let $u : G \to G'$ be a morphism of locally diagonalisable S-group schemes and let $K = \mathrm{Ker}\, u$. Then:*

(1) *K is locally diagonalisable and $K \to G$ is a closed immersion.*
(2) *The quotient $Y = G/K$ exists and is a locally diagonalizable S-group.*
(3) *One has $u = i \circ p$, where $p : G \to Y$ is affine and faithfully flat, and i is a closed immersion.*
(4) *Setting $H = i(Y)$, the quotient G'/H exists, is locally diagonalisable, and is a cokernel of u.*

Therefore, the category of locally diagonalisable S-groups is abelian.

PROOF. Since all assertions are local on S, we may assume that $G = D_S(M)$ and $G' = D_S(M')$, for some finitely generated abelian groups M, M'. Then, by the biduality theorem 3.4 and Cor. 6.3, combined with Prop. 7.5, we have

$$\mathrm{Hom}_{S\text{-Gr}}(G, G') = \mathrm{Hom}_{S\text{-Gr}}(M'_S, M_S) = \mathrm{Loc}(S, \mathrm{Hom}_{\mathrm{grp}}(M', M)).$$

Then, again, S is partitioned into open and closed subsets over which u comes from a morphism of groups $f : M' \to M$.[9] Setting $P' = \mathrm{Ker}\, f$ and $N' = P = f(M')$ and $N = \mathrm{Coker}\, f$, we have exact sequences

$$0 \longrightarrow P' \longrightarrow M' \longrightarrow N' \longrightarrow 0,$$

$$0 \longrightarrow P \longrightarrow M \longrightarrow N \longrightarrow 0.$$

Then the result follows from Prop. 7.4, with $K = D_S(N)$, $Y = D_S(P) = D_S(N') = H$ and $G'/H = D_S(P')$. □

REMARK 7.7. The enlargement, when S is not connected, of the category of diagonalisable S-groups to that of *locally* diagonalisable S-groups was necessary in order to obtain an abelian category. For example, if k_1, k_2 are fields, $S_i = \mathrm{Spec}\, k_i$ and S is the sum of S_1 and S_2, one may consider the morphism $f : \mathbb{G}_{m,S} \to \mathbb{G}_{m,S}$ which is the identity on S_1 and the trivial morphism on S_2, then $\mathrm{Ker}\, f$ is the locally diagonalisable group which is the trivial group over S_1 and $\mathbb{G}_m$ over S_2. It is not diagonalisable.

Using the same technique of faithfully flat descent as the one needed for the proof of assertion (1) of propositions 4.4 and 4.5 (see the next lecture), one can extend the previous proposition to the case of S-groups of multiplicative type. For this, we need to adopt the definition of [**SGA3**$_2$], X, Def. 1.1:

DEFINITION 7.8. A group scheme H over S is said to be **of multiplicative type** if for each $s \in S$ there exists an affine open neighbourhood U of s, a surjective flat morphism $U' \to U$, with U' affine, and a (finitely generated) abelian group M such that $H \times_S U' \simeq D(M)_{U'}$. Further, one says that H is :

- **quasi-isotrivial** if one may choose the maps $U' \to U$ to be étale;

[8] This explanation, nicer than the one in N.D.E. (3) of [**SGA3**$_2$], Exp. VIII, was given orally by Joseph Oesterlé in his lectures [**Oes14**].

[9] This reduction was omitted in [**SGA3**$_2$], VIII, Cor. 3.4.

- **isotrivial** if there exists a surjective **finite** étale map $S' \to S$ such that $H \times_S S' \simeq D(M)_{S'}$.
- locally isotrivial (resp. locally trivial) if each $s \in S$ admits an affine open neighbourhood U such that $H_U = H \times_S U$ is isotrivial (resp. diagonalisable).

REMARK 7.9. Then there is a partition of S into open and closed subschemes over which the type of H is constant. So, in most results we can restrict ourselves to groups of multiplicative type *of a given type* M, and the more general definition only brings complications in the statements or hypotheses. However when the base S is not connected this generality is needed to ensure that the category groups of multiplicative type has kernels (and is in fact abelian).

PROPOSITION 7.10. *Let $u : G \to G'$ be a morphism of S-group schemes of multiplicative type and let $K = \operatorname{Ker} u$. Then:*

(1) *K is of multiplicative type and $K \to G$ is a closed immersion.*
(2) *The quotient $Y = G/K$ exists and is of multiplicative type.*
(3) *One has $u = i \circ p$, where $p : G \to Y$ is affine and faithfully flat, and i is a closed immersion.*
(4) *Setting $H = i(Y)$, the quotient G'/H exists, is of multiplicative type, and is a cokernel of u.*

Therefore, the category of S-groups of multiplicative type is abelian.

Notes for this Lecture

The exactness of the functor D and the fact that the category is abelian are proved in Exp. VIII, Th. 3.1 and Prop. 3.4 in the (locally) diagonalisable case, and in Exp. IX, Prop. 2.7 in the general case; see also [**Oes14**], 5.3 and 6.5.

LECTURE 4

Faithfully flat descent

8. Faithfully flat descent

We give an overview of descent theory, trying to emphasize the main ideas and to avoid unnecessary formalism. The two main notions are that of **universal effective epimorphisms** and **morphisms of effective descent**.[1] Beware that *effective* does not have the same meaning in both expressions: we will see below that a universal effective epimorphism is the same thing as a morphism of descent, whereas the effectiveness of descent is a further property.

In this section, $\mathscr{C}$ denotes a category with fiber products. (In our applications, it will be the category of schemes.)

DEFINITION 8.1. Consider a morphism $p : X \to Y$ in $\mathscr{C}$. One says that:

(1) p is an **epimorphism** if for every morphism $Y \to Z$, the induced map $\mathrm{Hom}(Y, Z) \to \mathrm{Hom}(X, Z)$ is injective.[2]

(2) p is a **universal epimorphism** if for every morphism $T \to Y$ in $\mathscr{C}$ the morphism $p_T : X \times_Y T \to T$ obtained by base change is an epimorphism (in this case each p_T is again a universal epimorphism).

(3) p is an **effective epimorphism** if, denoting by p_1, p_2 the two projections from $X \times_Y X$ to X, the following diagram in $\mathscr{C}$:

$$X \times_Y X \underset{p_2}{\overset{p_1}{\rightrightarrows}} X \xrightarrow{p} Y \tag{8.1}$$

is **exact**, which means that for every object Z of $\mathscr{C}$ the following diagram of sets:

$$\mathrm{Hom}(Y, Z) \xrightarrow{p^*} \mathrm{Hom}(X, Z) \underset{p_2^*}{\overset{p_1^*}{\rightrightarrows}} \mathrm{Hom}(X \times_Y X, Z) \tag{8.2}$$

is **exact**, which in turn means that the map $f \mapsto f \circ p$ is a bijection from $\mathrm{Hom}(Y, Z)$ onto the set $\{g \in \mathrm{Hom}(X, Z) \mid g \circ p_1 = g \circ p_2\}$.

(4) p is a **universal effective epimorphism** if for every morphism $T \to Y$ in $\mathscr{C}$ the morphism $p_T : X \times_Y T \to T$ obtained by base change is an effective epimorphism (in this case each p_T is a universal effective epimorphism).

Firstly, let us record here the following easy remark and lemma.

REMARK 8.2. It is clear that if $p : X \to Y$ and $q : Y \to Z$ are epimorphisms (resp. universal epimorphisms), so is $q \circ p$. Further, if p and $p' : X' \to Y'$ are universal epimorphisms, so is $p \times p' : X \times X' \to Y \times Y'$, since it is the composition of the two morphisms obtained by base change: $X \times X' \xrightarrow[\mathrm{id}_X \times p']{} X \times Y' \xrightarrow[p \times \mathrm{id}_{Y'}]{} Y \times Y'$.

LEMMA 8.3. *If $p : X \to Y$ admits a section σ, then p is a universal effective epimorphism.*

[0] version of August 20, 2023

[1] And the companion notions of *equivalence relations* and *descent data*.

[2] For example, an epimorphism in the category of sets is just a surjective map.

Proof. Since having a section is preserved by base change, it suffices to prove that p is an effective epimorphism. It is clear that p^* is injective, since $\sigma^* \circ p^* = \mathrm{id}$. Let τ be the morphism $X \to X \times_Y X$, $x \mapsto (x, \sigma p(x))$. Let $g : X \to Z$ such that $g \circ p_1 = g \circ p_2$, then $g = g \circ p_1 \circ \tau = g \circ p_2 \circ \tau = g \circ \sigma \circ p$, i.e. $g = p^*(f)$ with $f = g \circ \sigma$. □

To be concrete, let us enunciate immediately the following fundamental result (see [**SGA1**], VIII, Th. 5.2 or [**BLR**], §6.1, Th. 6).

Theorem 8.4. *In the category of schemes, every faithfully flat, quasi-compact morphism is a universal effective epimorphism.*

Now, let us fix a morphism $p : S' \to S$ in our category $\mathscr{C}$. We want to study the base-change functor $\mathscr{C}_{/S} \to \mathscr{C}_{/S'}$ which sends every object X over S to the object $X \times_S S'$ over S'. The goal of *descent theory* is to give conditions on an arbitrary S'-object X' which would ensure that $X' \simeq X \times_S S'$ for some S-object X.

Notation 8.5. Denote by S''_1 (resp. S''_2) the scheme $S'' = S' \times_S S'$ regarded as a S'-scheme via the first projection p_1 (resp. second projection p_2).

Further,[3] for $i = 1, 2, 3$, denote by S'''_i the scheme $S''' = S' \times_S S' \times_S S'$ regarded as a S'-scheme via the projection to the i-th factor. For $i < j$ in $\{1, 2, 3\}$, denote by $\mathrm{pr}_{ji} : S''' \to S''$ the projection to the factors i and j. Further, for every S'-object X', set $X''_i = X' \times_{S'} S''_i$ for $i = 1, 2$ and $X'''_i = X' \times_{S'} S'''_i$ for $i = 1, 2, 3$.

Definition 8.6. One says that p is a **morphism of descent** if the following property is satisfied: for all objects X, Y over S, if we set $X' = X \times_S S'$ and $X'' = X \times_S S''$ and define Y', Y'' similarly, then the following diagram of sets is exact:

$$\mathrm{Hom}_S(X, Y) \xrightarrow{p^*} \mathrm{Hom}_{S'}(X', Y') \underset{p_2^*}{\overset{p_1^*}{\rightrightarrows}} \mathrm{Hom}_{S''}(X'', Y''). \tag{8.3}$$

Proposition 8.7. *$p : S' \to S$ is a morphism of descent if and only if it is a universal effective epimorphism.*[4]

Proof. Let X, Y be arbitrary S-objects. The diagram (8.3) identifies with the diagram:

$$\mathrm{Hom}_S(X, Y) \xrightarrow{p^*} \mathrm{Hom}_S(X', Y) \underset{p_2^*}{\overset{p_1^*}{\rightrightarrows}} \mathrm{Hom}_S(X'', Y). \tag{8.4}$$

Since $X'' = X \times_S S''$ identifies with the fiber product $X' \times_X X'$ where $p_X : X' \to X$ is obtained from p by the base change $X \to S$, we see that the exactness of the second diagram, for all Y and each given X, means that p_X is an effective epimorphism. Thus we see that p is a morphism of descent if and only if p is a universal effective epimorphism. □

Before we introduce the notion of *descent datum*, we need to introduce that of *equivalence relation.*

Definition 8.8. An **equivalence relation**[5] on an object X is a subfunctor of $X \times X$, which is represented by an object R (equivalently, one is given a monomorphism $R \hookrightarrow X \times X$) such that, for every object T of $\mathscr{C}$, the set $R(T)$ is the graph of an equivalence relation on $X(T) \times X(T)$.
In this case, one denotes by p_1, p_2 the restrictions to R of the two projections from $X \times X$ to X.

Here are two important examples.

[3]This is not used in the definition of morphism of descent, but this will be used later.

[4]One may wonder why introducing a new name for an already existing notion. The reason will appear later.

[5]By hypothesis, $\mathscr{C}$ has a final object e and the unadorned fiber product $\times$ is taken over e. When $\mathscr{C} = (\mathrm{Sch}_{/S})$ we will write explicitly $\times_S$ and we say that R is an equivalence relation on X "over S".

EXAMPLE 8.9. To every morphism $f : X \to Y$ in $\mathscr{C}$ is associated the equivalence relation $R_f = X \times_Y X$ on X. In this case, $R_f \to X \times X$ is an *immersion*.[6] For every object T, one has $R_f(T) = \{(x, x') \in X(T) \times X(T) \mid f(T)(x) = f(T)(x')\}$.

DEFINITION 8.10 (**Free actions**). Let X be an object of $\mathscr{C}$ and H a group-object in $\mathscr{C}$ acting on X, say on the right. One says that H acts *freely* if for every object T, the group $H(T)$ acts freely on $X(T)$. In this case, the morphism $X \times H \to X \times X$ defined on arbitrary T-points by $(x, h) \mapsto (x, xh)$ is a monomorphism and is an equivalence relation R_H on X.[7]

Now, let us study more closely the base change functor $\mathscr{C}_{/S} \to \mathscr{C}_{/S'}$. Observe that for each S-object X the S'-object $X' = X \times_S S'$ has the following three properties:

(1) There is a canonical isomorphism of S''-schemes $\varphi : X''_1 \simeq X \times_S S'' \simeq X''_2$.

(2) The pull-back of φ via the diagonal map $\delta : S' \to S''$ is $\mathrm{id}_{X'}$.

(3) The pull-backs $\mathrm{pr}_{ji}^*(\varphi)$ all identify with the canonical isomorphisms $X'''_i \simeq X \times_S S'' \simeq X'''_j$; in particular they satisfy the cocyle condition:

$$\mathrm{pr}_{31}^*(\varphi) = \mathrm{pr}_{32}^*(\varphi) \circ \mathrm{pr}_{21}^*(\varphi). \tag{8.5}$$

Thus, we see that these conditions are necessary for X' to come from a S-object. This motivates the following definition.

DEFINITION 8.11. **(1) A descent datum** on a S'-object X' relative to $S' \to S$ is an S''-isomorphism $\varphi : X''_1 \xrightarrow{\sim} X''_2$ which satisfies the cocycle condition (8.5). (This implies that $\delta^*(\varphi) = \mathrm{id}_{X'}$, see below.)

A more intuitive way to formulate this is as follows (see [**TDTE1**], §A.1 (c), p. 190-05 or [**BLR**], §6.1, p. 133). For any S-object T and any S-morphism $(t_1, t_2) : T \to S''$, we have a T-isomorphism

$$\varphi_{t_2,t_1} : X'_{t_1} \xrightarrow{\sim} X'_{t_2} \tag{8.6}$$

where $X'_{t_i} = X' \times_{S',t_i} T$ (i.e. T is over S' via t_i), and these isomorphisms are subject to the transitivity condition below, when t_1, t_2, t_3 are S-morphisms $T \to S'$:

$$\varphi_{t_3,t_1} = \varphi_{t_3,t_2} \circ \varphi_{t_2,t_1}. \tag{8.7}$$

In particular, for $t_1 = t_2 = t_3 = t$ one obtains $\varphi_{t,t} = \varphi_{t,t} \circ \varphi_{t,t}$ hence (since $\varphi_{t,t}$ is an isomorphism) $\varphi_{t,t} = \mathrm{id}_{X_t}$. This implies that $\varphi_{t_1,t_2} = \varphi_{t_2,t_1}^{-1}$. Further, applying this to the identity morphism $S' \to S'$, one obtains that $\delta^*(\varphi) = \mathrm{id}_{X'}$.

(2) Moreover, let q_1 be the first projection from $X''_1 = X' \times_{S'} S''_1$ to X' and q_2 the composition of φ and the first projection of X''_2. Then the morphism

$$X''_1 \xrightarrow{(q_1,q_2)} X' \times_S X'$$

is a monomorphism, since its composition with $X' \to S'$ is the isomorphism $X''_1 \simeq X' \times_S S'$, and the above interpretation of the cocycle condition shows that (q_1, q_2) is an **equivalence relation** on X' over S.[8] Namely, for any S-scheme T and pair of points $(x_1, x_2) \in \mathrm{Hom}_S(T, X' \times_S X')$ mapping to the pair $(t_1, t_2) \in \mathrm{Hom}_S(T, S' \times_S S')$, one has $x_1 \sim x_2$ if and only if $x_2 = \varphi_{t_2,t_1}(x_1)$. This is reflexive since $\varphi_{t,t} = \mathrm{id}$, symmetric since $\varphi_{t_1,t_2} = \varphi_{t_2,t_1}^{-1}$, and transitive since $\varphi_{t_3,t_1} = \varphi_{t_3,t_2} \circ \varphi_{t_2,t_1}$.

[6]That is, a closed immersion followed by an open immersion, see [**EGA**], I 5.3.10 and the correction III_2, Err_{10} to I 5.3.9.

[7]If there exists a quotient $p : X \to Y = G/H$ then $R_H = R_p = X \times_Y X$, as we shall see later.

[8]See [**StaPr**], Tag 024E (Lemma 30.1 in Chapter Simplicial Spaces) for a proof using only the cocycle condition and cartesian diagrams.

(3) There is an obvious notion of morphism of S'-objects with descent data relative to $S' \to S$. So we may introduce the category $\mathrm{Desc}(S'/S)$ of S'-objects with descent data.

Then, comparing with Def. 8.6 we obtain the following:

COROLLARY 8.12. *The morphism $p : S' \to S$ is a morphism of descent if and only if the base change functor $p^* : \mathscr{C}_{/S} \to \mathrm{Desc}(S'/S)$ is* **fully faithful**, *i.e. induces bijections between the* Hom*-sets.*

To illustrate the concept of descent data, consider the following example in the category of schemes.

EXAMPLE 8.13. Let $S' \to S$ be a Galois covering, with group Γ. This means, assuming for convenience that Γ acts on the right, that the morphism $\mu : S' \times \Gamma \to S'' = S' \times_S S'$ given by $\mu(s', \gamma) = (s', s'\gamma)$, for every $T \to S$ and $s' \in S'(T)$, is an isomorphism. Then we have isomorphisms:

$$\begin{array}{ccccc} S' \times \Gamma \times \Gamma & \xrightarrow{\sim} & S' \times_S S' \times \Gamma & \xrightarrow{\sim} & S' \times_S S' \times_S S' \\ (s', \gamma_1, \gamma_2) & \mapsto & (s', s'\gamma_1, \gamma_2) & \mapsto & (s', s'\gamma_1, s'\gamma_1\gamma_2). \end{array}$$

Thus, any T-point of S''' can be written uniquely as $(s', s'\gamma_1, s'\gamma_1\gamma_2)$.

Now, let $f : X' \to S'$ and assume given a right action of Γ on X' compatible with its action on S': this means that the obvious diagram is commutative, and at the level of arbitrary T-points this is expressed by $f(x'\gamma) = f(x')\gamma$. Then we have the S''-isomorphism:

$$\varphi : \quad X \times_{S'} S''_1 \xrightarrow{\sim} X' \times_{S'} S''_2, \qquad (x', s', s'\gamma) \mapsto (x'\gamma, s', s'\gamma)$$

and its pull-back $\mathrm{pr}_{21}^*(\varphi) : X' \times_{S'} S'''_1 \xrightarrow{\sim} X' \times_{S'} S'''_1$, as well as the two other pull-backs, are given in terms of arbitrary T-points by the diagram below:

$$\begin{array}{ccc} (x', s', s'\gamma_1, s'\gamma_1\gamma_2) & \xmapsto{\mathrm{pr}_{21}^*(\varphi)} & (x'\gamma_1, s', s'\gamma_1, s'\gamma_1\gamma_2) \\ & \searrow^{\mathrm{pr}_{31}^*(\varphi)} & \downarrow{\scriptstyle \mathrm{pr}_{32}^*(\varphi)} \\ & & (x'\gamma_1\gamma_2, s', s'\gamma_1, s'\gamma_1\gamma_2). \end{array}$$

So we see that the cocycle condition follows from (and is in fact equivalent to) the associativity condition $(s'\gamma_1)\gamma_2 = s'(\gamma_1\gamma_2)$.

DEFINITION 8.14. Let $p : S' \to S$ be a morphism of descent.

(1) On a S'-object X', a descent datum relative to p is said to be **effective** if X' (together with its descent datum) comes from a S-object X. (Necessarily unique, since the functor p^* is fully faithful.)

(2) One says that p is a **morphism of effective descent** if every descent datum relative to p on a S'-object X' is effective.

(3) Given a full subcategory of $\mathcal{D}$ of $\mathrm{Desc}(S'/S)$, for example the subcategory $\mathrm{QAff}(S'/S)$ of $\mathrm{Desc}(S'/S)$ consisting of schemes quasi-affine over S', one says that:

p is an **morphism of effective descent for the category $\mathcal{D}$**

if every descent datum relative to p on an object X' of $\mathcal{D}$ is effective. For example, we will see later that "A faithfully flat quasi-compact morphism $S' \to S$ is a morphism of effective descent for $\mathrm{QAff}(S'/S)$".

One has the following important lemma.

LEMMA 8.15. *Consider morphisms $U \xrightarrow{v} T \xrightarrow{u} S$.*

(1) *If $u \circ v$ is a universal effective epimorphism, so is u.*
(2) *If $u \circ v$ is a morphism of effective descent, so is u.*
(3) *If $u : U \to T$ and $v : T \to S$ are universal effective epimorphisms, resp. morphisms of effective descent, so is $v \circ u$.*

PROOF. (1) Suppose that $u \circ v$ is a universal effective epimorphism and consider the diagram

$$\begin{array}{ccccc}
S & \xleftarrow{\;u\;} & T & \leftleftarrows & T \times_S T \\
{\scriptstyle u\circ v}\uparrow & & \uparrow & & \uparrow \\
U & \longleftarrow & U \times_S T & \leftleftarrows & U \times_S T \times_S T \\
\upuparrows & & \upuparrows & & \upuparrows \\
U \times_S U & \longleftarrow & U \times_S U \times_S T & \leftleftarrows & U \times_S U \times_S T \times_S T
\end{array}\quad .$$

Columns 1,2,3 are exact since $u \circ v$ is an universal effective epimorphism. Row 2 is exact, since $U \times_S T \to U$ is an effective epimorphism (as it has a section over U, see Lemma 8.3) and so is row 3. Then a diagram-chasing shows that row 1 is exact, hence u is an effective epimorphism. As the hypotheses are stable under any base change $S' \to S$, it follows that u is a universal effective epimorphism.

(2) Now, suppose that $u \circ v$ is a morphism of effective descent. Since a morphism of descent is the same thing as a universal effective epimorphism (by Prop. 8.7), (1) gives already that u is a morphism of descent. Hence, given a T-object Y, we only have to show that any descent datum on Y relative to $u : T \to S$ is effective.

