On Path Logics with Synchronization

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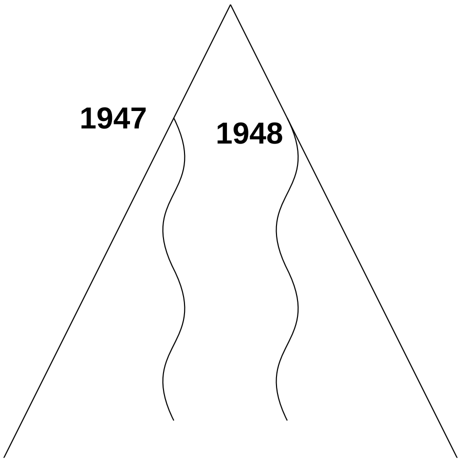
RWTH AACHEN

Workshop for Thiagu “Perspectives in Concurrency”
Chennai, December 2008

Synchronizing Paths: An Example



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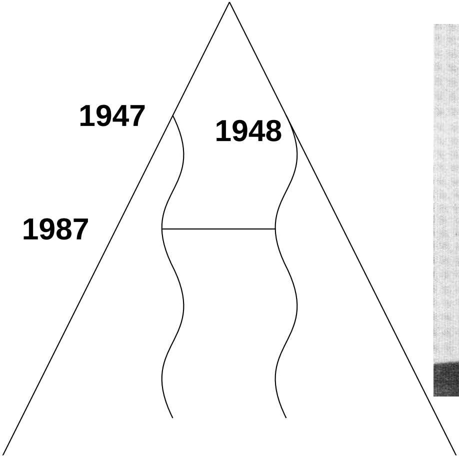

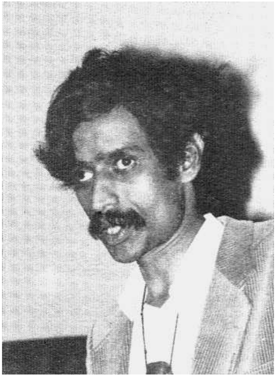

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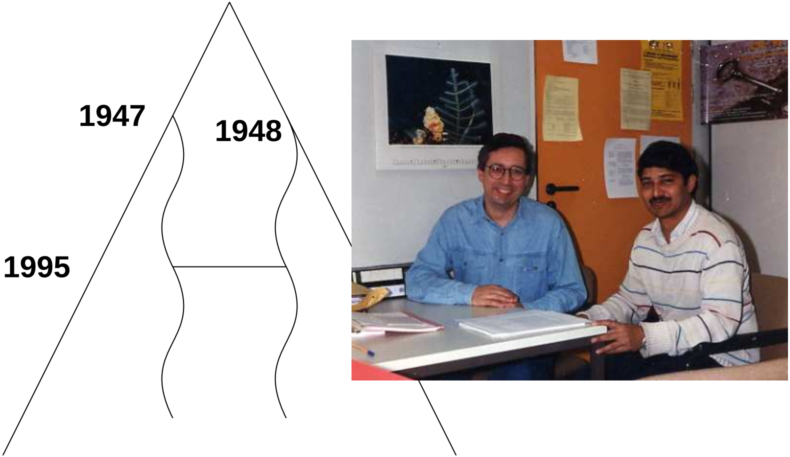

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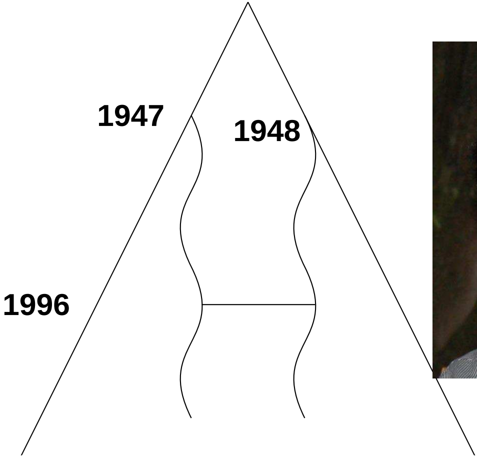