Since $f = u \circ v$ is a morphism of effective descent, there exists a S-object X such that $v^*(Y) \simeq (u \circ v)^*(X)$ as objects of $\mathrm{Desc}(U/S)$. It remains to show that $Y \simeq u^*(X)$ as objects of $\mathrm{Desc}(T/S)$. Since f is a universal effective epimorphism, so is the morphism $f_T : U \times_S T \to T$ obtained by base change. Further, $f_T^*(Y) \simeq f_T^*(X_T)$ as objects of $\mathrm{Desc}(U_T/T)$ (since f_T^* factors through v^*) and it follows that there exists a unique T-isomorphism $Y \simeq u^*(X)$, which respects the descent data. This proves (2).

(3) Suppose that u and v are universal effective epimorphisms and consider the diagram

$$\begin{array}{ccccc}
S & \xleftarrow{\;u\;} & T & \leftleftarrows & T \times_S T \\
 & & \uparrow{\scriptstyle v} & & \uparrow{\scriptstyle v \times_S v} \\
 & & U & \leftleftarrows & U \times_S U \\
 & & \upuparrows & \nearrow & \\
 & & U \times_T U & &
\end{array}\quad .$$

By hypothesis, the first line and column are exact, and $v \times_S v$ is an epimorphism by Remark 8.2. The conclusion follows by diagram-chasing: for every object X of $\mathscr{C}$, if an element x_U of $X(U)$ has the same images in $X(U \times_S U)$, it has also the same images in $X(U \times_T U)$, hence it comes from an element x_T of $X(T)$ since the first column is exact. We want to prove that x_T comes from an element of $X(S)$. As the first row is exact, it suffices to see that x_T has the same images in $X(T \times_S T)$ and as $v \times_S v$ is an epimorphism, it suffices to see that x_T has the same images in $X(U \times_S U)$, which is true because these are the images of x_U. This proves the first assertion of (3). Then one sees easily that if u, v are morphisms of effective descent, so is $u \circ v$. □

From now on, we take $\mathscr{C} = (\mathrm{Sch})$. Let us then give another example of morphism of effective descent.

LEMMA 8.16. *Let (U_i) be an open cover of a scheme S and let $T = \coprod_i U_i$. Then $\pi : T \to S$ is a morphism of effective descent.*[9]

PROOF. Note that $T \times_S T = \coprod_{i,j} U_i \times_S U_j \simeq \coprod_{i,j} U_i \cap U_j$, and the first (resp. second) projection $T \times_S T$ correspond to the inclusion of each $U_i \cap U_j$ into U_i (resp. U_j).

Let Y be a T-scheme endowed with a descent datum relative to $T \to U$. Then the $Y_i = Y \times_T U_i$ are open subschemes of Y, and the descent datum consists of isomorphisms on the

[9]It is even a *universal* morphism of effective descent, since for any $S' \to S$, the $U_i' = U_i \times_S S'$ form an open cover of S'.

intersections, which satisfy the cocyle condition. Hence the Y_i glue together to give a scheme X over S, whose pullback to T is Y. □

REMARK 8.17. π is not quasi-compact if there exists a point of S contained in infinitely many U_i's. For example, if k is an algebraically closed field, $S = \mathbb{A}^1_k$ and we take the covering by the $S - \{\lambda\}$, for λ running through the set of closed points of S.

Now, to illustrate the previous basic results, let us sketch the proof of the following theorem, which complements theorem 8.4 (and is in fact used in the proof of the latter).

THEOREM 8.18. *Let $p : S' \to S$ be a faithfully flat, quasi-compact morphism. Let $X' \to S'$ be quasi-affine. Then every descent datum on X' relative to $S' \to S$ is effective.*

PROOF. Firstly, assume the theorem proved when S is affine. For arbitrary S, let (U_i) be a covering of S by affine open subschemes. Consider the following cartesian diagram:

$$\begin{array}{ccc} T' = \coprod_i U_i' & \xrightarrow{\pi'} & S' \\ {\scriptstyle p_T}\downarrow & & \downarrow{\scriptstyle p} \\ T = \coprod_i U_i & \xrightarrow{\pi} & S. \end{array}$$

We have seen (Lemma 8.16) that π is a morphism of effective descent, and by assumption p_T is a morphism of descent, the descent being effective in $\mathrm{QAff}(S'/S)$. Hence, using point (3) and then points (1,2) of Lemma 8.15, we obtain that the same is true for $\pi \circ p_T$ and then for p.

So, it suffices to prove the theorem when S is affine. Then, since p is quasi-compact, S' is covered by a finite number of affine open subsets, their sum is an affine scheme S_1 and the induced morphism $S_1 \to S$ is flat and surjective, hence faithfully flat. As $S_1 \to S$ factors through p, it follows from points (1,2) of Lemma 8.15 that it suffices to prove the theorem when both S and S' are affine.

So, consider a faithfully flat map of rings $A \to A'$ and set $A'' = A' \otimes_A A'$. For any A'-module M', denote by $p_1^*(M')$ the A''-module $A'' \otimes_{A'} M'$, where A'' is regarded as A'-algebra via $a' \mapsto a' \otimes 1$, and define $p_2^*(M')$ similarly. Then, when $X' = \operatorname{Spec} R'$ for some A'-algebra R', the theorem follows from the following proposition, applied to $M' = R'$.

PROPOSITION 8.19. *In the category of A'-modules, every descent datum relative to $A \to A'$ is effective. That is, if M' is a A'-module endowed with an isomorphism of A''-modules $\varphi : p_1^*(M') \xrightarrow{\sim} p_2^*(M')$ satisfying the cocycle condition, then $M = \{x \in M' \mid \varphi(1 \otimes x) = 1 \otimes x\}$ is a A-submodule of M' such that $A' \otimes_A M = M'$.*

For the proof, we refer to [**SGA1**], VIII, 1.4–1.6. For the extension to the case where X' is only quasi-affine over S', we refer to [**SGA1**], VIII, Cor. 7.9 or [**BLR**], §6, Th. 6.1. □

Let us give more criteria for effective descent, that will be used in the sequel. We start with the following:

DEFINITION 8.20. Let $R \underset{q_2}{\overset{q_1}{\rightrightarrows}} X$ be an equivalence relation on X over S. It induces an equivalence relation on the topological space X: two points x_1, x_2 of X are equivalent if there exists a point z of R such that $q_1(z) = q_2(z)$. One says that a subset V of X is **saturated** if it is stable by this equivalence relation. This amounts to saying that $q_1^{-1}(V) = q_2^{-1}(V)$.

For any subset U of X, one sees that $V = q_2(q_1^{-1}(U))$ is the smallest saturated set containing U (hence $V = q_1(q_2^{-1}(U))$ also); one calls it the *saturation* of U.

LEMMA 8.21. *Let $p : S' \to S$ be faithfully flat and quasi-compact. Let X' be an object of $\mathrm{Desc}(S'/S)$. Assume that X' is covered by saturated open subsets (V_i') such that the descent datum on each V_i' is effective. Then so is the descent datum on X'.*

PROOF. Recall first that if $g : Y' \to Y$ is a faithfully flat quasi-compact morphism, the topology of Y is the quotient of the one of Y', that is, $g(V)$ is open in Y for every *saturated* open subset V of Y'.

Now, by hypothesis, there exist S-schemes (V_i) such that $V'_i \simeq V_i \times_S S'$ in $\mathrm{Desc}(S'/S)$. For all i, the projection $V'_i \to V_i$ is faithfully flat and quasi-compact (being a pull-back of $S' \to S$) hence has the property recalled above. For all i, j, $V'_i \cap V'_j$ is a saturated open subset of V'_i and V'_j, hence its images $V_{ij}(i)$ in V_i and $V_{ij}(j)$ in V_j are open.

Since p^* is full and faithful, the glueing data on the V'_i descend and allow us to glue the V_i by identifying the open subsets $V_{ij}(i)$ and $V_{ij}(j)$. This gives a S-scheme X such that $X \times_S S' \simeq X'$ in $\mathrm{Desc}(S'/S)$ (because the descent data relative to $S' \to S$ coincide on each V'_i). ☐

The next lemma and proposition are used, in later lectures, in the proofs of Prop. 4.5 and of Th. 11.5.

LEMMA 8.22. *Let S be affine and $f : S' \to S$ be faithfully flat and locally of finite presentation, but not necessarily quasi-compact. Then there exist an affine scheme S'' and a faithfully flat morphism of finite presentation $S'' \to S$ which factors through f.*

PROOF. Let $(S'_i)_{i\in I}$ be a covering of S' by affine open subsets; each is of finite presentation over S. The hypothesis imply that f is open, hence the $f(S'_i)$ form an open covering of S. As S is affine, hence quasi-compact, there exists a finite subset J of I such that S is covered by the $f(S'_j)$, for $j \in J$. Then $S'' = \coprod_{j\in J} S'_j$ is affine, of finite presentation over S, and the morphism $S'' \to S$ is flat and surjective, hence faithfully flat. ☐

PROPOSITION 8.23. *Let $S' \to S$ be faithfully flat and locally of finite presentation and let X' be a S'-scheme such that the morphism $X' \to S'$ is separated, locally of finite presentation and locally quasi-finite. Then every descent datum on X' relative to $S' \to S$ is effective.*

PROOF. As in the proof of Th. 8.18 we may reduce to the case where S is affine. Then, by the previous lemma we may assume that S' is affine, too.

Assume first that X' is quasi-compact. Then the morphism $X' \to S'$ is separated, of finite presentation and quasi-finite hence, by [**EGA**], IV_3, Th. 8.11.2 (or [**SGA1**], VIII, Th. 6.2 when S' is notherian), $X' \to S'$ is quasi-affine, and hence the descent datum is effective by Th. 8.18.

Now, in general, let U' be an affine open subset of X' and let $V' = q_1(q_2^{-1}(U))$ be its saturation. Recall that $q_1 : X' \times_{S'} S''_1 \to X'$ is obtained by base change from the first projection $p_1 : S'' \to S'$. As p is faithfully flat of finite presentation and affine, so is q_1; in particular q_1 is open and affine, and the same is true for q_2. Therefore, the open subscheme $q_2^{-1}(U')$ of X'' is affine, hence quasi-compact, and therefore V' is open and quasi-compact. By the previous argument, it is quasi-affine over S' hence the descent datum on V' is effective. Finally, as X' is covered by the various saturated open subsets V', the descent datum on X' is effective by Lemma 8.21. ☐

9. Notes for this Lecture

Lemma 8.3 is proved in Exp. IV, Prop. 1.12. Then Proposition 8.7 is Exp. IV, Prop. 2.3, while Corollary 8.12 is taken in Exp. IV, Def. 2.2 as the definition of "morphism of descent", while it is observed there that this depends only on the isomorphism φ in the descent datum, and not on the cocycle condition. (This same definition is given in many places.)

Assertion (1) and the first part of assertion (3) of Lemma 8.15 are proved in Exp. IV, Prop. 1.8 and Lemma 1.7 respectively. Part (2) is mentioned in [**SGA1**], Exp. VIII, proof of Th. 1.1 (top of p. 155), refering to [**Gir64**], but the lecturer has been unable to locate this statement in *loc. cit.*

Lemma 8.16 occurs, for example, in [**SGA1**], VIII, first paragraph of the proof of Th. 1.1, while Lemma 8.21 is [**SGA1**], VIII, Prop. 7.2. Then Lemma 8.22 is contained in [**SGA3**$_1$], IV, Prop. 6.3.1 (iv), and Prop. 8.23 is [**SGA3**$_2$], X, Lemme 5.4.

Results on MT-groups obtained by descent

10. Representability of $D(G)$ when G is a twisted constant or MT-group

Firstly, we want to complete the proof of Propositions 4.4 and 4.5. In both cases, we have a contravariant functor F : (Schemes/S) $\to$ (Groups) given by $T \mapsto \mathrm{Hom}_{T\text{-Gr}}(G_T, I_T)$, where $I = \mathbb{G}_{m,S}$ and G is the given S-group scheme (either twisted constant or of multiplicative type). More generally, let be given S-schemes X, Y and consider the functor

$$F = \underline{\mathrm{Hom}}_S(X, Y): \quad (\text{Schemes}/S) \to (\text{Sets}), \qquad T \mapsto \mathrm{Hom}_{T\text{-Sch}}(X_T, Y_T).$$

We want to give conditions ensuring that F is representable by a S-scheme.

Firstly, this F has the following property. Let T be a S-scheme and (U_i) a covering of T by open subsets; one has $U_i \cap U_j = U_i \times_T U_j$, denote it by U_{ij}. To give a morphism of T-schemes $X_T \to Y_T$ is the same thing as giving morphisms $f_i : X_{U_i} \to Y_{U_i}$ which agree on the intersections $X_{U_i} \cap X_{U_j} = X \times_T U_{ij}$, i.e. such that $\mathrm{pr}_1^*(f_i) = \mathrm{pr}_2^*(f_j)$ for all i, j, where $\mathrm{pr}_1, \mathrm{pr}_2$ denote the projections from $U_i \times_T U_j$ to the first and second factor respectively. Thus we have an exact diagram of sets:

$$\tag{10.1} \begin{array}{ccccc} \mathrm{Hom}_{T\text{-Sch}}(X_T, Y_T) & \longrightarrow & \prod_i \mathrm{Hom}_{U_i\text{-Sch}}(X_{U_i}, Y_{U_i}) & \rightrightarrows & \prod_{i,j} \mathrm{Hom}_{U_{ij}\text{-Sch}}(X_{U_{ij}}, Y_{U_{ij}}) \\ \| & & \| & & \| \\ F(T) & \longrightarrow & \prod_i F(U_i) & \overset{\mathrm{pr}_1^*}{\underset{\mathrm{pr}_2^*}{\rightrightarrows}} & \prod_{i,j} F(U_i \times_T U_j). \end{array}$$

Definition 10.1. Let $\mathscr{C}$ denote the category of S-schemes and $\widehat{\mathscr{C}}$ that of contravariant functors $\mathscr{C} \to$ (Sets). One says that a functor $F \in \widehat{\mathscr{C}}$ having the previous property is a **local functor**, or a **sheaf for the Zariski topology**.

Remark. Setting $T' = \coprod_i U_i$, the second line of (10.1) can be written as $F(T) \longrightarrow F(T') \overset{\mathrm{pr}_1^*}{\underset{\mathrm{pr}_2^*}{\rightrightarrows}} F(T' \times_T T')$.

To illustrate, let us give here the following lemma (a more general result will be proved later).

Lemma 10.2. *Let F be a* **local functor** (Schemes/S) $\to$ (Sets). *Suppose there exists an open covering (S_i) of S such that $F_i = F \times_S S_i$ be representable by a S_i-scheme X_i. Then F is representable by a S-scheme X.*

Proof. Both $X_i \times_S S_j$ and $X_j \times_S S_i$ represent the restriction of F to $S_{ij} = S_i \times_S S_j$ hence, by Yoneda lemma, there exists a unique isomorphism of S_{ij}-schemes

$$\varphi_{ji} : \quad X_i \times_S S_j \xrightarrow{\sim} X_j \times_S S_i.$$

Then one has isomorphisms of schemes over $S_{ijk} = S_i \times_S S_j \times S_k$:

$$\begin{array}{ccccc} X_i \times_S S_j \times S_k & \xrightarrow{\varphi_{ji} \times \mathrm{id}_{S_k}} & X_j \times_S S_i \times_S S_k & = & X_j \times_S S_k \times_S S_i \\ \| & & & & \downarrow{\scriptstyle \varphi_{kj} \times \mathrm{id}_{S_i}} \\ X_i \times_S S_k \times_S S_j & \xrightarrow{\varphi_{ki} \times \mathrm{id}_{S_j}} & X_k \times_S S_i \times_S S_j & = & X_k \times_S S_j \times_S S_i \end{array}$$

[0]version of August 21, 2023

and as all these objects represent the restriction of F to S_{ijk}, this diagram commutes, i.e. the φ_{ji} satisfy the cocyle condition $\varphi_{kj} \circ \varphi_{ji} = \varphi_{ki}$. Therefore the X_i glue together to give a S-scheme X such that $X \times_S S_i = X_i$ for each i. It remains to prove that X represents F.

For every scheme Y over S_i, one has

$$(*) \qquad F(Y) = F_i(Y) = \mathrm{Hom}_{S_i}(Y, X \times_S S_i) = \mathrm{Hom}_S(Y, X) = h_X(Y).$$

Next, for every scheme Y over S, the $Y_i = Y \times_S S_i$ form an open covering of Y; set $Y_{ij} = Y_i \times_Y Y_j = Y \times_S S_{ij}$. As F (resp. h_X) is a local functor then, taking $(*)$ into account, both $F(Y)$ and $h_X(Y)$ identify with the equalizer of the double-arrow:

$$\begin{array}{ccc} \prod_i F(Y_i) & \rightrightarrows & \prod_{i,j} F(Y_{ij}) \\ \| & & \| \\ \prod_i h_X(Y_i) & \rightrightarrows & \prod_{i,j} h_X(Y_{ij})\,. \end{array}$$

This proves that X represents F. □

Moreover, our functor $F = \underline{\mathrm{Hom}}_S(X, Y)$ has the following additional property.

PROPOSITION 10.3. *Let $p : T' \to T$ be faithfully flat and quasi-compact. Denoting by* $\mathrm{pr}_1, \mathrm{pr}_2$ *the two projections from $T'' = T' \times_T T'$ to T', the following diagram is exact:*

$$(10.2) \qquad \begin{array}{ccccc} \mathrm{Hom}_{T\text{-Sch}}(X_T, Y_T) & \longrightarrow & \mathrm{Hom}_{T'\text{-Sch}}(X_{T'}, Y_{T'}) & \rightrightarrows & \mathrm{Hom}_{T''\text{-Sch}}(X_{T''}, Y_{T''}) \\ \| & & \| & & \| \\ F(T) & \longrightarrow & F(T') & \overset{\mathrm{pr}_1^*}{\underset{\mathrm{pr}_2^*}{\rightrightarrows}} & F(T' \times_T T'). \end{array}$$

PROOF. As we saw in the proof of Prop. 8.7, the first line of (10.2) identifies with the diagram:

$$\mathrm{Hom}_S(X_T, Y) \xrightarrow{p^*} \mathrm{Hom}_S(X_{T'}, Y) \overset{\mathrm{pr}_1^*}{\underset{\mathrm{pr}_2^*}{\rightrightarrows}} \mathrm{Hom}_S(X_{T''}, Y)$$

which is exact since p is a universal effective epimorphism. □

DEFINITION 10.4. A functor $F \in \widehat{\mathscr{C}}$ is called a **sheaf for the fpqc topology** if it is local and satisfies the conclusion of Proposition 10.3. For the sake of brevity; we will simply say fpqc-sheaf.

REMARK 10.5. If X, Y are S-group schemes, one obtains similarly that $\underline{\mathrm{Hom}}_{S\text{-Gr}}(X, Y)$ is a sheaf for the fpqc topology.

REMARKS 10.6. (1) For each $X \in \mathscr{C}$ one has the functor $h_X \in \widehat{\mathscr{C}}$ defined by $h_X(Y) = \mathrm{Hom}_{\mathscr{C}}(Y, X)$. By Yoneda lemma, for each $F \in \widehat{\mathscr{C}}$ and $X \in \mathscr{C}$ there is a natural isomorphism $\mathrm{Hom}_{\widehat{\mathscr{C}}}(h_X, F) = F(X)$. In particular, for $X, Y \in \mathscr{C}$ one has $\mathrm{Hom}_{\widehat{\mathscr{C}}}(h_X, h_Y) = h_Y(X) = \mathrm{Hom}_{\mathscr{C}}(X, Y)$. This shows that $\mathscr{C}$ identifies with a full subcategory of $\widehat{\mathscr{C}}$.

(2) Noting that $\underline{\mathrm{Hom}}_S(S, X) = h_X$ one obtains that each $X \in \mathscr{C}$ (identified with h_X) is a sheaf for the fpqc topology.

(3) The *larger* categories of sheaves for the fppf, étale, finite étale topologies consist of all local functors which satisfy the condition of Def. 10.4 only for faithfully flat morphisms f which are of finite presentation, resp. étale, resp. finite étale. Note that the finer the topology is (i.e. the less restrictions on f there are), the closer the resulting sheaves get to actual schemes.

Then, one has the following important result, which is "well-known to the experts" but not easy to find in the literature in this explicit form (see however [**SGA3**$_1$], IV, Prop. 3.5.2). We have included it as Lemma 1.7.2 in Exp. VIII of the new edition of [**SGA3**$_2$].

Proposition 10.7. *Let $F : (\mathrm{Sch}_{/S}) \to (\mathrm{Sets})$ be a* fpqc *sheaf. Assume there exists $S' \to S$ faithfully flat and quasi-compact such that $F' = F \times_S S'$ is representable by a S'-scheme X'.*

(1) *Then X' is endowed with a descent datum with respect to $S' \to S$.*
(2) *If this descent datum is effective, that is, if X' comes by base change from a S-scheme X, then X represents F.*
(3) *The descent datum is always effective is X' is* quasi-affine *over S'; more generally if X' is covered by saturated open subsets which are quasi-affine over S'.*

Proof. (1) We use the notation S''_i and S'''_i introduced in 8.5. The hypotheses imply that $F''_i = F' \times_{S'} S''_i$ is represented by $X''_i = X' \times_{S'} S''_i$. But $F''_1 = F \times_S S'' = F''_2$. Hence, by uniqueness of the representing scheme (Yoneda lemma), there exists a unique S''-isomorphism $\varphi : X''_1 \xrightarrow{\sim} X''_2$.

For $i < j$ in $\{1, 2, 3\}$, denote by $\mathrm{pr}_{ji} : S''' \to S''$ the projection to the factors i and j. Then, set $X'''_i = X' \times_{S'} S'''_i$ and denote by $\mathrm{pr}^*_{ji}(\varphi) : X'''_i \xrightarrow{\sim} X'''_j$ the isomorphism of S'''-schemes obtained from φ by base change. Then, one obtains a diagram of isomorphisms of S'''-schemes:

$$\begin{array}{ccc} X'''_1 & \xrightarrow{\mathrm{pr}^*_{21}(\varphi)} & X'''_2 \\ & \underset{\mathrm{pr}^*_{31}(\varphi)}{\searrow} & \big\downarrow \mathrm{pr}^*_{32}(\varphi) \\ & & X'''_3 \end{array}$$

and as these schemes represent the restriction of F to S''' they are uniquely isomorphic, hence one has the cocyle relation $\mathrm{pr}^*_{31}(\varphi) = \mathrm{pr}^*_{32}(\varphi) \circ \mathrm{pr}^*_{21}(\varphi)$, that is, φ is a descent datum on X' relative to $S' \to S$.

(2) Assume further that this descent datum is effective, i.e. that there exists a S-scheme X such that $X' \simeq X \times_S S'$. We prove that X represents F, as in Lemma 10.2: for every $Y \to S'$, one has

$$(**) \qquad F(Y) = F'(Y) = \mathrm{Hom}_{S'}(Y, X \times_S S') = \mathrm{Hom}_S(Y, X) = h_X(Y).$$

Next, for every $Y \to S$ set $Y' = Y \times_S S'$ and $Y'' = Y' \times_Y Y' \simeq Y \times_S S''$. Then $Y' \to Y$ is faithfully flat and quasi-compact, since $S' \to S$ is so. As F and h_X are sheaves for the fpqc topology, one deduces from $(**)$ that $F(Y)$ and $h_X(Y)$ both identify to the equalizer of the double arrow:

$$\begin{array}{ccc} F(Y') & \rightrightarrows & F(Y'') \\ \| & & \| \\ h_X(Y') & \rightrightarrows & h_X(Y''). \end{array}$$

This proves that X represents F.

(3) This follows from Th. 8.18 and Lemma 8.21. □

We can now complete the proof of propositions 4.4 and 4.5. Recall the hypotheses: M is a finitely generated abelian group and E, resp. H is a twisted constant group, resp. MT-group, of type M over S. Further, one assumes that H is quasi-isotrivial. We have to prove that the functors $D(E)$ and $D(H)$ are representable, respectively, by a MT-group and a twisted constant group of type M.

Proof. By Lemma 10.2 and the previous proposition, we only have to prove that if S is affine and if $p : S' \to S$ is a flat surjective morphism, with S' affine, such that the functor $D(E)_{S'}$ (resp. $D(H)_{S'}$) is represented by $X' = D(M)_{S'}$ (resp. $X' = M_{S'}$), then the descent datum on

X' is effective. In the first case, this follows immediately from the previous proposition, since $D(M)_{S'}$ is affine.

In the second case, using the hypothesis that H is quasi-isotrivial, we may assume that the above morphism $p : S' \to S$ is étale, hence locally of finite presentation. Further, in this case, $X' = M_{S'}$ is étale over S' hence separated, locally of finite presentation and locally quasi-finite over S'. So we conclude by Prop. 8.23 that the descent datum on X' is effective. □

Example 10.8. Illustrate this when S is the nodal cubic curve of Remark 4.10 by constructing H and E over S which become trivial over the principal $\mathbb{Z}$-bundle $P \to S$, but are not isotrivial over S. (To be done during the lecture).

11. Isotriviality over a locally noetherian normal base

We fix a base scheme S and an abelian group M. For the sake of completeness, let us record here the following theorem ([**SGA3**$_2$], X, Cor. 4.5).[1]

Theorem 11.1. *Let H be a* MT*-group over S of type M. If M is finitely generated, then H is quasi-isotrivial.*

From now on, we assume that M is **finitely generated** and that S is **locally noetherian**.

Lemma 11.2. *Let P be a quasi-isotrivial twisted constant scheme over S. Let Z be an open and closed subset of P.*

(1) *Let U be the set of those $s \in S$ such that the fiber Z_s is finite. Then U is open and closed in S and the map $Z_U \to U$ is finite.*
(2) *In particular, if S is connected and U non-empty, then $Z \to S$ is finite.*

Proof. By assumption, there exists a surjective étale map $f : S' \to S$ such that $P_{S'} = I_{S'}$ for some set I. We have a cartesian diagram:

$$\begin{array}{ccc} Z' & \longrightarrow & Z \\ \downarrow & & \downarrow \\ I_{S'} & \longrightarrow & P \\ \downarrow & & \downarrow \\ S' & \xrightarrow{f} & S \end{array}$$

and, since f is étale, the inverse image $f^{-1}(U)$ equals the set U' of $s' \in S'$ such that the fiber $Z'_{s'}$ is finite.[2] Further, since f is étale, S' is still locally noetherian, hence its connected components are open and closed. If C is such a component, the C_i are the connected components of I_C, hence Z'_C equals J_C for some subset $J = J(C)$ of I, and one sees that the points of C belong to U' if and only if $J(C)$ is finite, in which case the map $Z'_C \to C$ is finite.

Thus, U' is the union of those connected components C of S' such that $J(C)$ is finite, hence is open and closed in S', and the map $Z'_{U'} \to U'$ is finite. Since the topology of S is the quotient of that of S', one obtains that U is open and closed in S. Further, the map $Z'_{U'} \to U'$ is the pull-back via f of $Z_U \to U$ and, since the former is finite, so is the later (use e.g. Lemma 8.22 and [**EGA**], IV$_2$, Prop. 2.7.1). This proves assertion (1), and assertion (2) follows immediately. □

Definition 11.3. Recall that S is supposed locally noetherian. Then one says that S is **geometrically unibranch** (see [**EGA**] IV$_2$, 6.15.1 and paragraph before 6.15.14) if the normalization map $\widetilde{S} \to S_{\mathrm{red}}$ is radicial (hence a universal homeomorphism).[3]

[1]This theorem is one of the reasons why we restricted to finitely generated abelian groups M.
[2]Setting $s = f(s')$, one has $Z'_{s'} \simeq Z_s \otimes_{\kappa(s)} \kappa(s')$, where $\kappa(s)$ denotes the residue field of s.
[3]For example, this is the case if S is normal or if S is a cuspidal curve.

The important fact is that, in this case, the connected components of S are irreducible.[4] Further, if a morphism $P \to S$ is étale, then P is also locally noetherian and geometrically unibranch (see [**EGA**], IV_4, Prop. 17.7.5).

PROPOSITION 11.4. *Suppose that S is locally noetherian and geometrically unibranch. Let $p : P \to S$ be a quasi-isotrivial twisted constant scheme over S. Then the connected components of P are* finite *over S.*[5]

PROOF. Since S and P are locally noetherian and geometrically unibranch, their connected components are open and closed, and are irreducible. In particular, replacing S by one of its connected components, we may assume S irreducible; let η be its generic point. Let C be a connected component of P; it is irreducible, denote by ξ its generic point. As p is flat, one has $p(\xi) = \eta$. As C is the closure of ξ in P, it follows that $C \cap p^{-1}(\eta)$ is the closure of ξ in $p^{-1}(\eta)$. But since p is étale, hence locally quasi-finite, the fiber $p^{-1}(\eta)$ is discrete. Thus, for the open and closed subset C of P one has $C_\eta = \{\xi\}$, which is finite. Hence, by the previous lemma, C is finite over S. □

THEOREM 11.5. *Suppose that S is locally noetherian and geometrically unibranch. Let H be a* MT*-group of type M, which is quasi-isotrivial.*[6] *Then H is in fact isotrivial.*

PROOF. Set $G = D(M)_S$ and denote the functors $\underline{\mathrm{Hom}}_{S\text{-}\mathrm{Gr}}(G, H)$ and $\underline{\mathrm{Isom}}_{S\text{-}\mathrm{Gr}}(G, H)$ by E and I respectively.

Let $S' \to S$ be an étale map such that $H_{S'} \simeq D(M)_{S'}$. Since M is finitely generated we obtain, by Prop. 7.5, that

$$E_{S'} = \underline{\mathrm{Hom}}_{S'\text{-}\mathrm{Gr}}(D(M)_{S'}, D(M)_{S'}) = \underline{\mathrm{Hom}}_{S'\text{-}\mathrm{Gr}}(M_{S'}, M_{S'})$$

is represented by the constant scheme $\mathrm{End}(M)_{S'}$, and then that $I_{S'}$ is represented by the constant scheme $\mathrm{Aut}(M)_{S'}$. By the effectiveness result of Prop. 8.23, I is represented by a twisted constant scheme P over S.

Let C be a connected component of P. It is étale over S and, by the previous proposition, finite. Hence $p(C)$ is open and closed in S, hence $p(C) = S$ since S is connected. Thus $p : C \to S$ is étale, surjective and finite. Further, the diagonal map $C \to C \times_S C$ produces a section over C of $P_C = \mathrm{Isom}_{C\text{-}\mathrm{Gr}}(G_C, H_C)$, hence H_C is isomorphic with $D(M)_C$. This proves that H is isotrivial. □

12. Classification of isotrivial groups of multiplicative type

In this section, we assume that the base scheme S is **connected**.

DEFINITION 12.1. (1) An **étale covering** of S is a morphism $\pi : S' \to S$ which is étale, surjective and finite (in particular, affine). Then $\pi_* \mathcal{O}_{S'}$ is a locally free $\mathcal{O}_S$-algebra of rank n, and n is called the *degree* of the covering.

(2) The group Γ of S-automorphisms of S' is finite, of cardinality $\leq n$. If S' is connected and $|\Gamma| = n$, one says that $S' \to S$ is a **Galois covering** with group Γ.

REMARK 12.2. Let $\pi : E \to S$ be an étale covering. One knows that: [7]

(1) E has finitely many connected components $C_1, \dots, C_r$, each open and closed.

[4] Beware that without the locally noetherian hypothesis, there exists connected normal schemes which are not irreducible, see [**StaPr**], Tag 033O or Exercise 2.4.12 in [**Co14**].

[5] Contrast this with the connected principal $\mathbb{Z}$-bundle over a nodal curve of Remark 4.10.

[6] As we suppose that M is finitely generated, H is automatically quasi-isotrivial, by Th. 11.1.

[7] See e.g. [**Sza09**], Prop. 5.3.9.

Then each morphism $\pi_i : C_i \to S$ is still finite and étale; further, since S is connected and π is open and closed (being étale and finite), each π_i is surjective. So each $\pi_i : C_i \to S$ is a *connected* étale covering.

(2) Every *connected* étale covering $p : C \to S$ is dominated by a Galois covering, that is, there exists a Galois covering $\pi : S' \to S$, with Galois group Γ, and a S-morphism $q : S' \to C$ such that $\pi = p \circ q$.

Now, if H is an isotrivial S-group scheme of multiplicative type, there exists an étale covering $E \to S$ such that $H_E \simeq D(M)_E$ for some (finitely generated) abelian group M. By the previous remark, we may replace E by a Galois covering $S' \to S$ with Galois group Γ. For the sake of simplicity, let us further assume that $S = \operatorname{Spec} R$ is affine. Then $S' = \operatorname{Spec} R'$ for some Galois covering $R \to R'$ with group Γ.

Consider now the category $\mathrm{MT}(S'/S)$ of all S-groups H of multiplicative type which become diagonalisable over S'. It is anti-equivalent to the category of R-Hopf algebras A such that $A \otimes_R R'$ is isomorphic with $R'[M]$, for some finitely generated abelian group M. In this case, we have an action of Γ on $B = R'[M]$ by semi-linear automorphisms of R-Hopf algebra. This induces an action of Γ on M by group automorphisms because, by the proof of the biduality theorem 3.4 and the fact that S' is connected, we have: [8]

$$D(M_{S'})(S') = \operatorname{Hom}_{R'\text{-Hopf}}(R'[X, X^{-1}], R'[M]) = \operatorname{Loc}(S', M) = M.$$

Thus, base change from S to S' is a contravariant functor from $\mathrm{MT}(S'/S)$ to the category of finitely generated Γ-modules. Now, the gist of Galois descent theory is contained in Example 1.5 above, namely that a quasi-inverse is given by the functor taking such a Γ-module M to $H = \operatorname{Spec} R'[M]^\Gamma$.

So far, we have assumed $S = \operatorname{Spec} R$ for simplicity, so that $H_{S'} = \operatorname{Spec} B$, where $B = R'[M]$, in which case we know that the quotient of $H_{S'}$ by Γ exists and is $\operatorname{Spec} B^\Gamma$. But it is known in general (see [**SGA1**], V, Cor. 1.8) that if $\pi : X \to S$ is affine and Γ is a finite group of S-automorphisms of X, then the quotient $Y = X/\Gamma$ exists, and over any open affine subset $U = \operatorname{Spec} R$ of S one has $Y_U = \operatorname{Spec} B^\Gamma$, if one denotes by $\operatorname{Spec} B$ the affine scheme $\pi^{-1}(U)$. So we have obtained the:

THEOREM 12.3. *Let S be a* connected *base scheme and $S' \to S$ a Galois covering with group Γ. The category $\mathrm{MT}(S'/S)$ of S-groups of multiplicative type which split over S' is anti-equivalent to the category of finitely generated Γ-modules.*

REMARKS 12.4. Let $H \in \mathrm{MT}(S'/S)$ correspond to a Γ-module M. Denote by M^Γ and M_Γ the sets of invariants and coinvariants, that is, the largest submodule (resp. quotient module) on which the action of Γ is trivial; one has $M_\Gamma = M/N$, where N is the submodule generated by the elements $m - \gamma(m)$, for $m \in M$ and $\gamma \in \Gamma$. Then:

(1) H is diagonalisable if and only if the action of Γ on M is trivial.
(2) Regarding by $\mathbb{Z}$ as a Γ-module with trivial Γ-action, one has:

$$\operatorname{Hom}_{S\text{-Gr}}(H, \mathbb{G}_{m,S}) = \operatorname{Hom}_\Gamma(\mathbb{Z}, M) = M^\Gamma, \tag{12.1}$$

$$\operatorname{Hom}_{S\text{-Gr}}(\mathbb{G}_{m,S}, H) = \operatorname{Hom}_\Gamma(M, \mathbb{Z}) = \operatorname{Hom}(M_\Gamma, \mathbb{Z}). \tag{12.2}$$

(3) The natural pairing $M^\Gamma \times \operatorname{Hom}_\Gamma(M, \mathbb{Z}) \to \mathbb{Z}$ is not necessarily perfect, even if M is a free $\mathbb{Z}$-module: if M is the permutation representation $\mathbb{Z}[\Gamma]$ one has $M^\Gamma = \mathbb{Z}v$, where $v = \sum_{\gamma \in \Gamma} e_\gamma$, whilst $M \to M_\Gamma = \mathbb{Z}$ is given by $\sum_{\gamma \in \Gamma} n_\gamma e_\gamma \mapsto \sum_{\gamma \in \Gamma} n_\gamma$. Thus the image of the pairing is $d\mathbb{Z}$, where $d = |G|$.

REMARK 12.5. To illustrate Remark (3) above, consider the Deligne torus $H = \operatorname{Res}^{\mathbb{C}}_{\mathbb{R}} \mathbb{G}_{m,\mathbb{C}}$, which corresponds to the permutation module $M = \mathbb{Z}e_0 \oplus \mathbb{Z}e_1$, where τ swaps e_0 and e_1. One has $M^\Gamma = \mathbb{Z}(e_0 + e_1)$, whereas the

[8] A purely algebraic formulation is that the set of group-like elements in $R'[M]$ is exactly M.

kernel of $M \to M_\Gamma$ is $\mathbb{Z}(e_0 - e_1)$. The exact sequence $0 \longrightarrow M^\Gamma \longrightarrow M \longrightarrow M/M^\Gamma \longrightarrow 0$ corresponds to the exact sequence

$$1 \longrightarrow \mathbb{S}^1 \longrightarrow H \xrightarrow{N} \mathbb{G}_{m,\mathbb{R}} \longrightarrow 1 \tag{12.3}$$

where N is the norm homomorphism. It has no section; indeed $D(M_\Gamma)$ is the largest split subtorus of H and the cokernel of $M^\Gamma \to M_\Gamma$ is $\mathbb{Z}/2\mathbb{Z}$. However, the group of $\mathbb{R}$-points $H(\mathbb{R})$ splits as $\mathbb{S}^1(\mathbb{R}) \times \mathbb{R}_+^\times$.

Proposition 12.6. *Let k be a field and M a finitely generated abelian group. Then any k-group H of multiplicative type of type M is isotrivial, i.e. there exists a finite separable extension k' of k such that $H_{k'} \simeq D(M)_{k'}$.*

Proof. Set $H = \operatorname{Spec} \Lambda$. By hypothesis, there exists a k-algebra A and an isomorphism of Hopf algebras $\phi : A[M] \xrightarrow{\sim} A \otimes_k \Lambda$. We reduce first to the case where A is a finitely generated k-algebra.

Let $m_1, \dots, m_s$ be a set of generators of M, write firstly $\phi(m_i) = \sum_j a_{ij} \otimes r_{ij}$ and then $\phi^{-1}(r_{ij}) = \sum_{m \in M} \alpha_{ijm} m$ (all sums being finite), and let Λ_1 (resp. B) be the k-subalgebra of Λ (resp. of A) generated by the r_{ij}'s (resp. the a_{ij} and α_{ijm}'s). Using that k is a field, one obtains firstly that $\Lambda_1 = \Lambda$ (because $A \otimes_k (\Lambda/\Lambda_1) = 0$) and secondly that $B \otimes_k \Lambda$ is a subalgebra of $A \otimes_k \Lambda$. Clearly, ϕ maps $B[M]$ into $B \otimes_k \Lambda$ and ϕ^{-1} maps $B \otimes_k \Lambda = B \otimes_k \Lambda_1$ into $B[M]$. It follows that ϕ induces an isomorphism $B[M] \xrightarrow{\sim} B \otimes_k \Lambda$.

Next, let $\mathfrak{m}$ be a maximal ideal of B and $K = B/\mathfrak{m}$. On the one hand, $K[M] \xrightarrow{\sim} K \otimes_k \Lambda$. On the other hand, by the Nullstellensatz, K is a finite extension of k. Let k' be the separable closure of k in K and set $S = \operatorname{Spec} k$ and $S' = \operatorname{Spec} k'$. Set also $R = \operatorname{Spec} K$.

Now, consider the twisted constant group $E = D(H)_S$, it is étale over S. Denote by E' and E_R its pull-backs over S' and R. To emphasize the idea, we can now invoke the general result that since $R \to S'$ is radicial, the base change from étale S'-schemes to étale R-schemes is *fully faithful* (and even an equivalence of categories), see e.g. [**SGA1**], IX, Cor. 3.4 (and Th. 4.10). Since $E_R \simeq M_R$, one has $E' \simeq M_{S'}$ and hence, by Proposition 4.4, $H_{S'} \simeq D_{S'}(E') = D(M)_{S'}$.

In our simple case we can give a direct proof of the key step. Let C be a connected component of E'. Then $C = \operatorname{Spec} L$ for a field L finite and separable over k', hence $C_R = \operatorname{Spec}(L \otimes_k K)$ is a sum of finitely many $\operatorname{Spec}(K_i)$, where each K_i is a field separable over K. Further, each K_i equals K, since each connected components of E_R is equal to $R = \operatorname{Spec} K$. On the other hand, as K/k' is purely inseparable, C_R is irreducible (see e.g. [**EGA**] IV_2, Prop. 4.3.2). It follows that $L \otimes_k K = K$ and hence $L = k$. This proves that E' is trivial over S'. □

Notes for this Lecture

The representability of $D(R)$ (resp. $D(H)$) is proved in Exp. X, Prop. 5.3 (resp. Cor. 5.7).

Theorem 11.5 is proved in Exp. X, Th. 5.16.

The classification of isotrivial groups of multiplicative type is given in Exp. X, Prop. 1.1.

The fact that a group of multiplicative type over a field is isotrivial is proved in Exp. X, Prop. 1.4, but the proof there uses radicial descent for groups of multiplicative type, proved using cohomology in a much more general setting in Exp. IX, Cor. 5.4. That proof has been much simplified by Oesterlé ([**Oes14**], §§12–13).

Theorem 11.1, that we gave without proof, is proved in Exp. X, Cor. 4.5 as a corollary of the spreading theorem Th. 4.4. It also uses in an essential manner the algebrisation theorem IX, Th. 7.1.

LECTURE 6

A first look at maximal tori and Lie algebras

13. Motivation: tori in reductive group schemes

Definition 13.1. Let k be an algebraically closed field an G a connected affine algebraic group over k, that is, a connected affine smooth group scheme over k.

One knows that all maximal tori T of G have the same dimension; in fact they are all conjugate under $G(k)$. Their common dimension is called the *reductive rank* of G and denoted by $\mathrm{rk}_{\mathrm{red}}(G)$.

One also knows that there exists a largest *normal* smooth connected solvable (resp. unipotent) subgroup of G, it is called the *radical* (resp. *unipotent radical*) of G and is denoted by $\mathrm{rad}(G)$ (resp. $\mathrm{rad}^u(G)$).

One says that G is *reductive* (resp. *semi-simple*) if $\mathrm{rad}^u(G) = \{e\}$ (resp. $\mathrm{rad}(G) = \{e\}$). In this case, if K is a larger algebraically closed field, G_K is reductive (resp. semi-simple).

Definition 13.2. Let S be a base scheme. One says that a S-group scheme G is **reductive** (resp. **semi-simple**) if:

(1) G is affine and smooth, with connected fibers.
(2) Its geometric fibers are reductive (resp. semi-simple), that is, for every $s \in S$, denoting by $\overline{s}$ the spectrum of an algebraic closure $\overline{\kappa}(s)$ of the residue field $\kappa(s)$, the $\overline{\kappa}(s)$-group $G_{\overline{s}}$ is reductive (resp. semi-simple).

Definition 13.3. Let G be a smooth affine S-group scheme. A **maximal torus** of G is a closed subgroup scheme[1] T such that:

(1) T is a torus (in the sense of Def. 1.3).
(2) For every geometric point $\overline{s}$ of S, the subgroup $T_{\overline{s}}$ is a maximal torus of $G_{\overline{s}}$

Remark 13.4. (1) In particular, one will obtain that a smooth affine S-group scheme H such that all its fibers are tori, is itself a torus. This is far from obvious!

(2) Let k be an algebraically closed field of characteristic $\neq 2$ and let $S = \mathbb{A}^1_k$. The constant S-group $\{\pm 1\}_S$ is smooth and isomorphic with $\mu_{2,S}$. Let H be the open subgroup obtained by removing the non-neutral point over $s = 0$. Then H is a smooth affine S-group scheme and all its fibers are MT-groups, but H is not a MT-group. This shows that the assumption that the fibers are connected is important. (However, Cor. 4.8 of Exp. X proves that if H is a flat S-group scheme of finite presentation, such that all fibers H_s are MT-groups and their type (i.e. the corresponding abelian group) is locally constant function of s, then H is a MT-group over S.)

A fundamental point in the study of reductive group schemes is to prove that maximal tori exist locally in the étale topology. The proof consists in showing that certain functors F are representable and *formally smooth*, so that for every $s \in S$ there exists some étale neighbourhood S' of s such that $F(S') \neq \emptyset$ (Hensel Lemma).

[0]corrected version of Nov. 4, 2023

[1]If H is a MT-group over S, every monomorphism of S-groups $H \to G$ is closed immersion, see Cor. 21.2 in Lect. 9.

DEFINITION 13.5. One says that a contravariant functor $F : (\mathrm{Sch}) \to (\mathrm{Sets})$ is **formally smooth** if for every nilpotent ideal I in a ring A, the map $F(\operatorname{Spec} A) \to F(\operatorname{Spec}(A/I))$ is surjective.

14. A first glimpse of Lie algebras

In this section and later ones, we consider a square-zero ideal I in a ring Λ and denote by $S \supset S_0$ the spectra of Λ and Λ/I; they have the same underlying topological space. For every S-scheme X we denote by X_0 its pullback over S_0.

DEFINITION 14.1. **(1)** Let $S' \to S$ be a morphism of schemes and let X' be a S'-scheme. One denotes by $\prod_{S'/S} X'$ the functor sending each S-scheme T to $X'(T \times_S S')$. It is called the *Weil restriction* of X' from S' to S.