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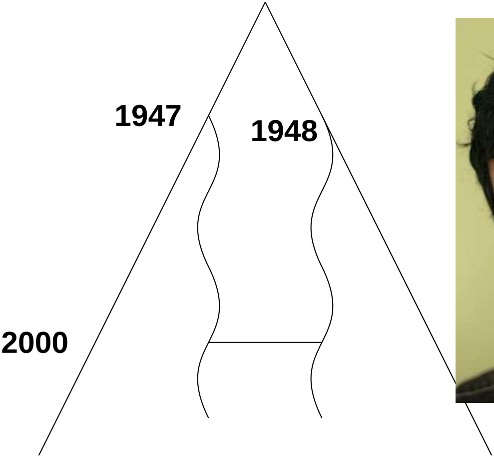

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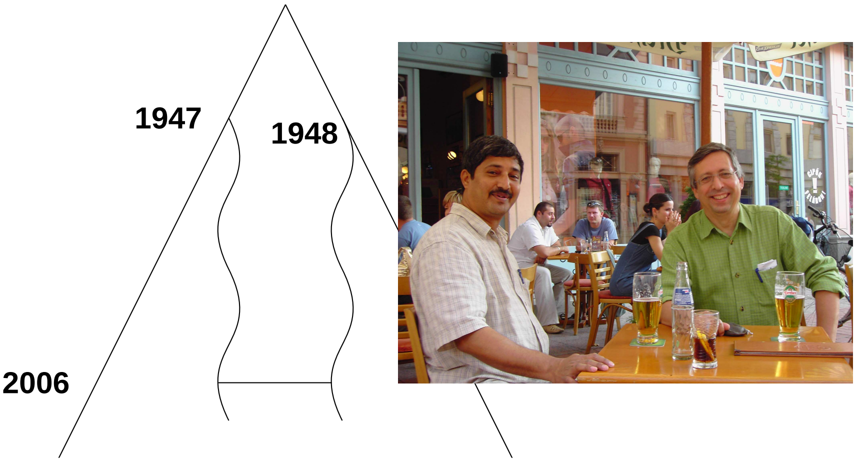


Second Motivation: 50 Years ago . . .

- **J. R. Büchi, C. C. Elgot. Decision problems of weak second-order arithmetics and finite automata, Abstract 553-112, *Notices Amer. Math. Soc.* 5 (1958), 834.**
- **B.A. Trakhtenbrot. Synthesis of logical nets whose operators are described of monadic predicates, Dokl. Akad. Naut. SSSR 118 (1958), 646-649 [in Russian].**

The weak monadic second-order theory of $(\mathbb{N}, +1)$ is decidable.

$$\forall X(\exists y X(y) \rightarrow \exists z(X(z) \wedge \neg X(z+1)))$$

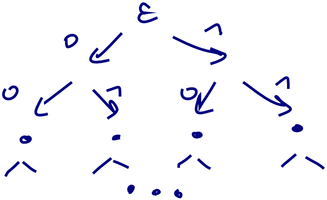
Some years later . . .

Büchi 1960:

The monadic second-order theory of $(\mathbb{N}, +1)$, short MTh$(\mathbb{N}, +1)$, is decidable.

Rabin 1969:

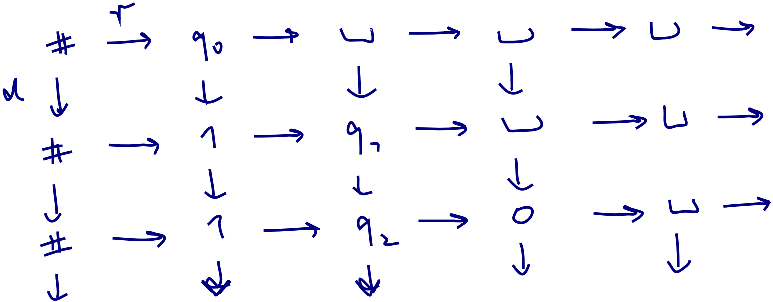
Let $\mathcal{T}_2 = (\{0,1\}^*, \cdot 0, \cdot 1)$. MTh$(\mathcal{T}_2)$ is decidable.



Two-Dimensional Grid

Infinite grid $G_2 = (\mathbb{N} \times \mathbb{N}, d, r)$

The EMSO-theory of G_2 is undecidable.



M-instructions: $q_0 \sqcup 1 R q_1$, $q_1 \sqcup 0 N q_2$

$M \mapsto \varphi_M = \exists X_{\#} \exists X_{\sqcup} \exists X_0 \exists X_1 \exists X_{q_0} \ldots$ (FO-formula)

Quantification over Binary Relations

Remark by Gödel (1931):

The second-order theory of $(\mathbb{N}+1)$ is undecidable.

Using quantification over finite binary relations:
Define addition in terms of successor
(and then multiplication in terms of addition).

$$x + y = z$$

iff

$$\forall R([R(0,x) \wedge \forall s,t(R(s,t) \rightarrow R(s+1,t+1))] \rightarrow R(y,z))$$

[Finite relations R suffice if we start with (y,z) and work downwards to $(0,x)$.]

Adding the double function δ

Double function $\delta : \mathbb{N} \longrightarrow \mathbb{N}$ is defined by $\delta(x) = 2x$.

Robinsion 1958:

The (weak) MSO-theory of $(\mathbb{N}, +1, \delta)$ is undecidable.

We follow a proof by Elgot and Rabin and show first:

**Suppose for $f : \mathbb{N} \longrightarrow \mathbb{N}$ the set $f^{-1}(m)$ is infinite for each m.
Then the weak MSO-theory of $(\mathbb{N}, +1, f)$ is undecidable.**

Using Infinite Pre-Images

Code a relation $R = \{(m_1, n_1), \ldots, (m_k, n_k)\}$

by a set $M_R = \{m'_1 < n'_1 < \ldots < m'_k < n'_k\}$

taking

$m'_1 =$ smallest s such that $f(s) = m_1$

$n'_1 =$ smallest $s > m'_1$ such that $f(s) = n_1$

$m'_2 =$ smallest $s > n'_1$ such that $f(s) = m_2$ etc.

Then
$R(x, y)$ iff
exists $x' \in M_R$ with $f(x') = x$ on an odd position of M_R such that for the next element y' in M_R we have $f(y') = y$.

Simulate a finite-relation quantifier by a finite-set quantifier.

δ Yields f with Infinite Pre-Images

First idea:

For an odd number r, map all products $r \cdot 2^i$ to r.

Define $g(n) =$ the odd number obtained from n by repeatedly dividing by 2 [and let $g(0) = 0$]

1. **g is WMSO-definable in $(\mathbb{N}, +1, \delta)$**
2. **For each odd m, the preimage $g^{-1}(m)$ is infinite.**
   (E.g., $5 = g(5) = g(10) = g(20) = g(40) = \ldots$)

Second idea: Now care for all numbers.

Let $f(n) = g(2n + 1)$

Then items 1. and 2. hold again, but 2. for all numbers.

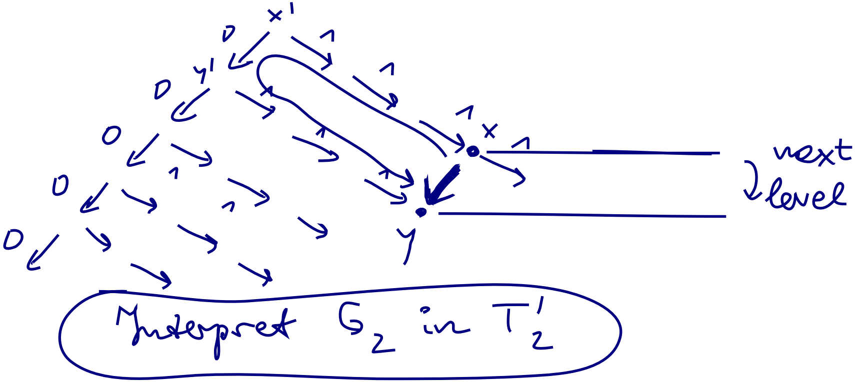
Consequence: Weak MTh$(\mathbb{N}, +1, \delta)$ is undecidable.