When $X' = X \times_S S'$ for some S-scheme X, one has for every S-scheme T:

$$\Big(\prod_{S'/S} X_{S'}\Big)(T) = \operatorname{Hom}_{S'}(T_{S'}, X_{S'}) = \operatorname{Hom}_S(T_{S'}, X) = X(T \times_S S').$$

(2) Now, let G be a smooth affine S-group scheme. We denote by G^+ the functor $\prod_{S_0/S} G_0$; that is, for any $T \to S$ a T-point of G^+ is a morphism of S-schemes $\phi : T_0 \to G$. Clearly, G^+ is a group functor and there is a canonical morphism of group functors $G \to G^+$, which sends an arbitrary point $T \to G$ to the point $T_0 \to G$ of G^+. Denote by[2] L'_G its kernel. Note that since G is smooth and I nilpotent, any morphism of schemes $f_0 : T_0 \to G$ lifts to a morphism of S-*schemes* $f : T \to G$, so we have an exact sequence of group functors:

$$1 \longrightarrow L'_G \longrightarrow G \xrightarrow{\pi} G^+ \longrightarrow 1. \tag{14.1}$$

If Y is a MT-group over S and $u_0 : Y_0 \to G$ is a morphism of S_0-group schemes, an important result in the sequel is that u_0 can be lifted to a morphism of S-**group schemes** $u : Y \to G$. We are going to describe the functor L'_G in order to prove this result.

Before going into this, let us illustrate some results with the following example.

EXAMPLE 14.2. Let k be a ring (if one wants, an algebraically closed fied) and consider the group scheme $G = \mathrm{GL}_{n,k}$ over $\operatorname{Spec} k$. Its Lie algebra $\mathrm{Lie}(G)$ is the free k-module $M_n(k)$; we define the functor $W(\mathrm{Lie}(G))$ on the category of k-algebras by $W(\mathrm{Lie}(G))(R) = \mathrm{Lie}(G) \otimes_k R$. Denoting the dual k-module $\mathrm{Lie}(G)^*$ by $\omega_{G/S}$, one can also say that $W(\mathrm{Lie}(G))$ is represented by the spectrum $\mathbb{V}(\omega_{G/S})$ of the symmetric algebra over k of $\omega_{G/S}$. .

Now, let ε be a square-zero variable and set $TG = \prod_{k[\varepsilon]/k} G_{k[\varepsilon]}$. The projection $k[\varepsilon] \to k$ sending ε to 0 induces a short exact sequence of group functors:

$$1 \longrightarrow W(\mathrm{Lie}(G)) \longrightarrow TG \xrightarrow{\pi} G \longrightarrow 1 \tag{14.2}$$

that is, for every k-algebra R, one has an exact sequence of groups:

$$1 \longrightarrow \mathrm{Lie}(G) \otimes_k R \longrightarrow G(R[\varepsilon]) \xrightarrow{\pi} G(R) \longrightarrow 1.$$

Here, the inclusion $k \hookrightarrow k[\varepsilon]$ is a section of $k[\varepsilon] \to k$ hence induces a morphism of group functors $G \to TG$ which is a section of π. Hence TG (which is the tangent bundle to G) is the semi-direct product of $\mathbb{V}(\omega_{G/S})$ by G, where G acts on $\omega_{G/S}$ via the so-called *coadjoint action*; in particular, TG is representable.

The point of this example is two-fold:

1. The additive group law on $\mathrm{Lie}(G)$ coming from its structure of k-module coincides with the group law on the kernel H of the morphism $TG \to G$.

[2] We use the notation L'_G as in Exp. III, where L_G denotes another functor.

(2) The action of G on H by conjugation coincides, under the previous identification, with the adjoint action of G on $\mathrm{Lie}(G)$.

Indeed, these assertions are easily verified in this case: a R-point of H is a matrix of the form $I_n + \varepsilon A$, for some $A \in M_n(R)$. The product of two such elements is:

$$(I_n + \varepsilon A_1)(I_n + \varepsilon A_2) = I_n + \varepsilon(A_1 + A_2).$$

Further, for any $B \in G(R) = \mathrm{GL}_n(R)$, one has $B(I_n + \varepsilon A)B^{-1} = I_n + \varepsilon BAB^{-1}$. These facts will remain true in the more general case consider below.

Notes for this Lecture

Reductive (or semi-simple) groups over an algebraically closed field and reductive (or semi-simple) group schemes are defined in Exp. XIX, 1.6.1 and 2.7.

Maximal tori are defined in Exp. XII, Def. 1.3 and studied in Exp. XII–XIV.

Weil restriction of scalars is defined in Exp. II, §1.

The functors G^+ and L'_G are defined in Exp. III, under more general hypotheses in Def. 0.1.1 and remarks 0.4–0.5, and then put together, in the simpler case where G is a S-group scheme, in Cor. 0.9

LECTURE 7

Infinitesimal liftings and Hochschild cohomology: the beauty of SGA3, Exp. III

15. Group cohomology and extensions

REMARK 15.1. For simplicity, we have written this section in the category $\mathscr{C}$ = (Sets), so that a $\mathscr{C}$-group G is just a usual group. However, the results remain valid if we replace (Sets) by an arbitrary category $\mathscr{C}$ with fiber products, G by a group-object in $\mathscr{C}$, and the abelian group V by a contravariant functor $F : \mathscr{C} \to$ (Abelian groups) on which G acts linearly, **provided that** the set-theoretic sections considered in the proof of Lemma 15.4 exist as morphisms in $\mathscr{C}$; see [**SGA3**$_1$], Exp. III, Section 1.

DEFINITION 15.2. Firstly, let G be an abstract group and V a G-module. The cohomology groups $H^i(G,V)$ are the cohomology groups of the following complex, where Hom denotes maps of sets:

$$0 \longrightarrow V \xrightarrow{d^0} \operatorname{Hom}(G,V) \xrightarrow{d^1} \operatorname{Hom}(G^2,V) \xrightarrow{d^2} \operatorname{Hom}(G^3,V) \xrightarrow{d^3} \cdots \tag{15.1}$$

where $d^0(v)$ is the map $g \mapsto gv - v$, then, given $f : G \to V$, $d^1(f)$ is the map $(g_1,g_2) \mapsto g_1 f(g_2) - f(g_1 g_2) + f(g_1)$, then, given $f : G^2 \to V$, $d^2 f$ is the map

$$(g_1,g_2,g_3) \mapsto g_1 f(g_2,g_3) - f(g_1g_2,g_3) + f(g_1,g_2g_3) - f(g_1,g_2).$$

More generally, for $n \geq 2$ and $f : G^{n-1} \to V$, $d^{n-1}(f)$ is the map

$$d^{n-1}f(g_1,\dots,g_n) = g_1 f(g_2,\dots,g_n) + \sum_{i=1}^{n-1}(-1)^i f(g_1,\dots,g_ig_{i+1},\dots,g_n) + (-1)^n f(g_1,\dots,g_{n-1}). \tag{15.2}$$

Clearly, $H^0(G,V) = V^G$ is the submodule of invariants. Then $H^1(G,V)$ is the quotient of the $\mathbb{Z}$-module $Z^1(G,M) = \{f : G \to V \mid f(g_1g_2) = g_1f(g_2) + f(g_1)\}$ of 1-cocycles by the submodule $B^1(G,V) = \{d^0(v) \mid v \in V\}$ of 1-coboundaries. Consider the semi-direct product $E_0 = V \rtimes G$ and, for each $f \in Z^1(G,V)$, denote by σ_f the automorphism of E_0 defined by $\sigma_f(u,g) = (u - f(g), g)$.

LEMMA 15.3. *Then $f \mapsto \sigma_f$ is a group isomorphism between $Z^1(G,V)$ and the group of automorphisms of E_0 which restrict to the identity on V and on the quotient E/V; and for each $v \in V$ the coboundary $d^0(v)$ corresponds under this isomorphism to the inner automorphism $(u,g) \mapsto v(u,g)v^{-1} = (u + v - gv, g)$.*

PROOF. The proof is easy and left to the reader. □

In the next lemma, V is just an abelian group, without a given structure of G-module.

LEMMA 15.4. *Each exact sequence of groups:*

$$1 \longrightarrow V \longrightarrow E \xrightarrow{\pi} G \longrightarrow 1 \tag{15.3}$$

[0] version of August 30: four typos corrected.

makes V into a G-module and defines a class $c(E) \in H^2(G,V)$.

This class is zero if and only if there exists a morphism of groups $\tau : G \to E$ such that $\pi \circ \tau = \mathrm{id}_G$, that is, if and only if E is a semi-direct product of G and V.

In this case, the set of all such τ' is $\tau + Z^1(G,V)$, and the set of all such τ' up to conjugacy by the elements of V is $\tau + H^1(G,V)$.

Proof. Let s be a set-theoretic section of π. For each $g \in G$, consider the automorphism $c(g)$ of V defined $c(g)(v) = s(g)\, v\, s(g)^{-1}$. Since V is *abelian*, one sees that:

(1) Any other section s' of π defines, for each $g \in G$, the same automorphism $c(g)$ of V.

(2) The resulting map $c : G \to \mathrm{Aut}(V)$ is a morphism of group. Indeed, for $g_1, g_2 \in G$, one has $\pi(s(g_1g_2)) = g_1g_2 = \pi(s(g_1)s(g_2))$ hence the element

$$\widetilde{s}(g_1,g_2) = s(g_1g_2)s(g_2)^{-1}s(g_1)^{-1} \tag{15.4}$$

is in V and hence for any $v \in V$ one has $c(g_1g_2)(v) = c(g_1)\big(c(g_2)(v)\big)$. Thus, setting $gv = c(g)(v)$, one obtains that V is a G-module.

Now, to any set-theoretic section s of π one associates the function $\widetilde{s} : G^2 \to V$ defined in (15.4) above. One checks easily that $\widetilde{s}$ is a 2-cocycle, i.e. it belongs to the $\mathbb{Z}$-module:

$$Z^2(G,M) = \{f : G^2 \to V \mid g_1 f(g_2,g_3) + f(g_1,g_2g_3) - f(g_1,g_2) = 0\}.$$

Indeed, in E one has the equality:

$$\begin{aligned} d^2\widetilde{s}(x,y,z) &= (x \cdot \widetilde{s}(y,z))\, \widetilde{s}(x,y)^{-1}\, \widetilde{s}(xy,z)^{-1}\, \widetilde{s}(x,yz) \\ &= s(x)s(yz)s(z^{-1})s(y^{-1})s(x^{-1})s(x)s(y)s(xy)^{-1}s(xy)s(z)s(xyz)^{-1}s(xyz)s(yz)^{-1}s(x^{-1}) = e. \end{aligned}$$

Next, denote by $B^2(G,V) = \mathrm{Im}(d^1)$ the submodule of 2-coboundaries. If s, s' are two set-theoretic sections of π, there exists $f : G \to V$ such that $s'(g) = f(g)^{-1}s(g)$ for all g, and one has in E the equalities:

$$\begin{aligned} \widetilde{s'}(x,y) &= s'(xy)s'(y^{-1})s'(x^{-1}) = f(xy)^{-1}s(xy)s(y^{-1})f(y)s(x^{-1})f(x) \\ &= f(xy)^{-1}\widetilde{s}(x,y)s(x)f(y)s(x^{-1})f(x) = \widetilde{s}(x,y)f(xy)^{-1}(x \cdot f(y))f(x), \end{aligned}$$

where in the last equality we used that V is a normal abelian subgroup of E. Writing additively the group law of V, one has $f(xy)^{-1}(x \cdot f(y))f(x) = xf(y) - f(y) + f(x) = d^1f(x,y)$. This shows that $\widetilde{s'} = \widetilde{s} + d^1 f$. This proves two things:

(1) The image of $\widetilde{s}$ in $H^2(G,V) = Z^2(G,V)/B^2(G,V)$ does not depend on the choice of the set-theoretic section s; it it the class $c(E)$ of the extension.

(2) A section s' is a group homomorphism if and only if $\widetilde{s'} = 0$. Since, with the notation above, $\widetilde{s'} = \widetilde{s} + d^1 f$ for some f, this is the case if and only if $c(E) = 0$.

Now, assume $c(E) = 0$ and let τ be a section of π which is a group homomorphism. By the above, any other such τ' has the form $f\tau$, with $f \in Z^1(G,V)$. Further, for any $v \in V$ one has:

$$v^{-1}\tau'(g)v = v^{-1}\big(\tau'(g)v\tau'(g)^{-1}\big)\tau'(g) = (d^0v)(g)\tau'(g).$$

Thus the set of τ' up to conjugacy by the elements of V identifies with $\tau + H^1(G,V)$. □

Now, let $\phi : Y \to G$ be a morphism of groups. Using ϕ we can form the group $E_\phi = E \times_G Y$ and pull-back the exact sequence (15.3) to obtain the following exact sequence of groups:

$$1 \longrightarrow V \longrightarrow E_\phi \xrightarrow{\pi_\phi} Y \longrightarrow 1 \tag{15.5}$$

where V is sent into E_ϕ via the inclusion into E and via the unit morphism to Y. Note that the resulting action of Y on V is the same as the one derived from the G-action through the morphism $\phi : Y \to G$.

Further, any map of sets $\psi : Y \to E$ lifting ϕ defines a set-theoretic section of π_ϕ. Denoting by pr_1 the first projection $E_\phi \to E$, one sees easily that the map $\tau \mapsto \mathrm{pr}_1 \circ \tau$ is a bijection from the set of sections of π_ϕ to the set of liftings of ϕ; further, under this bijection the sections and liftings which are group homomorphisms correspond to each other.

Therefore, the question of lifting ϕ to a group homomorphism $\psi : Y \to E$ is reduced to the question of finding a splitting of (15.5). By the previous discussion with G replaced by Y, we obtain the:

COROLLARY 15.5. *Consider the exact sequence* (15.3) *and let* $\phi : Y \to G$ *be a morphism of groups.*

(1) *Suppose that* $H^2(Y, V) = 0$. *Then* ϕ *lifts to a morphism of groups* $\psi : Y \to E$.
(2) *Suppose further that* $H^1(Y, V) = 0$. *Then any two such lifting are conjugate by an element of* V*, that is, by an element of* E *whose image in* G *is the identity.*

The following lemma will be useful in the next section.

LEMMA 15.6. *Let* N *be an abelian group. Let* $\mathrm{Ind}(N) = \mathrm{Hom}(G, N)$ *be the* induced G-*module, where* G *acts on a function* $\phi : G \to N$ *by* $(g\phi)(g') = \phi(g'g)$ *for all* $g, g' \in G$. *Then* $\mathrm{Ind}(N)$ *is* acyclic, *i.e.* $H^i(G, \mathrm{Ind}(N)) = 0$ *for all* $i > 0$.

PROOF. Set $P = \mathrm{Ind}(N)$. For $n > 0$ let $s^n : \mathrm{Hom}(G^{n+1}, P) \to \mathrm{Hom}(G^n, P)$ be given by

$$s^n(f)(g_1, \ldots, g_n)(g) = f(g, g_1, \ldots, g_n)(e).$$

Then, for $f \in \mathrm{Hom}(G^n, P)$ one has

$$\begin{aligned} s^n d^n f(g_1, \ldots, g_n)(g) = d^n f(g, g_1, \ldots, g_n)(e) &= f(g_1, \ldots, g_n)(g) - f(gg_1, g_2, \ldots, g_n)(e) \\ &+ \sum_{i=1}^{n-1} (-1)^{i+1} f(g, \ldots, g_i g_{i+1}, \ldots, g_n)(e) + (-1)^{n+1} f(g, g_1, \ldots, g_{n-1})(e), \end{aligned}$$

$$\begin{aligned} d^{n-1}s^{n-1} f(g_1, \ldots, g_n)(g) = f(gg_1, g_2, \ldots, g_n)(e) &+ \sum_{i=1}^{n-1} (-1)^i f(g, \ldots, g_i g_{i+1}, \ldots, g_n)(e) \\ &+ (-1)^n f(g, g_1, \ldots, g_{n-1})(e) \end{aligned}$$

and hence $f = s^n d^n f + d^{n-1} s^{n-1} f$. Thus, if $d^n f = 0$ then $f = d^{n-1} s^{n-1} f$, hence $H^n(G, P) = 0$ for $n > 0$. □

REMARK 15.7. For simplicity, we have worked with G-modules, i.e. modules over the group ring $\mathbb{Z}[G]$. If Λ is any commutative ring, the same discussion applies to $\Lambda[G]$-modules V, and then the cohomology groups $H^i(G, V)$ are Λ-modules.

16. Hochschild cohomology

In this section, Λ is a ring and G is a flat[1] affine group scheme over $S = \operatorname{Spec} \Lambda$, given by the Λ-Hopf algebra A. Denote by $c : A \to A \otimes A$ its comultiplication (we write $\otimes$ instead of $\otimes_\Lambda$) and by ε it counit. Let L be a Λ-module endowed with a structure of G-module, that is, we are given a Λ-linear coaction $\mu_L : L \to L \otimes A$ satisfying the conditions seen in Def. 2.1.

DEFINITION 16.1. The Hochschild complex of L is

(16.1) $$0 \longrightarrow L \xrightarrow{d^0} L \otimes A \xrightarrow{d^1} L \otimes A \otimes A \xrightarrow{d^2} L \otimes A \otimes A \otimes A \xrightarrow{d^3} \cdots$$

[1] Flatness ensures that the category of A-comodules is abelian, see e.g. [**SGA3**$_1$], Exp. I, Cor. 4.7.2.1.

where $d^0(v) = \mu_L(v) - v \otimes 1$ and, for $n \geq 1$:

$$d^n(v \otimes a_1 \otimes \cdots \otimes a_n) = \mu_L(v) \otimes a_1 \otimes \cdots \otimes_n + \sum_{i=1}^{n} (-1)^i v \otimes a_1 \otimes \cdots c(a_i) \otimes \cdots \otimes a_n + (-1)^{n+1} v \otimes a_1 \otimes \cdots \otimes a_n \otimes 1. \tag{16.2}$$

Its cohomology groups are denoted by $H^i(G, L)$.

REMARK 16.2. Let $\mathscr{C}$ denote the category of affine schemes over S and let $W(L)$ be the functor such that $W(L)(T) = L \otimes \mathcal{O}(T)$ for any object T of $\mathscr{C}$. Then $G(T)$ acts linearly on $W(L)(T)$: for any $g \in G(T)$, i.e. any Λ-algebra morphism $g : A \to \mathcal{O}(T)$ and $x \otimes 1 \in L \otimes \mathcal{O}(T)$, one has $g(x \otimes 1) = (\mathrm{id}_L \otimes g)\mu_L(x)$.

Further, one has $\mathrm{Hom}_{\mathscr{C}}(G^n, W(L)) = L \otimes A^{\otimes n}$. Thus we see that Hochschild cohomology can be viewed as the cohomology of groups in $\mathscr{C}$. The results of Section 15 about lifting of homomorphims remain valid, provided that there exist morphisms of schemes that replace the set-theoretic sections used in Section 15. This will be ensured by the hypothesis that the group-scheme G is *smooth* and that we are concerned with *infinitesimal liftings*.

Analogously to Lemma 15.6, one has:

LEMMA 16.3. *For a Λ-module N, let $\mathrm{Ind}(N) = N \otimes A$ regarded as A-comodule via $\mathrm{id}_N \otimes c : N \otimes A \to N \otimes A \otimes A$. Then $H^i(G, \mathrm{Ind}(N)) = 0$ for all $i > 0$.*

PROPOSITION 16.4. *Suppose that $G = D(M)_S$ is a diagonalisable group. Then $H^i(G, L) = 0$ for $i > 0$ and any G-Λ-module L.*

PROOF. [2] Let L a G-module. Recall that $L = \bigoplus_{m \in M} L_m$, where $L_m = \{x \in L \mid \mu_L(x) = x \otimes m\}$. Let $(p_m)_{m \in M}$ denote the corresponding family of projectors. Note first that the map $\mu_L : L \to \mathrm{Ind}(L)$ is a morphism of A-comodules: indeed, by one of the defining properties of comodules, the diagram below is commutative:

$$\begin{array}{ccc} L & \xrightarrow{\mu_L} & \mathrm{Ind}(L) \\ \downarrow{\scriptstyle \mu_L} & & \downarrow{\scriptstyle \mathrm{id}_L \otimes c} \\ L \otimes A & \xrightarrow{\mu_L \otimes \mathrm{id}_A} & \mathrm{Ind}(L) \otimes A. \end{array}$$

Now the map $r : \mathrm{Ind}(L) \to L$ sending each finite sum $\sum_m x_m \otimes m$ to $\sum_m p_m(x_m)$ is a retraction of μ_L, because writing $x = \sum_m p_m(x)$ we have $\mu_L(x) = \sum_m p_m(x) \otimes m$ and hence $r(\mu_L(x)) = x$. Further, r is a morphism of A-comodules, that is the diagram below is commutative:

$$\begin{array}{ccc} \mathrm{Ind}(L) & \xrightarrow{r} & L \\ \downarrow{\scriptstyle \mathrm{id}_V \otimes c} & & \downarrow{\scriptstyle \mu_L} \\ \mathrm{Ind}(L) \otimes A & \xrightarrow{r \otimes \mathrm{id}_A} & L \otimes A. \end{array}$$

Indeed, an element $y = \sum_m x_m \otimes m$ of $\mathrm{Ind}(L)$ is sent by $\mathrm{id}_V \otimes c$ to $\sum_m x_m \otimes m \otimes m$ which goes by $r \otimes \mathrm{id}_A$ to $\sum_m p_m(x_m) \otimes m$, which is also $\mu_L(r(y))$. Thus L is a G-module direct summand of $\mathrm{Ind}(L)$ and since the latter is acyclic, so is L. ☐

REMARK 16.5. Let $\Lambda \to \Lambda'$ be a flat map, set $L' = L \otimes \Lambda'$ and let $S' = \mathrm{Spec}\,\Lambda'$ and $G' = G_{S'}$. Let $C^\bullet(G, L)$ denote the Hochschild complex of L. Then $C^\bullet(G, L) \otimes \Lambda'$ identifies with the Hochschild complex $C^\bullet(G', L')$ hence, since $\Lambda \to \Lambda'$ is flat, one has $H^i(G, L) \otimes \Lambda' \simeq H^i(G', L')$ for all $i \geq 0$.

[2] This neat proof is taken from [**DG70**], §II.3, Prop. 4.2.

Corollary 16.6. *Let G be a* MT*-group of type M over S and L a G-Λ-module. Then $H^i(G, L) = 0$ for $i > 0$.*

Proof. By hypothesis, there exist faithfully flat morphisms $U_i' \to U_i$, where the U_i and U_i' are affine and the U_i form an open covering of $S = \operatorname{Spec} \Lambda$, such that $G_{U_i'} \simeq D(M)_{U_i'}$. Since S is affine, hence quasi-compact, it is covered by finitely many of the U_i. Then the disjoint sum S' of the corresponding U_i' is affine and faithfully flat over S, and $G_{S'} \simeq D(M)_{S'}$. Set $\Lambda' = \mathcal{O}(S')$ and $L' = L \otimes \Lambda'$. By the previous remark and proposition, one has $0 = H^i(G_{S'}, L') \simeq H^i(G, L) \otimes \Lambda'$. Since Λ' is faithfully flat over Λ, it follows that $H^i(G, L) = 0$ for $i > 0$. $\square$

17. Infinitesimal liftings and Hochschild cohomology

In this section, we fix a nilpotent ideal I in a ring Λ and denote by $S \supset S_0$ the spectra of Λ and Λ/I; they have the same underlying topological space. For every S-scheme X we denote by X_0 its pullback over S_0.