T_2 with Equal-Level Predicate E

$E(u, v) \quad :\Leftrightarrow \quad |u| = |v|$

$T_2' := (\{0,1\}^*, \cdot 0, \cdot 1, E)$

Remark: The MSO-theory of T_2' is undecidable.



Weakening MSO-Logic

Over acyclic graphs (like T_2 and G_2):

- **Path logic = MSO where set quantifiers are restricted to paths**
- **Chain logic = MSO where set quantifiers are restricted to subsets of paths (“chains”)**
- **Variants: weak MSO, finite-path logic, finite-chain logic**

CTL and CTL* are encompassed by path logic.

We speak of the path-theory, finite-path theory etc. of a structure when set quantification is restricted accordingly.

Aim of Talk

1. **For the grid prove undecidabílity for a weaker logic than MSO:**
   The finite-path theory of G_2 is undecidable.
2. **For trees with E, show decidability by weakening MSO:**
   The chain theory of T_2' and also of any regular tree T with equal-level predicate is decidable.
3. **Claim 2 fails when we pass to the larger class of algebraic trees.**
4. **Claim 2 fails when we add set quantification over single levels.**
5. **Claim 2 stays true when we add "nice" time predicates.**

Path Logic over the Grid

Theorem (or better: Remark)

The finite-path theory of G_2 is undecidable.

Idea:

Transform 2-counter machine M into finte-path sentence φ_M

such that

M stops when started with counters $(0,0)$ iff $G_2 \models \varphi_M$

M-configuration:
(instruction label, value of counter 1, value of counter 2)

Proof by Picture

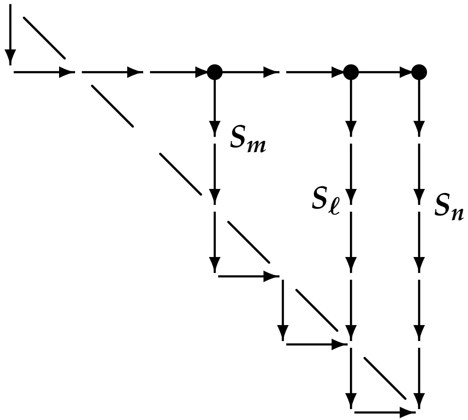
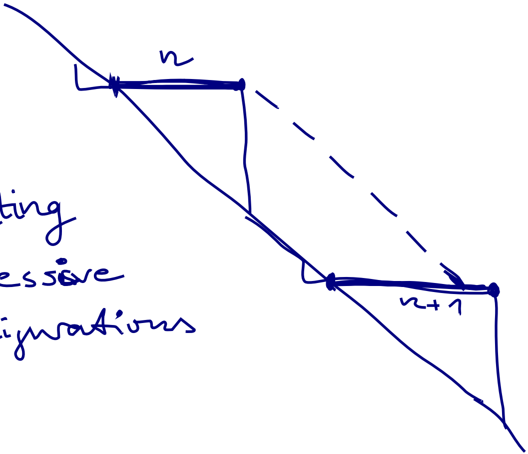


Figure: Coding configuration $(\ell, m, n) = (4, 2, 5)$.

Second Picture



Relating successive configurations

Chain Logic over Regular Trees

Theorem (1991)

The chain theory of a regular binary tree with equal level predicate is decidable.

Idea:

Code a chain C in (T_2, E, P)
by a pair (α_C, β_C) of ω-words over $\{0,1\}$:

α_C is the sequence $d_0 d_1 d_2 \ldots$ of "directions"

$\beta_C(i) = 1$ iff $d_0 \ldots d_{i-1} \in C$

A third sequence γ_C signals membership of the reached vertices in P

Lemma

We can rewrite any chain formula $\varphi(X_1, \ldots, X_n)$
speaking about the structure (T_2, E, P)

as an MSO-formula over $(\mathbb{N}, +1)$ $\varphi'(X_1, Y_1, Z_1 \ldots, X_n, Y_n, Z_n)$
such that

$$(T_2, E, P) \models \varphi[C_1, \ldots, C_n]$$
iff $(\mathbb{N}, +1) \models \varphi'[\alpha_{C_1}, \beta_{C_1}, \gamma_{C_1}, \ldots, \alpha_{C_n}, \beta_{C_n}, \gamma_{C_n}]$

Proof by induction over the formulas

Then use that the MSO-theory of $(\mathbb{N}, +1)$ is decidable (Büchi).

Next Step: Algebraic Trees

Let $P_0 = \{0^i 1^i \mid i \geq 0\}$.

The finite-chain theory of the algebraic tree (T_2, E, P_0) with equal level predicate is undecidable.

Use the result of R.M. Robinson on undecidability of the weak MSO-theory of $(\mathbb{N}, +1, (x \mapsto 2x))$

Another Proof by Picture

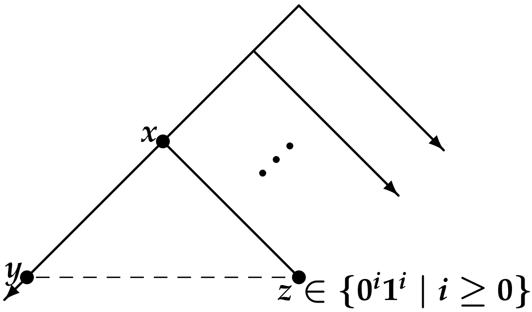


Figure: Computing $y = 2x$.

Adding Quantifiers Over the Levels

Idea by example formula

“the level of node x is met by an even number of paths X such that X has a certain property”

Add MSO quantifiers restricted to submodels consisting of the vertices of a fixed level, respectively.

We speak of FO$(+1,<)$, MSO$(+1,<)$, etc. on the levels.

Theorem

(a) The chain theory of a regular tree with E and FO$(+1,<)$ on the levels is decidable.

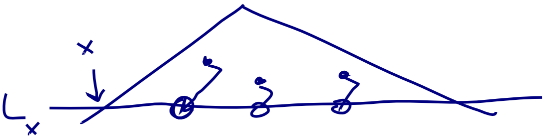
(b) Even the finite-path theory of the unlabelled binary tree with E and MSO$(S,<)$ on the levels is undecidable.

MSO Quantification Restricted to a Level

Simulation of weak set quantification by set quantification over a level

Code finite set S by set $C_S \subseteq$ level L beyond S

For vertex v let $v' = v01^* \cap L$ and set $C_S = \{v' | v \in S\}$



Translation: $\exists S$ such that $\varphi(S)$

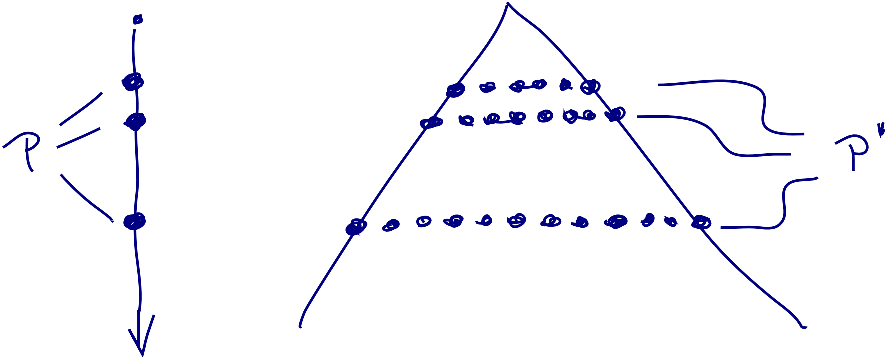
$\mapsto$ **$\exists x$ such that all $y \in S$ are $< L_x$ and $\varphi'(C_S)$**