Further, we fix **affine S-group schemes G and Y, with G smooth and Y flat and such that Y_0 is of multiplicative type**. The goal of this section is to prove the following theorem, where $\operatorname{int}(g)$ denote the automorphism of conjugation by g.

Theorem 17.1. *Let $u_0 : Y_0 \to G_0$ be a morphism of S_0-group schemes. Then:*

1. *There exists a morphism of S-group schemes $u : Y \to G$ that lifts u_0.*
2. *If v is another such morphism, there exists $g \in \operatorname{Ker}(G(S) \to G(S_0))$ such that $v = \operatorname{int}(g) \circ u$.*
3. *More generally, if $v : Y \to G$ is a morphism of S-group schemes and $g_0 \in G(S_0)$ is such that $v_0 = \operatorname{int}(g_0) \circ u_0$, there exists a lifting g of g_0 such that $v = \operatorname{int}(g) \circ u$.*

Let n be the smallest positive integer such that $I^n = 0$. Assume first that $n = 2$ and that assertions (1) and (2) are proved for $n = 2$. Let $v : Y \to G$ be a morphism of S-group schemes and $g_0 \in G(S_0)$ such that $v_0 = \operatorname{int}(g_0) \circ u_0$. Since G is smooth, g_0 lifts to an element $g' \in G(S)$; set $u' = \operatorname{int}(g') \circ u$. Then $v_0 = u_0'$ hence, by (2), there exists $g'' \in \operatorname{Ker}(G(S) \to G(S_0))$ such that $v = \operatorname{int}(g'') \circ u'$. Then $v = \operatorname{int}(g''g') \circ u$, and $g''g'$ is a lifting of g_0. This proves (3) for $n = 2$.

Now, let $n \geq 2$, assume the theorem proved for all ideals J such that $J^n = 0$ and let I be such that $I^{n+1} = 0$. Set $J = I^2$ and $S_J = \operatorname{Spec}(\Lambda/J)$, then $J^n = 0$ and the image $\overline{I}$ of I in Λ/J satisfies $\overline{I}^2 = 0$. By the case $n = 2$ and the induction hypothesis, u_0 lifts to a morphism of S_J-group schemes $u_J : Y_J \to G_J$ and u_J lifts to a morphism of S-group schemes $u : Y \to G$. This proves (1).

Further, let $v : Y \to G$ be a morphism of S-group schemes and $g_0 \in G(S_0)$ such that $v_0 = \operatorname{int}(g_0) \circ u_0$. By the case $n = 2$ and the induction hypothesis, g_0 lifts to an element $g_J \in G(S_J)$ such that $v_J = \operatorname{int}(g_J) \circ u_J$ and g_J lifts to an element $g \in G(S)$ such that $v = \operatorname{int}(g) \circ u$. This proves (3), and of course (2) is the special case $g_0 = e$. Thus, it suffices to prove the following proposition. From now on we assume that $I^2 = 0$.

Proposition 17.2. *Suppose that $I^2 = 0$ and let $u_0 : Y_0 \to G_0$ be a morphism of S_0-group schemes. Then:*

1. *There exists a morphism of S-group schemes $u : Y \to G$ that lifts u_0.*
2. *If v is another such morphism, there exists $g \in \operatorname{Ker}(G(S) \to G(S_0))$ such that $v = \operatorname{int}(g) \circ u$.*

Recall now the notation introduced in Definition 14.1. In particular, we have an exact sequence of group functors:

$$1 \longrightarrow L'_G \longrightarrow G \xrightarrow{\pi} G^+ \longrightarrow 1. \tag{17.1}$$

We are going to describe the functor L'_G in order to prove Proposition 17.2.

We have the smooth affine group-scheme $G = \operatorname{Spec}(A)$, where A is a Λ-Hopf algebra. Let $c : A \to A \otimes A$ and $\varepsilon : A \to \Lambda$ be the comultiplication and augmentation (or counit) maps, which correspond to the multiplication of G and to the unit section $S \to G$. Recall that $\Lambda_0 = \Lambda/I$ and that the pull-back of a S-scheme X to $S_0 = \operatorname{Spec} \Lambda_0$ is denoted by X_0.

DEFINITION 17.3. Let $\mathfrak{m} = \operatorname{Ker} \varepsilon$; since G is smooth over S, $\mathfrak{m}/\mathfrak{m}^2$ is a locally free Λ-module of finite rank (equal to the relative dimension d of G over S). Further, one has $(\mathfrak{m}/\mathfrak{m}^2) \otimes \Lambda_0 \simeq \mathfrak{m}_0/\mathfrak{m}_0^2$ with obvious notation.

By definition, $\operatorname{Lie}(G/S)$ is the Zariski tangent space to G along the unit section, i.e. it is the Λ-module $\operatorname{Lie}(G/S) = \operatorname{Hom}_\Lambda(\mathfrak{m}/\mathfrak{m}^2, \Lambda)$. It is locally free of rank d. Similarly for $\operatorname{Lie}(G_0/S_0)$ over Λ_0.

DEFINITION 17.4. The left action of G on itself by inner automorphisms, that is, $\operatorname{int}(g)(g') = gg'g^{-1}$, induces a structure of left A-comodule $\mu : V \to A \otimes V$ on $V = A$, which corresponds to the linear right action of G on A given $(\phi \cdot g)(g') = \phi(gg'g^{-1})$ for $\phi \in A$ and arbitrary R-points $g, g' \in G(R)$. Clearly, $\mathfrak{m} = \operatorname{Ker} \varepsilon$ is stable by this G-action, as well as $\mathfrak{m}^2$, hence there is a natural right action of G on the cotangent space $\mathfrak{m}/\mathfrak{m}^2$ which is called the *coadjoint action.* The induced action on the dual space $\operatorname{Lie}(G/S)$ is called the *adjoint action.*[3]

LEMMA 17.5. *Let $T = \operatorname{Spec} B$ for some Λ-algebra B. Consider the B_0-module $F(T) = \operatorname{Hom}_{\Lambda_0}(\mathfrak{m}_0/\mathfrak{m}_0^2, IB)$.*

(1) *$L'_G(T)$ is the set of Λ-algebras morphisms of the form $\phi = \varepsilon + D$, with $D \in F(T)$.*
(2) *The resulting identification $L'_G(T) = F(T)$ respects the group laws and the conjugation action of G on $L'_G(T)$ corresponds to the action on $F(T)$ induced by the coadjoint action on $\mathfrak{m}/\mathfrak{m}^2$.*
(3) *If T is* **flat** *over S, setting* $\boxed{L_0 = \operatorname{Hom}_{\Lambda_0}(\mathfrak{m}_0/\mathfrak{m}_0^2, I)}$ *one has* $\boxed{F(T) = L_0 \otimes_{\Lambda_0} B_0.}$

PROOF. (1) By definition, $L'_G(T)$ is the set of algebra morphisms $\phi : A \to B$ which reduce to ε modulo I. Thus, for any $a \in A$, we can write $\phi(a) = \varepsilon(a) + D(a)$, with $D(a) \in IB$. One has $A = \Lambda 1 \oplus \mathfrak{m}$ and $\phi(1) = 1 = \varepsilon(1)$, so we may consider D as a Λ-linear map $\mathfrak{m} \to IB$. Since $(IB)^2 = 0$, the condition that ϕ be a morphim of algebras becomes:

$$\varepsilon(a_1a_2) + D(a_1a_2) = \phi(a_1a_2) = \phi(a_1)\phi(a_2) = \varepsilon(a_1a_2) + \varepsilon(a_1)D(a_2) + \varepsilon(a_2)D(a_1),$$

which is equivalent to

$$D(a_1a_2) = \varepsilon(a_1)D(a_2) + \varepsilon(a_2)D(a_1). \tag{17.2}$$

One expresses this equality by saying that D is an ε-derivation $A \to IB$. This implies that D vanishes on $\mathfrak{m}^2$. Conversely, one sees that any Λ-linear map $\mathfrak{m}/\mathfrak{m}^2 \to IB$ defines a map D as above. This proves the first equality below, and the second follows since IB is annihilated by I:

$$L'_G(T) = \operatorname{Hom}_\Lambda(\mathfrak{m}/\mathfrak{m}^2, IB) = \operatorname{Hom}_{\Lambda_0}\big((\mathfrak{m}/\mathfrak{m}^2) \otimes \Lambda_0, IB\big).$$

Finally, since $(\mathfrak{m}/\mathfrak{m}^2) \otimes \Lambda_0 \simeq \mathfrak{m}_0/\mathfrak{m}_0^2$, one obtains assertion (1).

(2) Let $\phi \in \mathfrak{m}$. Since $A = \Lambda 1 \oplus \mathfrak{m}$ one can write uniquely:

$$c(\phi) = \lambda 1 \otimes 1 + \phi_1 \otimes 1 + 1 \otimes \phi_2 + \sum_i \psi_i \otimes \theta_i \tag{17.3}$$

with ϕ_1, ϕ_2 and the ψ_i, θ_i in $\mathfrak{m}$. Since $\phi = (\mathrm{id} \otimes \varepsilon)c(\phi) = (\varepsilon \otimes \mathrm{id})c(\phi)$, and $\varepsilon(\phi) = 0$, one obtains successively that $\lambda = 0$ and $\phi_1 = \phi = \phi_2$.

[3]Over a base field, the adjoint action is considered as the primary object and the coadjoint action is its dual, but over an arbitrary base one has to note that the action on $\mathfrak{m}/\mathfrak{m}^2$ (which is $\varepsilon^*(\Omega_{G/S})$) comes first.

Now, let $g_1, g_2 \in L'_G(T)$ and write $g_i = \varepsilon + D_i$ for $i = 1, 2$. Recall that the product $g_1 g_2$ is $m_B \circ (g_1 \otimes g_2) \otimes c$, where m_B is the multiplication of B. It follows from (17.3) that for any $\phi \in \mathfrak{m}$ one has

$$(g_1 g_2)(\phi) = D_1(\phi) + D_2(\phi) + \sum_i D_1(\psi_i) D_2(\theta_i) = (D_1 + D_2)(\phi)$$

where in the second equality we have used that $(IB)^2 = 0$. This proves the first part of (2), i.e. that under the identification $g_i \leftrightarrow D_i$ the group law of $L'_G(T)$ transforms into the addition law of the B_0-module $F(T)$.

Next, let $g \in G(T)$ and $g_1 = \varepsilon + D_1 \in L'_G(T)$. For any $\phi \in \mathfrak{m}$, let $\overline{\phi}$ denote its image in $\mathfrak{m}/\mathfrak{m}^2$. Then $\mathrm{int}(g)(g_1)$ sends ϕ to

$$\phi\big(c(g)(g_1)\big) = (\phi \cdot g)(g_1) = g_1(\phi \cdot g) = (\varepsilon + D_1)(\overline{\phi} \cdot g) = D_1(\overline{\phi} \cdot g)$$

and on the right-hand side this is the action of G on $F(T)$ induced by the coadjoint action on $\mathfrak{m}/\mathfrak{m}^2$. This completes the proof of (2).

(3) Suppose that B is flat over Λ. Then we have isomorphims $I \otimes B_0 = I \otimes B \xrightarrow{\sim} IB$ and hence, since B_0 is flat over Λ_0 and $\mathfrak{m}_0/\mathfrak{m}_0^2$ is locally free of finite rank, one obtains

$$F(T) \simeq \mathrm{Hom}_{\Lambda_0}(\mathfrak{m}_0/\mathfrak{m}_0^2, I) \otimes_{\Lambda_0} B_0 \simeq \mathrm{Lie}(G_0/S_0) \otimes_{\Lambda_0} I \otimes_{\Lambda_0} B_0.$$

Thus, setting $L_0 = \mathrm{Hom}_{\Lambda_0}(\mathfrak{m}_0/\mathfrak{m}_0^2, I) \simeq \mathrm{Lie}(G_0/S_0) \otimes_{\Lambda_0} I$, one has $F(T) = L_0 \otimes_{\Lambda_0} B_0$. □

We can now prove Proposition 17.2

Proof of proposition 17.2. Let $Y = \mathrm{Spec}\, B$ be a flat affine group scheme over S, with Y_0 of multiplicative type and suppose given a morphism of S-group functors $\phi : Y \to G^+$, i.e. a morphism of S_0-groups $u_0 : Y_0 \to G_0$. Since G is smooth and I nilpotent, there exists a morphism of S-*schemes* $s : Y \to G$ lifting u_0.

As in Section 15, we can use ϕ to form the S-group functor $E = G \times_{G^+} Y$ and pull-back the short exact sequence (17.1) to obtain the following short exact sequence of S-group functors:

$$1 \longrightarrow L'_G \longrightarrow E \xrightarrow{\pi} Y \longrightarrow 1. \qquad (17.4)$$

The morphism of S-*schemes* $s : Y \to G$ is a section of π. Proceeding as in Section 15, we obtain a morphism $\widetilde{s} : Y^2 \to L'_G$ defined for arbitrary points $y_1, y_2 \in Y(T)$ by

$$\widetilde{s}(T)(y_1, y_2) = s(y_1 y_2) s(y_1)^{-1} s(y_2)^{-1}.$$

This is an element of $L'_G(Y^2)$, which equals $L_0 \otimes B_0 \otimes B_0$ since B is flat over Λ. By Yoneda lemma, the fact that $\widetilde{s}(T)$ is a cocycle for group cohomology, for any T, translates into the fact that $\widetilde{s}$ defines a class in the Hochschild cohomology group $H^2(Y_0, L_0)$. But the latter is 0 by Cor. 16.6 since Y_0 is of multiplicative type. Therefore, u_0 can be lifted to a morphism of S-group schemes $u : Y \to G$.

Then, as in Section 15, any other such morphism v has the form $v = fu$, where f is a morphism $Y \to L'_G$, i.e. an element of $L'_G(Y) = L_0 \otimes B_0$, which is a 1-cocyle. Since $H^1(Y_0, L_0) = 0$, one obtains by Corollary 15.5 that $v = \mathrm{int}(g) \circ u$ for some $g \in \mathrm{Ker}(G(S) \to G(S_0))$. This completes the proof of Proposition 17.2. □

Notes for this Lecture

The cohomology of groups in a category is defined in Exp. I, §5.1. Then Lemma 15.4 and Corollary 15.5 are proved in Exp. III, Prop. 1.2.4 but are standard results in group cohomology, as well as Lemma 15.6.

Hochschild homology is defined in Exp. I, §5.3, where Prop. 16.4 is proved as Th. 5.3.3 (whereas Lemma 16.3 is contained in the proof of 5.3.1.1). The extension to groups of multiplicative type (Cor. 16.6) is Exp. IX, Th. 3.1.

Theorem 17.1 corresponds to Theorems 3.2 and 3.6 of Exp. IX, whose proofs rely on Exp. III, Th. 2.1 and Cor. 2.5.

LECTURE 8

MT-groups over a complete noetherian ring

18. MT-groups over infinitesimal thickenings

Remark 18.0. Beware that assertion (1) of the proposition below cannot be derived from Th. 17.1 because in that theorem there is a smoothness assumption on the target group, whereas a MT-group is not necessarily smooth: μ_n is not smooth over S is some residual characteristic of S divides n.

Proposition 18.1. *Let S be a scheme and S_0 a closed subscheme having the same underlying topological space. Then:*

1. *The functor $H \longmapsto H_0 = H \times_S S_0$ from the category of* MT*-groups over S to the analogous one over S_0 is* fully faithful.
2. *It induces an* equivalence *between the subcategory of* quasi-isotrivial*, resp.* isotrivial*,* MT*-groups over S and the analogous one over S_0*

Proof. (1) Let H, G be MT-groups over S. Let us prove that the map

$$u : \mathrm{Hom}_{S\text{-Gr}}(H,G) \longrightarrow \mathrm{Hom}_{S_0\text{-Gr}}(H_0,G_0)$$

is bijective. As this question is local over S, we may assume S affine. Then there exists a faithfully flat morphism $S' \to S$, with S' affine, such that the pullbacks H' and G' are diagonalisable. Let H'' and G'' denote the pullbacks over $S'' = S' \times_S S'$. Let $S'_0 = S \times_S S_0$ and define similarly H'_0, G'_0, etc. One has then a commutative diagram with exact rows:

$$\begin{array}{ccccc}
\mathrm{Hom}_{S\text{-Gr}}(H,G) & \longrightarrow & \mathrm{Hom}_{S'\text{-Gr}}(H',G') & \rightrightarrows & \mathrm{Hom}_{S''\text{-Gr}}(H'',G'') \\
\downarrow u & & \downarrow u' & & \downarrow u'' \\
\mathrm{Hom}_{S_0\text{-Gr}}(H_0,G_0) & \longrightarrow & \mathrm{Hom}_{S'_0\text{-Gr}}(H'_0,G'_0) & \rightrightarrows & \mathrm{Hom}_{S''_0\text{-Gr}}(H''_0,G''_0),
\end{array}$$

hence to prove that u is bijective, it suffices to do so for u' and u''. We are therefore reduced to the case where H and G are diagonalisable, say $G = D(M)_S$ and $H = D(N)_S$. Then $G_0 = D(M)_{S_0}$ and $H_0 = D(M)_{S_0}$. By Cor. 6.3 and the proof of Prop. 7.5 in Lecture 3, we obtain a commutative diagram:

$$\begin{array}{ccc}
\mathrm{Hom}_{S\text{-Gr}}(D(N)_S, D(M)_S) & \xrightarrow{\ u\ } & \mathrm{Hom}_{S_0\text{-Gr}}(D(N)_{S_0}, D(M)_{S_0}) \\
\| & & \| \\
\mathrm{Loc}_{\mathrm{grp}}(M \times S, N) & \xrightarrow{\ D(u)\ } & \mathrm{Loc}_{\mathrm{grp}}(M \times S_0, N)
\end{array}$$

where $\mathrm{Loc}_{\mathrm{grp}}(M \times S, N)$ is the set of maps $M \times S \to N$ which are additive in the first variable and locally constant in the second, and where $D(u)$ is the map induced by the inclusion $S_0 \to S$. Since S_0 and S have the same underlying topological space, $D(u)$ is bijective, hence so is u. This proves (1).

(2) Let H_0 be a quasi-isotrivial (resp. isotrivial) MT-group over S_0. We have to prove that there exists a quasi-isotrivial (resp. isotrivial) MT-group H over S such that $H \times_S S_0 \simeq H_0$.

[0]Version of Sep. 1, 2023. This is the original version prepared for the lecture. The shortcuts discussed during the lecture were not legitimate, see Remark 18.0, so we revert to the original text.

By hypothesis, there exists a surjective étale (resp. finite étale) morphism $S_0' \to S_0$ such that the pullback H_0' is a diagonalisable group $D(M)_{S_0'}$. Now, recall that the functor $X \longmapsto X_0 = X \times_S S_0$ is an equivalence between the category of schemes étale over S and that of schemes étale over S_0; see [**SGA1**], Exp. I, 8.3 when S is locally noetherian, and EGA IV_4, 18.1.2 in general. Thus, there exists a surjective (resp. finite étale) étale morphism $S' \to S$ such that $S_0' = S' \times_S S_0$.

Then $H' = D(M)_{S'}$ is such that $H' \times_{S'} S_0' = H_0'$. Define as usual S'', S''' and note $S'' \times_S S_0 \simeq S_0' \times_{S_0} S_0'$ and similarly for S_0'''. As $H_0' = H_0 \times_{S_0} S_0'$, it is endowed by with a descent datum relative to $S_0' \to S_0$. Applying the result of (1) to the pairs (S'', S_0'') and (S''', S_0'''), one obtains that this descent datum comes from a descent datum on H' relative to $S' \to S$. Since H' is affine over S', this descent datum is effective, by Theorem 8.18 of Lecture 4. Thus there exists a S-group scheme H such that $H \times_S S' = H' = D_{S'}(M)$, and hence H is a quasi-isotrivial (resp. isotrivial) MT-group over S.

Further, the pullbacks of $H \times_S S_0$ and H_0 by $S_0' \to S_0$ are isomorphic. Since $S_0' \to S_0$ is surjective and étale, it is a morphism of descent (see e.g. Lemme 8.22), hence the previous isomorphism comes from an isomorphism $H \times_S S_0 \simeq H_0$. This completes the proof of (2). □

Remark 18.2. Suppose for simplicity that $S = \operatorname{Spec} \Lambda$ is affine. Then a closed subscheme $S_0 = \operatorname{Spec}(\Lambda/I)$ has the same underlying space topological space if and only if I is a *nilideal*, i.e. for every $a \in I$ there exists an integer n such that $a^n = 0$. If Λ is not noetherian, I need not be nilpotent: for example, let k be a field, A the polynomial ring over k in infinitely many variables $(X_i)_{i \in \mathbb{N}^*}$ and Λ the quotient of A by the relations $X_i^{i+1} = 0$. Then the maximal ideal of Λ is a nilideal which is is not nilpotent.

For simplicity, assume now that $S = \operatorname{Spec} \Lambda$ and $S_0 = \operatorname{Spec}(\Lambda/I)$. Under the additional assumption that I be nilpotent, one obtains the following stronger result.

Theorem 18.3. *Suppose that $S = \operatorname{Spec} \Lambda$ and $S_0 = \operatorname{Spec}(\Lambda/I)$, with I* nilpotent. *Let H be a flat S-group scheme such that H_0 is a quasi-isotrivial, resp. isotrivial,* MT-*group over S_0. Then H is a quasi-isotrivial, resp. isotrivial,* MT-*group over S.*

Proof. Suppose that H_0 is a quasi-isotrivial (resp. isotrivial) MT-group over S_0. Proceeding as in the previous proof, we obtain a surjective étale (resp. finite étale) morphism $S' \to S$ such that $H_0' \simeq D(M)_{S_0'}$. We want to prove that $H \simeq D(M)_{S'}$. So, replacing S by S', we are reduced to the case where $H_0 = D(M)_{S_0}$.

Set $G = D(M)_S$. Then we have an isomorphism $u_0 : H_0 \xrightarrow{\sim} G_0$. Let us show[1] that u_0 lifts uniquely to a morphism of S-group schemes $u : H \to G$. By Cor. 6.3 one has

$$\operatorname{Hom}_{S\text{-Gr}}(H, G) = \operatorname{Hom}_{S\text{-Gr}}(M_S, \underline{\operatorname{Hom}}_{S\text{-Gr}}(H, \mathbb{G}_{m,S})) = \operatorname{Hom}_{\mathrm{grp}}(M, \operatorname{Hom}_{S\text{-Gr}}(H, \mathbb{G}_{m,S})) \tag{18.1}$$

the second equality coming from $\operatorname{Hom}_{S\text{-Gr}}(M_S, Y) = \operatorname{Hom}_{\mathrm{grp}}(M, Y(S))$ for any S-group scheme Y. Then, we have a commutative diagram:

$$\begin{array}{ccc} \operatorname{Hom}_{S\text{-Gr}}(H, G) & = & \operatorname{Hom}_{\mathrm{grp}}(M, \operatorname{Hom}_{S\text{-Gr}}(H, \mathbb{G}_{m,S})) \\ \downarrow & & \downarrow \\ \operatorname{Hom}_{S_0\text{-Gr}}(H_0, G_0) & = & \operatorname{Hom}_{\mathrm{grp}}(M, \operatorname{Hom}_{S_0\text{-Gr}}(H_0, \mathbb{G}_{m,S_0})) \end{array}$$

where the vertical maps are induced by the base change $S_0 \to S$. Since H is flat over S and H_0 of multiplicative type and $G = \mathbb{G}_{m,S}$ is smooth and *commutative* (so that the inner automorphisms are trivial), Theorem 17.1 ensures that the map

$$\operatorname{Hom}_{S\text{-Gr}}(H, \mathbb{G}_{m,S}) \to \operatorname{Hom}_{S_0\text{-Gr}}(H_0, \mathbb{G}_{m,S_0})$$

is *bijective.* Therefore $u_0 : H_0 \to G_0$ lifs to a unique morphism of S-group schemes $u : H \to G$.

[1]Again, we cannot invoke directly Th. 17.1 because H is not necessarily smooth. This is why the duality functor D is used, in order to be in a situation where the target group is $\mathbb{G}_m$, which is smooth (and commutative).

Moreover, u is an isomorphism. Indeed, since u_0 is an isomorphism, it suffices to see that for each $h \in H$, the ring homomorphism $\phi : \mathcal{O}_{G,u(h)} \to \mathcal{O}_{H,h}$ is bijective. Let C and K denote its cokernel and kernel. By assumption, $\phi_I = \phi \otimes (\Lambda/I)$ is bijective. It follows that C satisfies $C = IC$, hence $C = 0$ since I is nilpotent. Then, since $\mathcal{O}_{H,h}$ is flat over Λ, the kernel of ϕ_I is K/IK. It follows, as above, that $K = IK$ and hence $K = 0$. This completes the proof. □

Notes for this Lecture

Prop. 18.1 and Th. 18.3 are respectively Prop. 2.1 and Cor. 2.3 of Exp. X.

MT-groups over a complete noetherian ring

19. Projective limits of homomorphisms: the algebrisation theorem

Definition 19.1. Let I be an ideal in a ring A. One says that A is *separated and complete for the I-adic topology* if the natural ring homomorphism $A \to \varprojlim_n A/I^n$ is bijective. In other words, $\bigcap_{n\in\mathbb{N}} I^n = \{0\}$ and every sequence $(a_n)_{n\in\mathbb{N}}$ of elements of A such that $a_{n+1} - a_n \in I^{n+1}$ converges to an element a of A (equivalently, every series $\sum_{n\geq 0} b_n$ with $b_n \in I^n$ converges to an element b of A).

Theorem 19.2. *Let A be a noetherian ring, with an ideal I such that A is separated and complete for the I-adic topology. Set $S = \mathrm{Spec}(A)$ and $S_n = \mathrm{Spec}(A/I^{n+1})$ for $n \geq 0$.*

Let G be an affine S-group scheme and H an isotrivial MT-group over S. *For each n, we denote by G_n, H_n their pullbacks to S_n.*

(1) *The canonical map $\theta : \mathrm{Hom}_{S\text{-Gr}}(H, G) \longrightarrow \varprojlim_n \mathrm{Hom}_{S_n\text{-Gr}}(H_n, G_n)$ is bijective.*

(2) *Suppose further that* G is flat over S at each point of G_0 and G_0 is an isotrivial MT-group over S_0. *Then the map* $\mathrm{Hom}_{S\text{-Gr}}(H, G) \longrightarrow \mathrm{Hom}_{S_0\text{-Gr}}(H_0, G_0)$ *is bijective.*

Proof. (1) Suppose first the result proved when H is diagonalisable. In the general case, there exists by hypothesis a surjective *finite* étale morphism $A \to A'$ such that the pull-back H' of H to $S' = \mathrm{Spec}\, A'$ is diagonalisable. Then so are H'' and H'_n, H''_n, with obvious notation. Moreover, since A' and A'' are finite over A, they are separated and complete for the topology defined by the ideals IA' and IA''. Thus, assuming the result proved in the diagonalisable case, the second and third vertical maps in the commutative diagram below are bijective:

$$\begin{array}{ccccc}
\mathrm{Hom}_{S\text{-Gr}}(H,G) & \longrightarrow & \mathrm{Hom}_{S'\text{-Gr}}(H',G') & \rightrightarrows & \mathrm{Hom}_{S''\text{-Gr}}(H'',G'') \\
\downarrow u & & \downarrow u' & & \downarrow u'' \\
\varprojlim_n \mathrm{Hom}_{S_n\text{-Gr}}(H_n,G_n) & \longrightarrow & \varprojlim_n \mathrm{Hom}_{S'_n\text{-Gr}}(H'_n,G'_n) & \rightrightarrows & \varprojlim_n \mathrm{Hom}_{S''_n\text{-Gr}}(H''_n,G''_n).
\end{array}$$

Further, $A \to A'$ is a morphism of descent (being faithfully flat and quasi-compact) hence the first row is exact, as well as the analogous row for a given n. Since projective limits are left exact, the bottom row is also exact. It follows that the first vertical map is bijective.

Thus, it suffices to prove the theorem when H is diagonalisable, say $H = D(M)_S$. Set $B = A[M]$ and let C be the A-Hopf algebra of the affine group scheme G. Denote by Δ_B and Δ_C their comultiplication maps. Denoting by $(e_m)_{m\in M}$ the canonical basis of $A[M]$, recall that $\Delta_B(e_m) = e_m \otimes e_m$.

For $n \in \mathbb{N}$, set $A_n = A/I^{n+1}$ and let B_n and C_n be obtained by base change. Note that $B_n = A_n[M]$. The morphisms of S-group schemes $H \to G$, resp. $H_n \to G_n$, correspond to the morphisms of A-Hopf algebras $\varphi : C \to B$, resp. $\varphi_n : C_n \to B_n$. Set $\widehat{B} = \varprojlim_n B_n$ and $\widehat{C} = \varprojlim_n C_n$ and let $\tau_B : B \to \widehat{B}$ and $\tau_C : C \to \widehat{C}$ be the canonical maps.

[0]version of Sept. 2, 2023, after the lecture.

Note first that one has $I^nB = \bigoplus_m I^n e_m$ for each n, hence $\bigcap_{n\in\mathbb{N}} I^nB = \{0\}$. That is, τ_B is injective.[1] Since $B \otimes_A B = A[M \times M]$, the same argument shows that $\tau_{B\otimes_A B}$ is injective too. The injectivity of τ_B immediately gives that the map θ is **injective**. Indeed, for any morphism of Hopf algebras $\varphi : C \to B$, the projective system $\theta(\varphi) = (\varphi_n)_{n\geq 0}$ of morphisms of Hopf algebras induces a morphism of algebras $\widehat{\varphi} : \widehat{C} \to \widehat{B}$ such that the diagram

$$\begin{array}{ccc} C & \xrightarrow{\varphi} & B \\ {\scriptstyle \tau_C}\downarrow & & \downarrow{\scriptstyle \tau_B} \\ \widehat{C} & \xrightarrow{\widehat{\varphi}} & \widehat{B} \end{array}$$

is commutative. Since τ_B is injective, this shows that the map $\theta : \varphi \mapsto (\varphi_n)_{n\geq 0}$ is injective.

Let us prove that θ is surjective. Let $(\varphi_n)_{n\geq 0}$ be a projective system of Hopf algebra morphisms $C_n \to B_n$. It induces a morphism of algebras $\widehat{\varphi} : \widehat{C} \to \widehat{B}$.

What we want is a morphism of Hopf algebras $C \to B$, but a difficulty is that taking the projective limit of the comultiplication maps

$$\Delta_{B_n} : \quad B_n = A_n[M] \to B_n \otimes B_n = A_n[M \times M]$$

gives a map $\widehat{\Delta_B} : \widehat{B} \to \widehat{B \otimes B}$. As noted in footnote (1) the latter algebra is the A-submodule of the product $A^{M\times M}$ consisting of families $(a_{m,m'})$ which tend to zero. Further, the projective system of morphisms $\widehat{B} \otimes \widehat{B} \to B_n \otimes B_n$ gives a morphism of algebras $\eta : \widehat{B} \otimes \widehat{B} \to \widehat{B \otimes B}$ but this morphism is not surjective in general. However, we have the following commutative diagram:

$$\begin{array}{ccccccccc}
C_n & & & & \xrightarrow{\quad\varphi_n\quad} & & & & B_n \\
\uparrow & \nwarrow & & & & & & \nearrow & \uparrow \\
 & & C & \xrightarrow{\tau_C} & \widehat{C} & \xrightarrow{\widehat{\varphi}} & \widehat{B} & & \\
{\scriptstyle \Delta_{C_n}}\downarrow & & {\scriptstyle \Delta_C}\downarrow & & \searrow{\scriptstyle \Psi} & & \downarrow{\scriptstyle \widehat{\Delta_B}} & & \downarrow{\scriptstyle \Delta_{B_n}} \\
 & & C \otimes C & \xrightarrow{\tau_C\otimes\tau_C} \widehat{C}\otimes\widehat{C} & \xrightarrow{\widehat{\varphi}\otimes\widehat{\varphi}} & \widehat{B}\otimes\widehat{B} \xrightarrow{\eta} & \widehat{B \otimes B} & & \\
 & \swarrow & & & & & & \searrow & \\
C_n \otimes C_n & & & & \xrightarrow{\quad\varphi_n\otimes\varphi_n\quad} & & & & B_n \otimes B_n
\end{array}$$

Set $\Phi = \widehat{\varphi} \circ \tau_C$ and let Ψ be the composed map indicated in the diagram:

$$C \xrightarrow{\Delta_C} C \otimes C \xrightarrow{\Phi\otimes\Phi} \widehat{B} \otimes \widehat{B} \xrightarrow{\eta} \widehat{B \otimes B}.$$

For each $f \in C$, $\Phi(f)$ is a family (a_m) of $\widehat{B}$ whose image by $\widehat{\Delta_B}$ is a family $(a_{m,m'})$ of $\widehat{B \otimes B}$ which satisfies the hypotheses of Lemma 19.3 below. Hence the support of the families (a_m) and $(a_{m,m'})$ are finite. Therefore $\Phi(C) \subset B$ and $\Psi(C) \subset B \otimes B$ (recall that $\tau_{B\otimes B}$ is injective) and we obtain the commutative diagram below:

$$\begin{array}{ccc} C & \xrightarrow{\quad\Phi\quad} & B \\ {\scriptstyle \Delta_C}\downarrow & & \downarrow{\scriptstyle \Delta_B} \\ C \otimes_A C & \xrightarrow{\Phi\otimes\Phi} & B \otimes B. \end{array}$$

[1] Moreover $\widehat{B}$ identifies with the A-submodule of the product A^M consisting of families $(a_m)_{m\in M}$ which tend to zero in the sense such that for each $n \in \mathbb{N}$, all but a finite number of the a_m belong to I^n.

This proves that Φ is a morphism of Hopf algebras $C \to B$, which reduces modulo I^{n+1} to the given φ_n. This completes the proof of assertion (1).

(2) By assertion (1), the map $\mathrm{Hom}_{S\text{-Gr}}(H, G) \longrightarrow \varprojlim_n \mathrm{Hom}_{S_n\text{-Gr}}(H_n, G_n)$ is bijective. Further, by Th. 18.3, each G_n is an isotrivial MT-group over S_n and hence, by assertion (1) of Prop. 18.1, each map $\mathrm{Hom}_{S_n\text{-Gr}}(H_n, G_n) \to \mathrm{Hom}_{S_0\text{-Gr}}(H_0, G_0)$ is bijective. This proves assertion (2). □

LEMMA 19.3. *Let A be a noetherian ring, M a set, $(a_{m,m'})$ a family of elements of A indexed by $M \times M$ such that:*

(1) $a_{m,m'} = 0$ *if* $m \neq m'$ *(i.e. the support of the family is contained in the diagonal of $M \times M$).*

(2) *There exist a finite number of elements $b^i, c^i \in A^M$ such that $a_{m,m'} = \sum_i b^i_m c^i_{m'}$ for every m, m'. (This means that the element $(a_{m,m'})$ of $A^{M\times M}$ is the image of the element $\sum_i b^i \otimes c^i$ under the canonical morphism $A^M \otimes_A A^M \to A^{M\times M}$).*

Then the support of the family $(a_{m,m'})$ is finite.

PROOF. By (1), the family $(a_{m,m'})$ is determined by the $a_m = a_{m,m}$. Define a homomorphism $u : A^{(M)} \to A^M$ as follows: for every $x = (x_m)_{m\in M} \in A^{(M)}$,

$$u(x)_{m'} = \sum_m a_{m',m}\, x_m.$$

Denote by (e_m) the canonical basis of $A^{(M)}$. By (1), one has simply $u(e_m) = a_m e_m$. On the other hand, by (2) one has

$$u(e_m)_{m'} = \sum_i c^i_m b^i_{m'}$$

hence the $u(e_m) = a_m e_m$ are contained in the finitely generated A-module $\sum_i Ab^i$. Since A is noetherian, they generate a finitely generated A-module. Since the e_m are linearly independent, it follows that $a_m = 0$ for all but a finite number of m. □

COROLLARY 19.4. *Let A, I, S, S_0 and H be as in* Th. 19.2, *suppose that G is a* smooth affine *S-group scheme and let $u_0 : H_0 \to G_0$ be a morphism of S_0-group schemes. Then:*

(1) *There exists a morphism of S-group schemes $u : H \to G$ that lifts u_0.*

(2) *If v is another such lifting, there exists $g \in \mathrm{Ker}\big(G(S) \to G(S_0)\big)$ such that $v = \mathrm{int}(g)\circ u$.*

PROOF. (1) Using Theorem 17.1, one can lift u_0 to a projective system of morphisms (u_n). Then assertion (1) of theorem 19.2 gives a morphism $u : H \to G$ lifting u_0.

(2) Let u, v be two liftings of u_0. By Theorem 17.1, one obtains a projective system of elements $g_n \in \mathrm{Ker}\big(G(S_n) \to G(S_0)\big)$ such that $v_n = \mathrm{int}(g_n) \circ u_n$ for all n. That is, we have a projective system of morphisms of algebras $g_n : C \to A/I^{n+1}$. Since A is separated and complete the sequence $g_n(c)$ converges, for each $c \in C$, to an element $g(c)$. This gives an element $g \in \mathrm{Ker}\big(G(S) \to G(S_0)\big)$ such that $v_n = (\mathrm{int}(g) \circ u)_n$ for all n. Since the morphism θ of Theorem 19.2 is injective, it follows that $v = \mathrm{int}(g) \circ u$. □

20. The density theorem

Recall that we have always assumed that MT-groups be of finite type, i.e. that the corresponding abelian group M be finitely generated. As this is important in the next theorem, we write this hypothesis explicitly.

REMARK 20.1. Let G be a commutative affine group scheme over S. For each $n \geq 1$, let ${}_nG$ be the kernel of the n-th power map r_n. As we have a cartesian diagram

$$\begin{array}{ccc} G & \xrightarrow{r_n} & G \\ \uparrow & & \uparrow \varepsilon \\ {}_nG & \longrightarrow & S \end{array}$$

and the unit section $\varepsilon : S \to G$ is a closed immersion (G being affine hence separated over S), one obtains that ${}_nG$ is a closed sugbgroup of G, hence is defined by a sheaf of ideals that we will denote by $\mathcal{I}_n$.

THEOREM 20.2. *Let G be a* MT-*group* of finite type *over S. For each $n \geq 1$, let ${}_nG$ be the kernel of the n-th power map and $\mathcal{I}_n$ the corresponding sheaf of ideals.*

(1) *Let Z be a* closed *subscheme of G containing all ${}_nG$ in the schematic sense, i.e. if $\mathcal{J}$ is the sheaf of ideals of $\mathcal{O}_G$ defining Z, the assumption is that $\mathcal{J} \subset \mathcal{I}_n$ for all n. Then $Z = G$, i.e. $\mathcal{J} = 0$.*

(2) *Let H be a* subgroup *scheme of G such that each ${}_nG$ is a closed subscheme of H. Then $H = G$.*

PROOF. [2] (1) Taking a covering of S by affine subsets, we may suppose that $S = \operatorname{Spec} A$ is affine. Denote then by I_n the ideal of $\mathcal{O}(G)$ corresponding to ${}_nG$. Let $A \to A'$ be a faithfully flat morphism such that the pullback G' of G to $S' = \operatorname{Spec}(A')$ is isomorphic to $D(M)_{S'}$ for some finitely generated abelian group M. Since the formation of kernels commutes with base change, we have ${}_nG' = ({}_nG) \times_S S'$ and hence, with obvious notation, $I'_n = I_n \otimes_A A'$.

We have to prove that any $f \in \mathcal{O}(G)$ belonging to all I_n is zero. Since the map $\mathcal{O}(G) \to \mathcal{O}(G) \otimes_A A' = \mathcal{O}(G')$ is injective, it suffices to prove the corresponding result over S'. Thus, replacing S by S' we may assume that $G = D(M)_S$, i.e. $\mathcal{O}(G) = A[M]$.

One has $M \simeq \mathbb{Z}^r \times Q$ for some finite abelian group Q of order q. Denote by B the Laurent polynomial ring $A[T_1^{\pm 1}, \ldots, T_r^{\pm 1}]$, then $A[M] \simeq B[Q]$. Let $f = \sum_{x \in Q} b_x\, x \in A[M]$ and suppose that f is zero in each quotient $\mathcal{O}({}_nG) = A[M/nM]$. Let m be the supremum of the absolute values of the exponents of the T_i's in the various b_x. Let n be a multiple of q which is $> 2m$. Then

$$A[M/nM] \simeq \big(A[T_1, \ldots, T_r]/(T_1^n - 1, \ldots, T_r^n - 1)\big)[Q]$$

and hence in $A[M/nM]$ the elements $T_1^{d_1} \cdots T_r^{d_r} b_x$, with $-m \leq d_i \leq m$ and $x \in Q$ are linearly independent over A. It follows that every coefficient of f is zero, hence $f = 0$. This proves (1).

(2) When S is the spectrum of a field k, one knows that every subgroup scheme is *closed*, hence (2) follows from (1) in that case. In the general case, let H be a subgroup scheme of G containing all the ${}_nG$. Being a subscheme means that H is a closed subscheme of an open subscheme U of G.

Then, on each fiber one has $H_s = G_s$. Thus H has the same underlying space as G, hence $U = G$ and H is a closed subscheme of G, and we conclude by (1) that $H = G$. □

REMARK 20.3. (1) Note that the assumptions and conclusions in the previous theorem are *schematic* and not purely topological. Let us illustrate this in the case where $S = \operatorname{Spec}(k)$ for an algebraically closed field of characteristic $p > 0$.

a) Let $G = \mu_{p,S}$; then $\mathcal{O}(G) = k[T]/(T^p - 1) \simeq k[T]/(T-1)^p$. Here the reduced scheme $G_{\mathrm{red}} = S$ has the same topological space as G, but is not equal to G.

[2] This neat proof is due to Joseph Oesterlé, see [**Oes14**], §8. In [**SGA3**$_2$], IX, Th. 4.7, Grothendieck proves the stronger result that the family of subschemes $({}_nG)_{n\geq 1}$ is *schematically dense* in G.

b) On the other hand, let $G = \mathbb{G}_{m,S} \times_S \mu_{p,S} = D(M)_S$, where $M = \mathbb{Z} \times \mathbb{Z}/p\mathbb{Z}$. The above proof shows that it suffices to consider the ${}_{n_i}G$ for a sequence of integers $(n_i)_{i\in\mathbb{N}^*}$ divisible by the order of the torsion group of M such that $\lim_{i\to\infty} n_i = +\infty$. Here we can take $n_i = p^i$; then the subgroup schemes ${}_{p^i}G \simeq \mu_{p^i,S} \times_S \mu_{p,S}$ have the same topological space as S, but any subgroup scheme of G containing them must equal G.

(2) One could be tempted to call "schematic density" the notion considered in the theorem. In fact, this terminology is used for a stronger property: one says that a family $(Y_i)_{i\in I}$ of subschemes of a scheme X is *schematically dense* if for every open subset U of X, every closed subscheme Z of U which contains all $Z_i \cap U$ must equal U. In [**SGA3**$_2$], Th. 4.5, Grothendieck proves a stronger version of Th. 20.2: the family $({}_nG)_{n\in\mathbb{N}^*}$ is schematically dense in G.

Remark 20.4. The theorem does not hold for the (non-finitely generated) abelian group $\mathbb{Q}$. Indeed, setting $G = D(\mathbb{Q})_S$, one has ${}_nG = \{1\}$ for all $n \geq 1$.

21. Free actions of MT-groups on schemes affine over S

This section was meant to be given in Lecture 6, after the results on faithfully flat descent of Section 9 and their applications in Section 10. In order to go quickly into deformation theory, we postponed it till now and perhaps, due to lack of time, this material will not be covered in an actual lecture. In a later reorganisation of this notes, this section will probably be moved to an earlier place.

Theorem 21.1. *Let H be a* MT*-group scheme over S acting freely, say on the right, on a scheme X affine over S. Then there exists a scheme Y affine over S, together with a faithfully flat, H-invariant, morphism $p : X \to Y$, which represents the quotient X/H.*

In particular, p makes X into a H_Y-torsor over Y, where $H_Y = H \times_S Y$.

Proof. See [**SGA3**$_2$], VIII Th. 5.1 together with IX, Prop. 2.3, or [**Oes14**], §10. □

Corollary 21.2. *Let $u : H \to G$ be a* monomorphism *of S-group schemes, where is H is a* MT*-group and G is affine over S. Then:*

(1) *u is a closed immersion.*

(2) *There exists a scheme Y affine over S, together with a faithfully flat morphism $p : G \to Y$, which represents the quotient G/H.*

(3) *Further, if H is a normal subgroup of G then Y has a structure of S-group scheme such that p is a morphism of group schemes.*

Proof. Assertions (1,2) are in Exp. IX, Cor 2.5, whereas assertion (3) follows from Exp. IV, Prop. 5.2.3. □

Notes for this Lecture

Lemma 19.3 is Lemma 7.2 of Exp. IX.

Assertion (1) of Th. 19.2 is Th. 7.1 of Exp. IX, while assertion (2) is Lemma 3.1 of Exp. X.

The proof of Th. 20.2 is that given by Oesterlé in [**Oes14**], §8. In [**SGA3**$_2$], IX, Th. 4.7, a stronger result is proved (with a much longer proof).

The references for the results of Section 21 are given in the text.