Add Properties of Time Instances

$P \subseteq \mathbb{N} \quad \mapsto \quad P' = \{v \in \{0,1\}^* \mid |v| \in P\}$

$(\mathbb{N}, +1, P) \quad \mapsto \quad (T_2, P')$ **(without E)**

$(\mathbb{N}, +1, P) \quad \mapsto \quad (T_2', P')$ **(with E)**



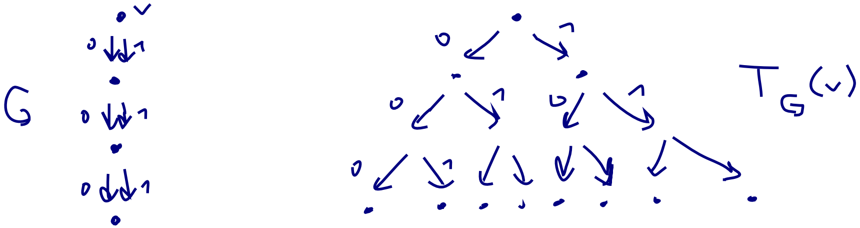
T_2 via Unfolding

If MTh$(\mathbb{N}, +1, P)$ is decidable, then so is MTh(T_2, P').

We invoke a result of Muchnik, Courcelle-Walukiewicz

relating a graph G with MSO-definable vertex v to the unfolding $T_G(v)$ of G from v.

If MTh(G) is decidable and v is an MSO-definable vertex of G, then MTh$(T_G(v))$ is decidable.



(T_2', P') via Interpretation

If MTh$(\mathbb{N}, +1, P)$ is decidable, then so is the chain theory of (T_2', P').

Extend the previous Lemma

We can rewrite any chain formula $\varphi(X_1, \ldots, X_n)$ speaking about the structure (T_2', P')

as an MSO-formula $\varphi'(X_1, Y_1, Z_1 \ldots, X_n, Y_n, Z_n)$ over $(\mathbb{N}, +1, P)$:

$$(T_2', P') \models \varphi[C_1, \ldots, C_n]$$
$$\text{iff } (\mathbb{N}, +1, P) \models \varphi'[\alpha_{C_1}, \beta_{C_1}, \gamma_{C_1}, \ldots, \alpha_{C_n}, \beta_{C_n}, \gamma_{C_n}]$$

Question: For which P is MTh$(\mathbb{N}, +1, P)$ decidable?

Successor Structure + Unary Predicate

Consider $(\mathbb{N}, +1, P)$

χ_P is the characteristic function of P

$\chi_{\mathbb{P}} = 0\,0\,1\,1\,0\,1\,0\,1\,0\,0\,\ldots$

Consequence of Büchi's analysis of MTh$(\mathbb{N}, +1)$:

For each monadic formula $\varphi(X)$ one can construct a Büchi (or Muller) automaton $\mathcal{A}_\varphi$ such that

$(\mathbb{N}, +1) \models \varphi[P]$ **iff** $\mathcal{A}_\varphi$ **accepts** χ_P.

Acceptance Problem $\mathrm{Acc}(P)$:
Given a Büchi autoamaton $\mathcal{A}$, does $\mathcal{A}$ accept χ_P?

Then

MTh$(\mathbb{N}, +1, P)$ is decidable iff $\mathrm{Acc}(P)$ is decidable.

MTh$(\mathbb{N}, +1, P)$ can be difficult

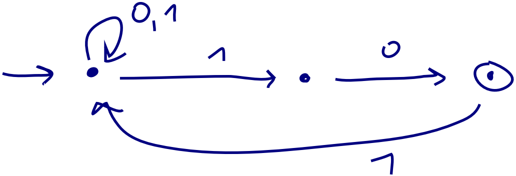
Let $R = \{r_0, r_1, r_2, \ldots\}$ be non-recursive but r.e.