LECTURE 10

Quasi-isotriviality of MT-groups of finite type

22. The spreading theorem over a complete noetherian local ring

LEMMA 22.1. *Let A be a noetherian ring, with an ideal I such that A is separated and complete for the I-adic topology. Set $S = \mathrm{Spec}(A)$ and $S_0 = \mathrm{Spec}(A/I)$.*

(1) *Every maximal ideal of A contains I.*

(2) *Therefore, if U is an open subset of S containing S_0, then $U = S$.*

PROOF. (1) Let $x \in I$. For every $a \in A$ the element $1 - ax$ is invertible, its inverse being $1+\sum_{n\geq 1}(ax)^n$ (this sum converges since $(ax)^n \in I^n$ and A is I-adically complete). Now suppose that there exists a maximal ideal $\mathfrak{m}$ such that $x \notin \mathfrak{m}$, then there exists $y \in A$ and $z \in \mathfrak{m}$ such that $yx = 1 - z$, hence $1 - yx = z$ belongs to $\mathfrak{m}$, contradicting the fact that $1 - yx$ is invertible. This proves (1).

(2) The complement of U is a closed set $V(J) = \{P \in \mathrm{Spec}(A) \mid P \supset J\}$. If it is not empty (i.e. if J is a proper ideal), it contains a maximal ideal $\mathfrak{m}$, which is impossible since all maximal ideals belong to $V(I) = S_0$. Thus $U = S$. $\square$

THEOREM 22.2. *Let A be a noetherian ring, with an ideal I such that A is separated and complete for the I-adic topology. Set $S = \mathrm{Spec}(A)$ and $S_0 = \mathrm{Spec}(A/I)$*

(1) *The functor $H \mapsto H_0 = H \times_S S_0$ is an* equivalence of categories*:*

$$\left\{\begin{array}{c}\textit{isotrivial } \mathrm{MT}\textit{-groups}\\ \textit{over } S\end{array}\right\} \xrightarrow{\sim} \left\{\begin{array}{c}\textit{isotrivial } \mathrm{MT}\textit{-groups}\\ \textit{over } S_0\end{array}\right\}$$

Now, let G be a finite type affine S-group, flat over S at each point of G_0, and such that G_0 is an isotrivial MT*-group over S_0.*

(2) *There exists a finite type isotrivial* MT*-group H over S and a morphism of S-groups $u : H \to G$ such that $u_0 : H_0 \to G_0$ is an isomorphism.*

(3) *If one assumes further that G is a* MT*-group over S then u is an isomorphism; hence the hypothesis that G_0 be isotrivial implies that G is so.*

(4) *In general, u is an open and closed immersion.*

PROOF. (1)[1] By assertion (2) of Th. 19.2 we know already that this functor is fully faithful. Now, let H_0 be an isotrivial MT-group over S_0. The proof that there exists an isotrivial MT-group H over S such that $H \times_S S_0 \simeq H_0$ is similar to that of assertion (2) of Prop. 18.1. Namely, there exists a surjective *finite étale* morphism $S_0' \to S_0$ such that the pullback H_0' is a diagonalisable group $D(M)_{S_0'}$. By [**EGA**] IV$_4$, Prop. 18.3.2, the functor $X \longmapsto X_0 = X \times_S S_0$ is an equivalence between the category of schemes *finite and étale* over S and the corresponding one over S_0. Thus, there exists a surjective finite étale morphism $S' \to S$ such that $S_0' = S' \times_S S_0$. Then $H' = D(M)_{S'}$ is such that $H' \times_{S'} S_0' = H_0'$.

Next, one obtains as in the proof of assertion (2) of Th. 18.1 that the descent datum on H_0' relative to $S_0' \to S_0$ comes from a descent datum on H' relative to $S' \to S$. Since H' is

[0]version of Sept. 4, 2023.

[1]A shorter proof of assertion (1) is given in Lemma 23.10below.

affine over S', this descent datum is effective, hence there exists a S-group scheme H such that $H \times_S S' = H' = D_{S'}(M)$, and hence H is an isotrivial MT-group over S. Further, $H \times_S S_0$ and H_0 become isomorphic over S'_0, hence they are isomorphic because $S'_0 \to S_0$ is a morphism of descent. This completes the proof of assertion (1).

Now, let G be a finite type affine S-group, flat over S at each point of G_0, and such that G_0 is an isotrivial MT-group over S_0. Let M be the abelian group corresponding to G_0; it is finitely generated since G, hence G_0, is of finite type.

By assertion (1), there exists an isotrivial MT-group H over S and an isomorphism of S_0-groups $u_0 : H_0 \xrightarrow{\sim} G_0$. By assertion (2) of Th. 19.2 we know that u_0 lifts uniquely to a morphism of S-groups $u : H \longrightarrow G$.

On the other hand, since the type of the fibers H_s is a *locally constant function* of s, there exists an open subset U of S containing S_0 such that H_U is of type M. By Lemma 22.1, the only open subset of S containing S_0 is S itself. Thus, H is of type M over S, in particular it is of finite type. This proves (2).

(3) Assume further that G is a MT-group over S. The same reasoning as above, applied to G instead of H, shows that G is of type M over S, in particular it is of finite type. Then, by Prop. 7.10 of Lecture 3, $K = \mathrm{Ker}(u)$ and $C = \mathrm{Coker}(u)$ are MT-groups over S, hence the type of their fibers is again a locally constant function of s. Since u_0 is an isomorphism, the type of K and of C is the trivial abelian group $\{0\}$ over S_0 and hence over S. Thus K and C are trivial and hence u is an isomorphism. This proves (3)

Let us prove (4). Let $S' \to S$ be a finite étale map such that $H' \simeq D(M)_{S'}$. It suffices to prove that u' is an open and closed immersion, because then u will be so ([**EGA**] IV_2, Prop. 2.7.1). So, replacing S by S', we may assume that $H = D(M)_S$.

Let us denote by u_n the pullback over S_n of the morphism $u : H \to G$. By assumption, u_n is an isomorphism; in particular it is flat. By the local criterion of flatness (see e.g. [**EGA**], IV_3, Lemma 11.3.10.2 or [**Mat86**], Th. 22.3) it follows that u is flat at any point of H_0, in particular at any point of the unit section of H_0. Now, one knows that the locus V of points of H where u is flat is open ([**EGA**] IV_3, Th. 11.1.1), hence its inverse image by the unit section $\varepsilon : S \to H$ is an open subset U of S containing S_0. By Lemma 22.1 one has $U = S$, hence $u : H \to S$ is flat near every point of the unit section. For every $s \in S$ it follows that u_s is flat, because over a field one can show, going to an algebraic closure and using translations, that a morphism between finite type groups is flat as soon as it is flat near the identity (see [$\mathbf{SGA3}_1$], VI_B, Prop. 1.3 and also VI_A, Lemma 2.5.3 for the stronger result without finiteness hypotheses). Thus, by the fibral criterion of flatness (see [**EGA**] IV_3, Cor. 11.3.11), $u : H \to G$ is *flat*.

Let $K = \mathrm{Ker}(u)$. As we have a cartesian diagram

$$\begin{array}{ccc} H & \xrightarrow{u} & G \\ \uparrow & & \uparrow \varepsilon \\ K & \longrightarrow & S \end{array}$$

and the unit section $\varepsilon : S \to G$ is a closed immersion (G being affine hence separated over S), one obtains that K is a closed sugbgroup of H, flat over S, and such that K_0 is trivial. Let us prove that K is trivial.

For each $n \in \mathbb{N}^*$ the n-torsion subgroup ${}_nG = D(M/nM)_S$ is finite over S, hence so is its closed subgroup ${}_nK$. Its pullback ${}_nK_0$ over S_0 is trivial, hence by Nakayama's lemma ${}_nK$ is trivial. In particular, for each $s \in S$ we have that ${}_n(K_s) = ({}_nK)_s$ is trivial. One knows that over a field every closed subgroup of a diagonalisable group is diagonalisable (see [$\mathbf{SGA3}_2$] IX, Prop. 8.1 or [**Oes14**], §5.4). Thus each fiber K_s is a diagonalisable group over $\kappa(s)$, and since ${}_n(K_s)$ is trivial for each n it follows from the density theorem 20.2 that K_s is trivial. Therefore, the unit section $\varepsilon : S \to K$ is an isomorphism on each fiber and hence, by Lemma 22.3 below, K is the trivial group. This proves that $u : H \to G$ is a monomorphism.

Since H is diagonalisable, Cor. 21.2 tells us that u is in fact a closed immersion. On the other hand, as it is flat and of finite presentation (because H and G are of finite type over the noetherian base S) it is open, and is therefore an isomorphism from H to an open and closed subgroup of G. This completes the proof of assertion (4). □

In the proof of assertion (4), we have used the lemma below, which is [**EGA**] IV_4, Cor. 17.9.5.

LEMMA 22.3. *Let $f : X \to Y$ be a morphism of S-schemes, where X, Y are locally of finite presentation over S and X is flat over S. If for each $s \in S$ the morphism $f_s : X_s \to Y_s$ is an open immersion, resp. an isomorphism, so is f.*

23. MT-groups of finite type over a henselian local ring

In this section, $(A, \mathfrak{m})$ denotes a local ring, $S = \operatorname{Spec} A$, s the closed point of S and $S_0 = \{s\}$.

DEFINITION 23.1. One says that $(A, \mathfrak{m})$ is **henselian** if it satisfies the following equivalent conditions:

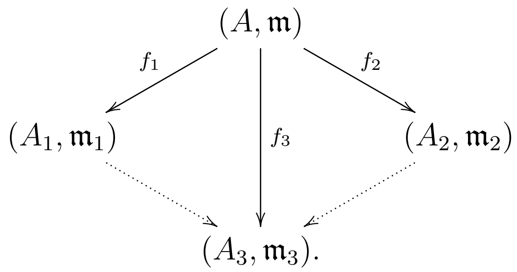
(1) Every finite A-algebra B decomposes as a product of local rings.
(2) For every morphisms $X \to S$ *finite* and $Y \to S$ *étale and separated*, the natural map $\operatorname{Hom}_S(X, Y) \to \operatorname{Hom}_{S_0}(X_0, Y_0)$ is bijective.
(3) For every smooth morphism $f : X \to S$ and every point $x \in X$ over s such that $\kappa(x) = \kappa(s)$, there exists a section $u : S \to X$ of f such that $u(s) = x$.

REMARK 23.2. Of course, the equivalence of the conditions is far from trivial. The first one is usually taken as the definition, see [**SGA3**$_2$], Exp. X, §4 and [**EGA**] IV_4, Def. 18.5.8 and Prop. 18.5.9 (ii). The equivalence with (2) is proved in [**EGA**] IV_4, Cor. 18.5.12 and that with (3) in *loc. cit.* Th. 18.5.17.

NOTATION 23.3. A morphism of local rings $f : (A, \mathfrak{m}) \to (B, \mathfrak{n})$ is *local* if $f^{-1}(\mathfrak{n}) = \mathfrak{m}$. In this case[2] it induces an extension of residue fields $A/\mathfrak{m} \hookrightarrow B/\mathfrak{n}$; if further this extension is trivial, we will say for brevity that f is *tlocal*. Beware that this is not standard terminology!

REMARK 23.4. Recall that a *flat* local morphism $(A, \mathfrak{m}) \to (B, \mathfrak{n})$ is *faithfully flat*.

For what follows, we refer to [**EGA**] IV_4, Th. 18.6.6 or [**StaPr**], Algebra, §155 (Tag 0BSK). One can prove that the set of tlocal étale morphims $f_1 : (A, \mathfrak{m}) \to (A_1, \mathfrak{m}_1)$ is *filtered*, that is, if f_2 is another such morphism, there exists a third one f_3 which dominates f_1 and f_2, i.e. such that f_3 factors through f_1 and f_2:



DEFINITION 23.5. Using this, one can construct the filtered inductive limit of these morphisms. One obtains a flat tlocal morphism $A \to \widetilde{A}$, where $\widetilde{A}$ is a local henselian ring with maximal ideal $\widetilde{A}\mathfrak{m}$, which is determined up to unique isomorphism by the universal property that, for every local henselian ring B,

$$\operatorname{Loc.Hom}(\widetilde{A}, B) = \operatorname{Loc.Hom}(A, B). \tag{23.1}$$

[2]Note that, for example, the inclusion $A \hookrightarrow K$ of a DVR in its field of fractions is **not** a local morphism.

One says that $\widetilde{A}$ is the **henselisation** of A.

From now on, suppose that A is **noetherian** and denote its $\mathfrak{m}$-adic completion by $\widehat{A}$. Then one knows that $A \longrightarrow \widehat{A}$ is faithfully flat (in particular, injective). Further, in this case one has the following proposition.

PROPOSITION 23.6. *Let $(A, \mathfrak{m})$ be a noetherian local ring. Then:*

(1) *$\widetilde{A}$ is noetherian and has the same $\mathfrak{m}$-adic completion than A.*
(2) *Thus, one has tlocal, flat morphisms $A \longrightarrow \widetilde{A} \longrightarrow \widehat{A}$ of local noetherian rings.*

REMARK 23.7. Under the assumption that A be noetherian, one can define informally $\widetilde{A}$ as follows. Set $k = A/\mathfrak{m}$ and $\widehat{S} = \operatorname{Spec} \widehat{A}$. We know already that $\widehat{A}$ is henselian. Let $(A, \mathfrak{m}) \to (A', \mathfrak{m}')$ be an étale tlocal morphism, corresponding to an étale map $f : S' \to S$ sending s' to s (with obvious notation). Since $\kappa(\widehat{s}) \otimes_k \kappa(s') = k$, there is a unique point x of $\widehat{S} \times_S S'$ mapping to $\widehat{s}$ and to s' and, by condition (2) of Def. 23.1, the morphism $\widehat{S} \times_S S' \to \widehat{S}$ admits a section sending s to x, which is necessarily étale (see [**EGA**] IV_4, Prop. 17.3.4). In other words, the morphism $\widehat{A} \to \widehat{A} \otimes_A A'$ admits a retraction τ which is a tlocal étale morphism and this gives a flat tlocal morphism $A' \to \widehat{A}$. Then $\widetilde{A}$ is the union of the images of these morphisms (since they form a filtered set, the union of their images is a subring).

NOTATION 23.8. For the rest of this section, we fix a noetherian local **henselian** ring $(A, \mathfrak{m})$, denote by A' its completion, by S, S' their spectra, and we set $S_0 = \operatorname{Spec}(k)$, where $k = A/\mathfrak{m}$.

REMARK 23.9. Consider the following diagram of categories and base-change functors:

$$\begin{array}{ccc}
\left\{\begin{array}{c}\text{Schemes finite and}\\ \text{étale over } S\end{array}\right\} & \xrightarrow{X \mapsto X \times_S S'} & \left\{\begin{array}{c}\text{Schemes finite and}\\ \text{étale over } S'\end{array}\right\} \\
& {}_{X \mapsto X \times_S S_0}\searrow^{\simeq} \quad {}^{\simeq}\swarrow_{X' \mapsto X' \times_{S'} S_0} & \\
& \left\{\begin{array}{c}\text{Schemes finite and}\\ \text{étale over } S_0\end{array}\right\}. &
\end{array}$$

The two oblique arrows are equivalence of categories, hence so is the horizontal one.

Note that S, S' and S_0 are connected. So, choosing a geometric point $\overline{s}$ over s, the observations above imply that fundamental (profinite) groups are isomorphic:

$$\pi_1(S, \overline{s}) \xleftarrow{\sim} \pi_1(S_0, \overline{s}) \xrightarrow{\sim} \pi_1(S', \overline{s}). \tag{23.2}$$

LEMMA 23.10. *The functor $H \mapsto H_0 = H \times_S S_0$ is an equivalence between the category of* **isotrivial** MT*-groups over S, resp. S', and the corresponding one over S_0.*

PROOF. Since S, S' and S_0 are connected, it follows from Th. 12.3 of Lecture 5 that the category of isotrivial MT-groups over S is anti-equivalent to the category of $\pi_1(S_0, \overline{s})$-modules M such that the kernel of $\pi_1(S, \overline{s}) \to \operatorname{Aut}(M)$ is an open subgroup, and similarly for S_0 and S'. Since the three fundamental groups are the same, the result follows.

Note that this argument also proves assertion (1) of Th. 22.2. □

PROPOSITION 23.11. *In the following diagram of categories and base-change functors, all arrows are equivalence of categories:*

$$\begin{array}{ccc}
\left\{\begin{array}{c}\textit{MT-groups}\\ \textit{finite over } S\end{array}\right\} & \xrightarrow{X \mapsto X \times_S S'} & \left\{\begin{array}{c}\textit{MT-groups}\\ \textit{finite over } S'\end{array}\right\} \\
& {}_{X \mapsto X \times_S S_0}\searrow^{\simeq} \quad {}^{\simeq}\swarrow_{X' \mapsto X' \times_{S'} S_0} & \\
& \left\{\begin{array}{c}\textit{MT-groups}\\ \textit{finite over } S_0\end{array}\right\}. &
\end{array}$$

Proof. Any MT-group H_0 finite over S_0 is, in particular, of finite type, hence isotrivial by Prop. 12.6 of Lecture 5, so it comes by change from a MT-group H over S, of the same type as H_0, hence finite over S.

Let H, G be finite MT-groups over S. We have a commutative diagram

$$\begin{array}{ccc} \mathrm{Hom}_{S\text{-Gr}}(H,G) & \longrightarrow & \mathrm{Hom}_{S_0\text{-Gr}}(H_0,G_0) \\ \| & & \| \\ \mathrm{Hom}_{S\text{-Gr}}(D(G),D(H)) & \longrightarrow & \mathrm{Hom}_{S_0\text{-Gr}}(D(G_0),D(H_0)) \end{array}$$

and, since $D(G), D(H)$ are *finite étale* group schemes, the bottom horizontal map is bijective (the unique lift u of a group morphism $u_0 : D(G_0) \to D(H_0)$ is a group morphism, as one sees by considering the diagram involving the group laws of $D(G_0)$ and $D(H)$). Therefore, the top horizontal map is bijective too.

This proves that the base-change from S to S_0 is fully faithful. The same argument applies to the base-change from S' to S_0, since S' is also local henselian with closed point s. Thus the two oblique arrows are equivalence of categories, hence so is the horizontal one. □

Proposition 23.12. *Recall the hypotheses of* 23.8. *Let H, G be* MT-*groups of finite type over S and let H', G' be their pull-backs over S'. Then the natural map below is bijective :*

$$\mathrm{Hom}_{S\text{-Gr}}(H,G) \to \mathrm{Hom}_{S'\text{-Gr}}(H',G'). \tag{23.3}$$

Proof. Set $S'' = S' \times_S S'$ and let $\mathrm{pr}_1, \mathrm{pr}_2$ be its two projections to S'. As $S' \to S$ is faithfully flat and quasi-compact, one has an exact diagram

$$\mathrm{Hom}_{S\text{-Gr}}(H,G) \longrightarrow \mathrm{Hom}_{S'\text{-Gr}}(H',G') \underset{\mathrm{pr}_2^*}{\overset{\mathrm{pr}_1^*}{\rightrightarrows}} \mathrm{Hom}_{S''\text{-Gr}}(H'',G'')$$

hence we see that the assertion is that for **every** morphism of S'-groups $f' : H' \to G'$, the two morphisms of S''-groups $\mathrm{pr}_1^*(f'), \mathrm{pr}_2^*(f)' : H'' \to G''$ coincide.

Note that f' induces for each $n \in \mathbb{N}^*$ a morphism of S'-groups $f'_n : {}_nH' \to {}_nG'$. But ${}_nH'$ and ${}_nG'$ are *finite* MT-groups over S' hence, by the previous proposition, f'_n comes from a morphism of S-groups $f_n : {}_nH \to {}_nG$ and hence satisfies $\mathrm{pr}_1^*(f'_n) = \mathrm{pr}_2^*(f'_n)$.

Now, since G'' is affine hence separated over S'', the locus where $\mathrm{pr}_1^*(f')$ and $\mathrm{pr}_2^*(f')$ coincide is a closed subscheme of H'', and by the previous paragraph, it contains all the subgroups ${}_nH''$. Hence, by the density theorem 20.2, $\mathrm{pr}_1^*(f') = \mathrm{pr}_2^*(f')$. This proves the proposition. □

Corollary 23.13. *Recall the hypotheses of* 23.8. *Let G be a* MT-*group of finite type over S. Then G is isotrivial.*[3]

Proof. As G_0 is a MT-group of finite type over $S_0 = \mathrm{Spec}(k)$ it is isotrivial, say of type M for some finitely generated abelian group M (see Prop. 12.6 in Lecture 5). By Lemma 23.10, there exists an *isotrivial* MT-group H over S and an isomorphism $u_0 : H_0 \simeq G_0$. Since S is connected, both H and G are of constant type M over S.

By assertion (2) of Theorem Th. 19.2, that is, by the algebrisation theorem Th. 19.2 (1) and by the lifting results of Th. 18.3 and Prop. 18.1 (which use duality to extend the cohomological result of Th. 17.1 to MT-groups, not necessarily smooth), we know that u_0 lifts uniquely to a morphism of S'-groups $u' : H' \longrightarrow G'$. By the previous proposition, u comes by base-change from a morphism of S-groups $u : H \longrightarrow G$.

[3]This extends assertion (3) of Th. 22.2 from the complete to the henselian case. There is also an extension of assertion (4), but the proof is more difficult, see Exp. X, Lemma 4.3 and Th. 4.4.

Now, both H and G are MT-groups of constant type M over S, with M finitely generated. Hence, by Prop. 7.10 of Lecture 3, $K = \mathrm{Ker}(u)$ and $C = \mathrm{Coker}(u)$ are MT-groups over S, and since S is connected they also have a constant type over S. Since u_0 is an isomorphism, the type of K and of C is the trivial abelian group $\{0\}$. Thus K and C are trivial and hence u is an isomorphism. □

24. Quasi-isotriviality of finitely generated MT-groups

From the previous corollary, one deduces the following theorem, valid over an arbitrary base scheme S.

THEOREM 24.1. *Let G be a* MT*-group of finite type over a scheme S. Then G is quasi-isotrivial, i.e. for every $s \in S$ there exist an open neighbourhood and a surjective étale morphism $U' \to U$ such that $G_{U'}$ is diagonalisable.*

SKETCH OF PROOF. This uses the principle, detailed in [**EGA**] IV_3, §8, that is $f : X \to Y$ is a morphism of schemes of finite presentation over the spectrum S of a ring Λ, and if Λ is the filtered inductive limit of a family of subrings $(\Lambda_i)_{i \in I}$, then, denoting by a subscript i the pull-backs over $S_i = \mathrm{Spec}\,\Lambda_i$, one has:

a) There exists an index $i \in I$ and a morphism $f_i : X_i \to Y_i$ such that f comes from f_i by base change.
b) If one considers a property (P) like being: an isomorphism, an open or closed immersion, an affine, quasi-affine, finite, quasi-finite, or proper morphism, then f has property (P) if and only if there exists an index $i_1 \geq i$ such that for every $j \geq i_1$ the morphism f_j obtained from f_i by base-change has property (P).