Define $\chi_P = 1\,0^{r_0}\,1\,0^{r_1}\,1\,0^{r_2}\,\ldots$ (recursive!)

$n \in R$ iff
$(\mathbb{N}, +1, P) \models \exists x \in P$ s.t. $x + n + 1$ is next element in P

So: There is a recursive P such that even Th$(\mathbb{N}, +1, P)$ is undecidable.

For $\mathbb{P}$ = primes, instances of the decision problem Acc$(\mathbb{P})$ are open problems of number theory.



The Elgot-Rabin-Method

Fix P.

Given $\mathcal{A}$, try to construct from χ_P a sequence χ' such that

1. **$\mathcal{A}$ accepts χ_P iff $\mathcal{A}$ accepts χ'**
2. **χ' is ultimately periodic.**

Basic example: $P = \mathrm{Fact} := \{n! \mid n \geq 0\}$

Obtain χ'
by compressing the 0-segments between successive 1's

Similar proofs are known for many more predicates, e.g.:

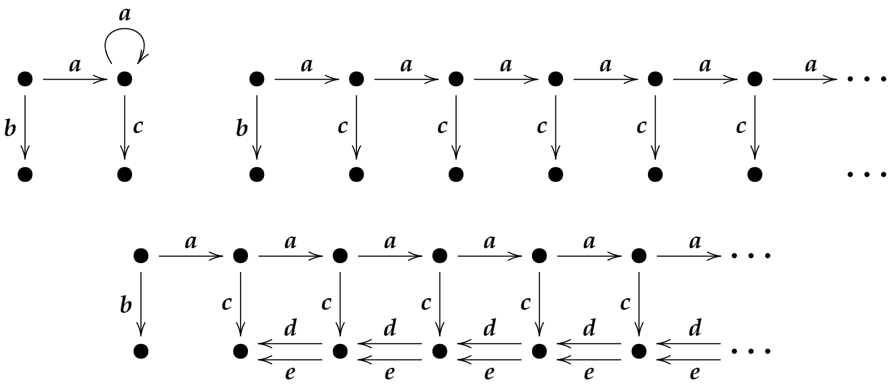
- $\{n^k \mid n \geq 0\}, \{k^n \mid n \geq 0\}$
- **hyperpowers of 2 (forming the set HP_2)**

Caucal's hierarchy

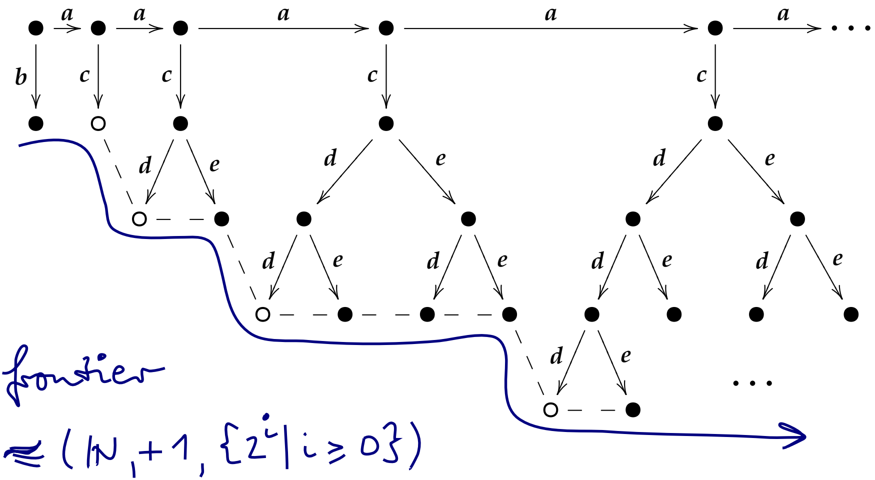
- $\mathcal{T}_0 =$ the class of finite trees
- $\mathcal{G}_n =$ the class of graphs which are MSO-interpretable in a tree of $\mathcal{T}_n$
- $\mathcal{T}_{n+1} =$ the class of unfoldings of graphs in G_n

Each structure in the Caucal hierarchy has a decidable MSO-theory.