Applying this firstly to the local ring $\mathcal{O}_{S,s}$, it suffices to prove the theorem over $\mathrm{Spec}\,\mathcal{O}_{S,s}$. Next, $\mathcal{O}_{S,s}$ is the inductive limit of local subrings which are localizations of finitely generated $\mathbb{Z}$-algebras, so we are reduced to the case where $A = \mathcal{O}_{S,s}$ is *noetherian.* Let $\widetilde{A}$ be its henselisation.

Then, by the previous corollary, there exists a finite étale morphism $\widetilde{A} \to A'$ such that the pull-back G' of G over $\mathrm{Spec}\,A'$ is diagonalisable. Since $\widetilde{A}$ is the filtered inductive limit of "étale neighbourhoods of s", one obtains from the previous principle an open neighbourhood U of s in S and a surjective étale morphism $U' \to U$ such that $G_{U'}$ is diagonalisable. This proves the theorem. □

Notes for this Lecture

In Theorem 22.2, assertions (1) and (3) are in Th. 3.2 of Exp. X, whereas assertion (4) is Th. 3.7 of Exp. X.

The proof of assertion (4) is taken from the proof of [**Co14**], Th. B.3.2; it is easier to understand than the proof of X, Th. 3.7, which relies on the results 6.1 to 6.6 of Exp. IX.

Remark 23.9, Lemma 23.10 and Prop. 23.11 are Exp. X, 4.0, Rem. 4.0.1 and Lemma 4.1.

Prop. 23.12 and Cor. 23.13 are taken from Exp. X, Lemma 4.3 and Th. 4.4, which prove a more general result, namely that the spreading theorem (assertion (4) of Th. 22.2) holds true over a local henselian ring. We have followed the more comprehensible proof of [**Co14**], Prop. B.3.4.

Theorem 24.1 is Exp. X, Cor. 4.5. The reductions to the local ring $\mathcal{O}_{S,s}$, then to a noetherian local ring and then to a noetherian local henselian ring are detailed in the proof of X, Th. 4.4 in the new edition of [**SGA3**$_2$], whose preliminary version is available on the lecturer's web page.

LECTURE 11

Representability of centralisers and transporters

25. Weil restriction

Let us fix a morphism of schemes $Z \to S$.

DEFINITION 25.1. For any Z-scheme Y, its **Weil restriction of scalars** from Z to S, denoted by $\mathcal{R}_{Z/S}(Y)$ (or sometimes $\prod_{Z/S} Y$) is the contravariant functor from $(\mathrm{Sch}_{/S})$ to (Sets) such that, for every S-scheme T:

$$\mathrm{Hom}_S\big(T, \mathcal{R}_{Z/S}(Y)\big) = \mathrm{Hom}_Z(T \times_S Z, Y).$$

REMARK 25.2. Note that $\mathcal{R}_{Z/S}(Y)$ is a sheaf for the fpqc topology. Indeed, let $T' \to T$ be a Zariski covering or a faithfully flat and quasi-compact morphism. Setting as usual $T'' = T' \times_T T'$, one has a commutative diagram:

$$\begin{array}{ccccc} \mathrm{Hom}_S(T, \mathcal{R}_{Z/S}(Y)) & \longrightarrow & \mathrm{Hom}_S(T', \mathcal{R}_{Z/S}(Y)) & \rightrightarrows & \mathrm{Hom}_S(T'', \mathcal{R}_{Z/S}(Y)) \\ \| & & \| & & \| \\ \mathrm{Hom}_Z(T \times_S Z, Y) & \longrightarrow & \mathrm{Hom}_Z(T' \times_S Z, Y) & \rightrightarrows & \mathrm{Hom}_Z(T'' \times_S Z, Y). \end{array} \tag{25.1}$$

Since the morphism $Z_{T'} \to Z_T$ obtained by base change is again a Zariski covering or faithfully flat and quasi-compact and since $Z_{T'} \times_{Z_T} Z_{T'} \simeq Z \times_S T''$, the second row is exact and hence so is the first row.

REMARK 25.3. If $S = \mathrm{Spec}(A)$ and $Z = \mathrm{Spec}(B)$, where B is a finite free A-module of rank d, it is easy to see that $\mathcal{R}_{Z/S}(\mathbb{A}^n_Z) \simeq \mathbb{A}^{nd}_S$, see [**BLR**], §7.6, Th. 4 for a more general result along these lines.

Here, we will be interested in the case where Y is a *closed subscheme* of Z, with applications to the representability of centralisers and transporters, see below. Let us start with the following definition.

DEFINITION 25.4. One says that the S-scheme Z is **essentially free** if there exists a covering (S_i) of S by affine open subsets S_i and for each i an affine and faithfully flat morphism $S'_i \to S_i$ such that $Z'_i = Z \times_S S'_i$ is covered by affine open subsets Z'_{ij} such that every $\mathcal{O}(Z'_{ij})$ is a projective module over $\mathcal{O}(S'_i)$.

EXAMPLE 25.5. If H is a S-group of multiplicative type, it is essentially free over S. Indeed, by assumption there exists a covering $(S_i)_{i \in I}$ of S by affine open subsets S_i and for each i an affine and faithfully flat morphism $S'_i \to S_i$ such that $H'_i = H \times_S S'_i$ is a diagonalisable group over S'_i of type M_i, for some abelian group M_i. Then $\mathcal{O}(H'_i) = \mathcal{O}(S'_i)[M_i]$ is a free $\mathcal{O}(S'_i)$-module.

LEMMA 25.6. (a) *If Z is essentially free over S, it is flat over S.*

(b) *If $S = \mathrm{Spec}(k)$, where k is a field, every S-scheme is essentially free.*

(c) *If Z is essentially free over S, then for any base-change morphism $X \to S$, the morphism $Z \times_S X \to X$ is essentially free.*

PROOF. Left to the reader. ☐

[0]version of Nov. 1, 2023.

REMARK 25.7. Suppose that $\tau : Y \hookrightarrow Z$ is a closed immersion. Then, for any S-scheme T one has:

$$\mathcal{R}_{Z/S}(Y)(T) = \begin{cases} \{\tau_T^{-1}\} & \text{if } \tau_T : Y_T \hookrightarrow Z_T \text{ is an isomorphism,} \\ \varnothing & \text{otherwise.} \end{cases}$$

Indeed, $\mathcal{R}_{Z/S}(Y)(T) = \mathrm{Hom}_Z(Z \times_S T, Y) = \mathrm{Hom}_{Z_T}(Z_T, Y_T)$ is the set of morphisms $f : Z_T \to Y_T$ such that $\tau_T \circ f = \mathrm{id}_{Z_T}$. Since τ_T is a closed immersion, such an f exists if and only if τ_T is an isomorphism, and then $f = \tau_T^{-1}$.

THEOREM 25.8. *Suppose that $Y \hookrightarrow Z$ is a closed immersion and that Z is essentially free over S.*

(i) *Then $\mathcal{R}_{Z/S}(Y)$ is represented by a closed subscheme C of S.*

(ii) *Further, if $Z \to S$ is quasi-compact and $Y \hookrightarrow Z$ is of finite presentation, then $C \hookrightarrow S$ is of finite presentation.*

PROOF. Set $F = \mathcal{R}_{Z/S}(Y)$. The proof is in four steps.

(1) Suppose firstly that $S = \mathrm{Spec}(A)$ and $Z = \mathrm{Spec}(B)$, where B is a projective A-module. Hence B is a direct summand of a free A-module L with basis $(e_\lambda)_{\lambda \in \Lambda}$. Let $\varphi_\lambda : L \to A$ be the coordinate forms with respect to this basis. Let E be a set of generators of the ideal J of B defining the closed subscheme $Y \subset Z$ and let I be the ideal of A generated by the $\varphi_\lambda(x)$, for $x \in E$ and $\lambda \in \Lambda$.

Now, let $T \to S$ be a morphism such that the closed immersion $Y_T \to Z_T$ is an isomorphism. Then, for any affine open subset $T' = \mathrm{Spec}(R)$ of T, one has a morphism of rings $f : A \to R$ and one obtains that the surjective morphism $B \otimes_A R \to (B/J) \otimes_A R$ is an isomorphism, which amounts to saying that for any $x \in E$ the image of $x \otimes 1$ in $B \otimes_A R$ or, equivalently, in $L \otimes_A R$ is zero. Since $x = \sum_\lambda \varphi_\lambda(x) e_\lambda$, the latter image is $\sum_\lambda e_\lambda \otimes f(\varphi_\lambda(x))$ and this is zero if and only if $f(\varphi_\lambda(x)) = 0$. Thus $\mathrm{Ker}(f)$ contains I and hence $T' \to S$ factors through the closed subscheme $C = \mathcal{V}(I)$. Since this is true for any open affine subset of T, one obtains thats $T \to S$ factors through C. Conversely, under this condition one has $Y_T = Z_T$. This proves the first assertion. Further, if J is finitely generated we may take E to be finite and as each $x \in E$ as only finitely many non-zero coordinates $\varphi_\lambda(x)$, it follows that I is finitely generated.

(2) Still with $S = \mathrm{Spec}(A)$, suppose now that Z is covered by affine open subsets Z_j such that each $B_j = \mathcal{O}(Z_j)$ is a projective A-module. For each j, set $Y_j = Y \cap Z_j$ and let the ideals $J_j \subset B_j$ and $I_j \subset A$ be defined as above. Then, for any S-scheme T, the base change of $Y \to Z$ is an isomorphism if and only if the same is true for each $Y_j \to Z_j$. It follows that F is represented by the intersection C of the closed subschemes $C_j = \mathcal{V}(I_j)$, defined by the ideal $I = \sum_j I_j$. Assume further that $Z \to S$ is quasi-compact, then Z is quasi-compact hence can be covered by finitely many open subsets Z_j. Therefore, if the closed immersion $Y \hookrightarrow Z$ is of finite presentation, so is the closed immersion $C \hookrightarrow S$.

(3) Suppose now that $S = \mathrm{Spec}(A)$ and there exists an affine and faithfully flat morphism $S' \to S$ such that $Z' = Z \times_S S'$ is covered by affine open subsets Z'_j such that every $\mathcal{O}(Z'_j)$ is a projective module over $\mathcal{O}(S')$. Then, by the previous step, $F_{S'}$ is represented by a closed subscheme C' of S'. It is endowed with a descent datum relative to S'/S (because $F_{S'}$ is) and, by [**SGA1**], Exp. VIII, Cor. 1.9, C' descends to a closed subscheme C of S; moreover, since F is a fpqc sheaf, C represents F (see the proof of Prop. 10.7 in Lecture 5). Further, if $Z \to S$ is quasi-compact and $Y \hookrightarrow Z$ is of finite presentation, then $C' \hookrightarrow S'$ is of finite presentation and hence so is $C \hookrightarrow S$, by [**EGA**] IV_2, Prop. 2.7.1.

(4) Finally, in the general case, with the notation of Def. 25.4, each functor $F_i = F \times_S S_i$ is represented by a closed subscheme C_i of S_i. Since F is a local functor, the C_i glue together to give a closed subscheme C of S, which represents F (see the proof of Lemma 10.2 in Lect. 5). Further, if $Z \to S$ is quasi-compact and $Y \hookrightarrow Z$ is of finite presentation, then each $C_i \hookrightarrow S_i$

is of finite presentation, hence $C \hookrightarrow S$ is locally of presentation, and being a closed immersion (hence quasi-compact and separated), it is of finite presentation. □

26. Transporters and centralisers

26.1. Two consequences of theorem 25.8. Let S be a base scheme and let G, Y, X, U denote S-schemes, with U *essentially free* over S.

PROPOSITION 26.1. *Let $P = G \times_S U$, let $a : P \to X$ be a S-morphism, let V be a closed subscheme of X and let $P' = P \times_X V$. Then $\mathcal{R}_{P/G}(P')$ is represented by a closed subscheme G' of G and for any G-scheme T one has:*

$$G'(T) = \{g \in G(T) \mid \text{ the morphism } a \circ (g \times \mathrm{id}_U) : T \times_S U \to X \text{ factors through } V\}.$$

Further, if $U \to S$ is quasi-compact and $V \to X$ is of finite presentation, then $G' \to G$ is of finite presentation.

PROOF. Since P' is a closed subscheme of P and $P = G \times_S U$ is essentially free over G, the first assertion follows from Theorem 25.8 applied to $S = G$, $Z = P$ and $Y = P'$. For any G-scheme T one has, by Remark 25.7,

$$G'(T) = \{g \in G(T) \mid T \times_G P' = T \times_G P\}.$$

Since $T \times_G P = T \times_S U$ and $T \times_G P' = T \times_S U \times_X V$, one obtains that $G'(T)$ is the set of those $g \in G(T)$ such that $a \circ (g \times \mathrm{id}_U)(T \times_S U) \subset V$.

Further, the second assertion follows from assertion (ii) of Th. 25.8. □

PROPOSITION 26.2. *Suppose that X is separated. Let $P = Y \times_S U$, let a_1, a_2 be two S-morphisms $P \to X$, let $a = a_1 \times a_2$ be the corresponding morphism $P \to X \times_S X$ and let P' be the pull-back of the diagonal of $X \times_S X$ by a. Then $\mathcal{R}_{P/Y}(P')$ is represented by a closed subscheme Y' of Y and for any Y-scheme T one has:*

$$Y'(T) = \{y \in Y(T) \mid \text{ the morphisms } a_i \circ (y \times \mathrm{id}_U) : T \times_S U \to X \text{ coincide, for } i = 1, 2\}.$$

Further, if $U \to S$ is quasi-compact and $X \to S$ is locally of finite type, the closed immersion $Y' \hookrightarrow Y$ is of finite presentation.

PROOF. As before, $P = Y \times_S U$ is essentially free over Y. Since $X \to S$ is separated, the diagonal $\Delta_{X/S}$ of $W = X \times_S X$ is a closed subscheme and hence P' is a closed subscheme of P. Thus, the first assertion follows from Theorem 25.8 applied to $S = Y$, $Z = P$ and $Y = P'$. For any T-scheme T one has, by Remark 25.7,

$$Y'(T) = \{y \in Y(T) \mid T \times_Y P' = T \times_Y P\}.$$

Since $T \times_Y P = T \times_S U$ and $T \times_Y P' = T \times_S U \times_W \Delta_{X/S}$, one obtains that $Y'(T)$ is the set of those $y \in Y(T)$ such that $a \circ (y \times \mathrm{id}_U)(T \times_S U) \subset \Delta_{X/S}$, i.e. such that $a_i \circ (y \times \mathrm{id}_U)$ coincide on $T \times_S U$, for $i = 1, 2$.

Finally, if $X \to S$ is separated and locally of finite type, the closed immersion $\Delta_{X/S} \hookrightarrow X \times_S X$ is of finite presentation (see [**EGA**] IV_1, Cor. 1.4.3.1), hence the second assertion follows from assertion (ii) of Th. 25.8. □

26.2. Transporters and centralisers. Suppose now that G is a S-group scheme acting on a S-scheme X. Let U, V be subschemes of X.

DEFINITION 26.3. (1) The *transporter* of U in V, denoted $\underline{\mathrm{Tran}}_G(U, V)$, is the subfunctor of G whose T-points are those $g \in G(T)$ such that $g(U_T) \subset V_T$.

(2) The *strict transporter* of U to V, denoted $\underline{\mathrm{Transt}}_G(U, V)$, is the subfunctor of G whose T-points are those $g \in G(T)$ such that $g(U_T) = V_T$.

(3) Denoting by $\phi : G \to G \times_S G$ the morphism sending an arbitrary T-point g to (g, g^{-1}), one sees that $\underline{\mathrm{Transt}}_G(U, V)$ is the inverse image by ϕ of $\underline{\mathrm{Tran}}_G(U, V) \times_S \underline{\mathrm{Tran}}_G(V, U)$. Therefore, if both $\underline{\mathrm{Tran}}_G(U, V)$ and $\underline{\mathrm{Tran}}_G(V, U)$ are represented by closed subschemes of G, so is $\underline{\mathrm{Transt}}_G(U, V)$.

Firstly, one deduces from Prop. 26.1 the following corollary.

Corollary 26.4. *Let G be a S-group scheme acting on a S-scheme X and let U, V be subschemes of X.*

(1) If V is a closed subscheme and U is essentially free over S, then $\underline{\mathrm{Tran}}_G(U, V)$ is represented by a closed subscheme of G.

(2) If U, V are closed subschemes and are essentially free over S, then $\underline{\mathrm{Transt}}_G(U, V)$ is represented by a closed subscheme of G.

For $X = G$, on which G acts by conjugation, one obtains:

(3) If H is a closed subscheme of G, essentially free over S, then $\underline{\mathrm{Norm}}_G(H)$ is represented by a closed subgroup scheme of G.

(4) In particular, if H is a closed S-subgroup scheme of multiplicative type, then $\underline{\mathrm{Norm}}_G(H)$ is represented by a closed subgroup scheme of G.

Proof. Assertion (4) follows from assertion (3) since any S-group of multiplicative type is essentially free over S (Example 25.5). □

Secondly, let us derive three consequences of Prop. 26.2.

Corollary 26.5. *If X is essentially free and separated over S, then the kernel of the action of G on X is represented by a closed S-subgroup scheme K of G.*

If further $X \to S$ is of finite type, the closed immersion $K \hookrightarrow G$ is of finite presentation.

Proof. One applies Prop. 26.2 to the given action $a_1 : G \times_S X \to X$ of G on X and to the trivial action $a_2 : (g, x) \mapsto x$. (For the second assertion, recall that finite type = quasi-compact and locally of finite type.) □

Replacing (Y, U, X) in Prop. 26.2 by (X, G, X), that is, applying Prop. 26.2 to the maps $a_1 : X \times_S G \to X$, $(x, g) \mapsto gx$ and $a_2 : (x, g) \mapsto x$, one obtains the:

Corollary 26.6. *If G is essentially free over S and X separated over S, the subfunctor of invariants X^G is represented by a closed subscheme of X.*

If further $G \to S$ is quasi-compact and $X \to S$ locally of finite type, the closed immersion $X^G \hookrightarrow X$ is of finite presentation.

Proof. Since G is essentially free over S then $P = X \times_S G$ is essentially free over X. And since $X \to S$ is separated, the diagonal $\Delta_{X/S}$ of $W = X \times_S X$ is a closed subscheme and hence P' is a closed subscheme of P. Thus, the first assertion follows from Theorem 25.8 applied to $S = X$, $Z = P$ and $Y = P'$. For any X-scheme T one has, by Remark 25.7,

$$X^G(T) = \{x \in X(T) \mid T \times_X P' = T \times_X P\}.$$

Since $T \times_X P = T \times_S G$ and $T \times_X P' = T \times_S G \times_W \Delta_{X/S}$, one obtains that $X^G(T)$ is the set of those $x \in X(T)$ such that $a_i \circ (x \times \mathrm{id}_G)$ coincide on $T \times_S G$, for $i = 1, 2$, which amounts to saying that for every $T' \to T$ and $g \in G(T')$, one has $g x_{T'} = x_{T'}$.

Further, the last assertion follows from the last assertion of Prop. 26.2. □

Definition 26.7. Let U be a subscheme of G. The *centraliser* of U in G, denoted by $\underline{\mathrm{Cent}}_G(U)$, is the S-subgroup functor of G defined for every S-scheme T by:

$$\underline{\mathrm{Cent}}_G(U)(T) = \{g \in G(T) \mid u g u^{-1} = g, \quad \forall u \in U(T)\}.$$

Thus, setting $P = G \times_S U$ and considering the S-morphisms $a_1 : P \to G$, $(g,u) \mapsto ugu^{-1}$ and $a_2 : (g,u) \mapsto g$, we can use Prop. 26.2 with $(Y,U,X) = (G,U,G)$. Thus, we obtain:

COROLLARY 26.8. *Suppose that $G \to S$ is separated and let U be a subscheme of G, essentially free over S.*

(1) *The centraliser $\underline{\mathrm{Cent}}_G(U)$ is represented by a closed subgroup scheme $C_G(U)$ of G.*

(2) *If further $U \to S$ is quasi-compact and $G \to S$ locally of finite type, the closed immersion $C_G(U) \hookrightarrow G$ is of finite presentation.*

(3) *If $G \to S$ is locally of finite type and if H is a subgroup scheme of multiplicative type, the closed immersion $C_G(H) \hookrightarrow G$ is of finite presentation.*

PROOF. (1) and (2) follow from Prop. 26.2. If H is a subgroup scheme of multiplicative type, then $H \to S$ is essentially free (Example 25.5) and affine, hence quasi-compact. Thus (3) follows from (2). $\square$

PROPOSITION 26.9. *Suppose that G is a smooth affine S-group and H is a subgroup scheme of multiplicative type. Then the closed subgroup scheme $C = C(H)$ is smooth.*

PROOF. By the previous corollary, we know already that $C \to S$ is of finite presentation, so it suffices to see that the functor $\underline{\mathrm{Cent}}_G(H)$ is formally smooth. Denoting by u the immersion $H \hookrightarrow G$, one has $\underline{\mathrm{Cent}}_G(H) = \underline{\mathrm{Cent}}_G(u)$. Let $S' = \mathrm{Spec}(A)$ be an affine scheme over S, let I be a nilpotent ideal of A and $S'_0 = \mathrm{Spec}(A/I)$. Let u_0 denote the pull-back of u to S'_0 and let $z \in C(S'_0)$. That is, z is an element of $G(S'_0)$ such that $\mathrm{int}(z) \circ u_0 = u_0$. Since G is smooth, z lifts to an element $x \in G(S')$. Set $v = \mathrm{int}(x) \circ u$, then $v_0 = u_0$. By Th. 17.1 in Lect. 7, there exists $g \in \mathrm{Ker}(G(S') \to G(S'_0))$ such that $\mathrm{int}(g) \circ v = u$. Set $y = gx$, then $\mathrm{int}(y) \circ u = u$ hence $y \in C(S')$, and the image of y in $G(S'_0)$ is z. This proves that C is smooth. $\square$

Notes for this Lecture

The content of this lecture appears in Exp. VIII, §6, n$^{\text{os}}$ 6.1 to 6.5 of [**SGA3**$_2$] and has also been reproduced in the new edition of [**SGA3**$_1$], Exp. VI$_B$, n$^{\text{os}}$ 6.2.1 to 6.2.5, following a footnote by Grothendieck at the beginning of Exp. VIII, §6: "*The natural place for this paragraph would be in Exp. VI$_B$*".

The assertion (ii) of Th. 25.8 (and the similar assertions in all subsequent results) was not in [**SGA3**$_2$] and was added by the lecturer in [**SGA3**$_1$], Exp. VI$_B$, where the hypothesis that $Z \to S$ be quasi-compact has been overlooked, unfortunately.

The representability of Weil restrictions is also discussed in [**BLR**], §7.6, where a result similar to Th. 25.8 is proved under the more restrictive hypothesis that $Z \to S$ be finite and locally free.

Proposition 26.9 is proved several times in SGA3. In Exp. XI, Cor. 5.3 (a), it is derived from the (hard) resultat that the functor of homomorphisms of S-groups $H \to G$ is representable. A simpler proof is given in Exp. XI, Th. 6.2 (iii), see also Cor. 9.8 in the additional section XII.9 in the new edition. Finally, the direct proof given above is taken from Exp. XIX, proof of Prop. 6.1.

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