A finite graph, a regular tree, a PD graph



Unfolding again



Discussion

$(\mathbb{N}, +1, \{2^n \mid n \geq 0\})$ **occurs in the Caucal hierarchy, and thus**

MTh$(\mathbb{N}, +1, \{2^n \mid n \geq 0\})$ **has a "uniform" decidability proof,**

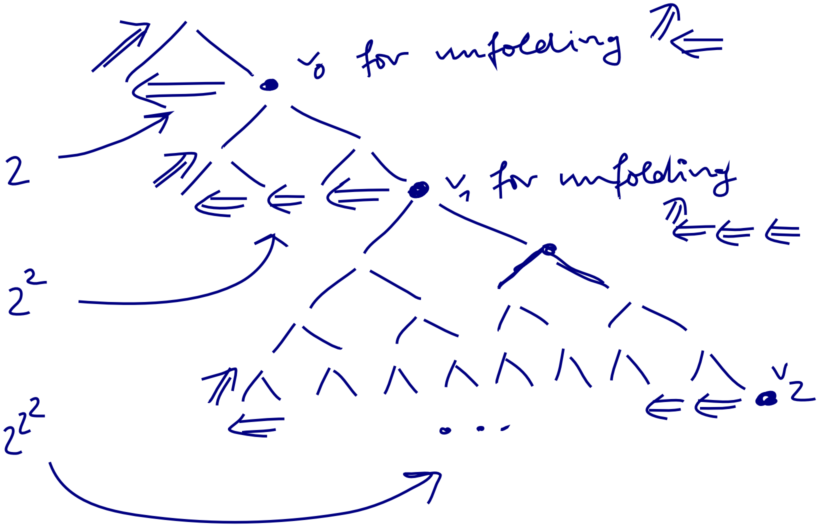
by global transformations of models, starting from a finite graph.

The analogous statement holds for the predicates

- **of** k**-th powers,**
- **of factorial numbers**

but not for the set HP_2 **of hyperpowers of 2.**

A limit model



Interpretation-Unfolding Schemes

The interpretation-unfolding scheme
IU$(\varphi(x), \psi_0(x,y), \psi_1(x,y,z), \psi_2(x,y,z))[t_0]$

- **selects by the formula $\varphi(x)$ a leaf v of t_0**
- **by ψ_0, ψ_1, ψ_2 defines in (t_0, v) a structure $\mathcal{A}_v = (A_v, R_v, S_v)$ (with binary R_v, S_v)**
- **by unfolding $\mathcal{A}_v$ from v generates a tree $t_0 \circ_v t_1$**

and in the limit defines the infinite product "$t_0 \circ t_1 \circ t_2 \ldots$"

The structure $(\mathbb{N}, +1, \mathrm{HP}_2)$ is constructible this way.

Details

[Work in progress with B. Morcrette]

1. **Three variants: Linear IU-schemes (one leaf selected only), pointed IU-schemes (one vertex selected), branching IU-schemes (several leaves possible).**
2. **By a branching IU-scheme one can generate the infinite grid from a finite tree — so branching IU-schemes are too strong; similarly for pointed IU-schmes.**
3. **Question: Under which assumptions (if any) can one guarantee decidability of MTh$(\mathcal{M})$ when $\mathcal{M}$ is generated by a linear IU-scheme from a finite tree?**

More information: W.Th., CSL 2008

Conclusion

- **Synchronization of paths is good for scientific collaboration.**

- **Restricted systems of MSO-logic are – even after half a century of investigation – an interesting, pleasing, and hopefully also useful subject of study.**