Ranking based Techniques for Disambiguating Büchi Automata

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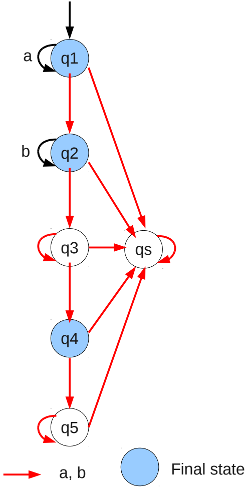
Non-deterministic Büchi automata over words (NBW)

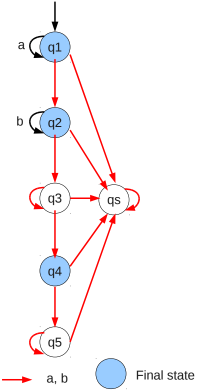
A 5−tuple $(\Sigma, Q, Q_0, \delta, F)$, where

- Σ : Input alphabet
- Q : Finite set of states
- $Q_0 \subseteq Q$: Initial states
- $\delta \subseteq Q \times \Sigma \times Q$: State transition relation
- F : Set of final/accepting states



Runs and acceptance

- A *run* of $\mathcal{A}$ on $\alpha \in \Sigma^\omega$ is a sequence $\rho : \mathbb{N} \rightarrow Q$ such that
  - $\rho(0) \in Q_0$
  - $\rho(i+1) \in \delta(\rho(i), \alpha(i))$



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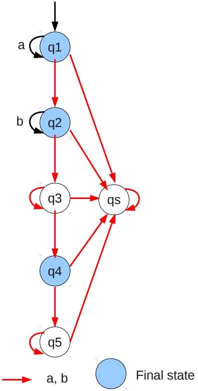

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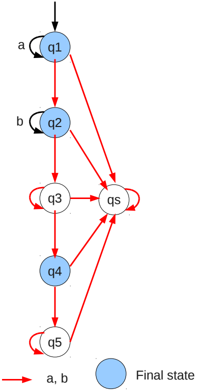
- ρ is accepting iff $\inf(\rho) \cap F \neq \emptyset$
- α is accepted by $\mathcal{A}$ ($\alpha \in L(\mathcal{A})$) iff there is an accepting run of $\mathcal{A}$ on α.



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$\mathcal{A}$ is *ambiguous* if there exists $\alpha \in L(A)$ such that there are ≥ 2 accepting runs of $\mathcal{A}$ on α Otherwise, $\mathcal{A}$ is *unambiguous*.

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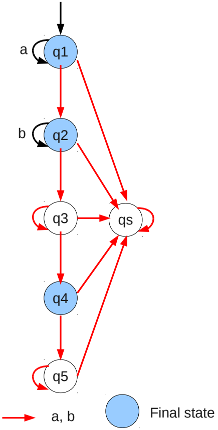
An ambiguous NBW



- $\alpha = ab^\omega$
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  - A word $\notin L(\mathcal{A})$ may have 0 or more final runs
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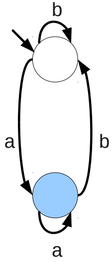
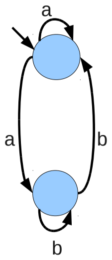


Deterministic (hence unambiguous) but not strongly unambiguous



Strongly unambiguous

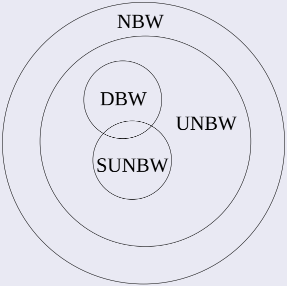
Containment relations

UNBW: Unambiguous NBW, SUNBW: Strongly unambiguous NBW, DBW: Deterministic Büchi automata over words

Expressive power-wise

$$\text{DBW} \subsetneq \text{NBW} \equiv \text{UNBW} \equiv \text{SUNBW}$$

Automata structure-wise



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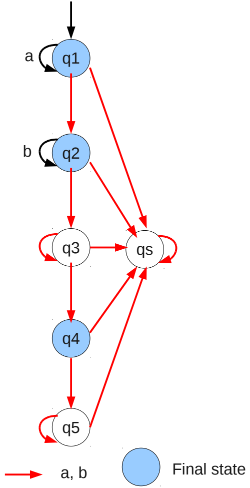
Our contribution

- Effective construction of UNBW, size bound $O(n^2.(0.76n)^n)$
  - Same as best known bound for NBW complementation!

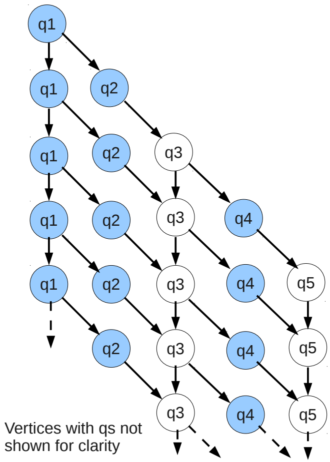
Why care about disambiguation?

- Of course, a theoretically interesting problem
- Can it lead to a better understanding of what kinds of NBW admit easy determinization?
- Practical application? Seek inputs from the audience.

Run DAGs



Run DAG for a^ω



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  - We use Kupferman-Vardi style rankings

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n: Number of states in NBW
V: Set of run DAG vertices
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- *Odd ranking*: Every path eventually trapped in an odd rank
- *Even ranking* otherwise

Example of KV-ranking

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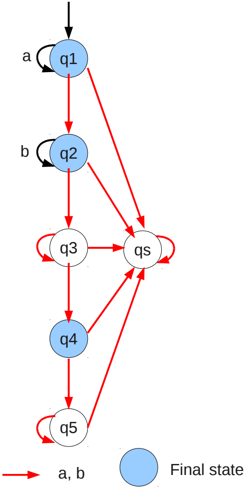
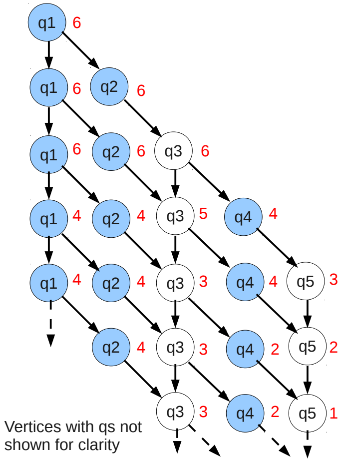
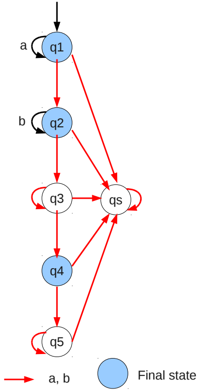




Ranking based complementation

Theorem (Kupferman-Vardi 2001)

An ω-word $\alpha \in \overline{L(\mathcal{A})}$ iff there is an odd ranking of the run DAG of $\mathcal{A}$ on α



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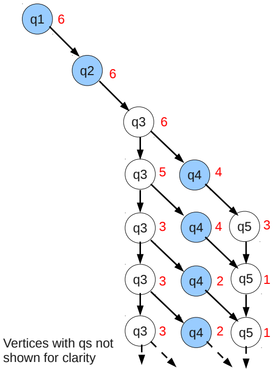

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  - Achieves same bound of $O(n^2, (0.76n)^n)$.

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We call this a **full ranking** of the run DAG.

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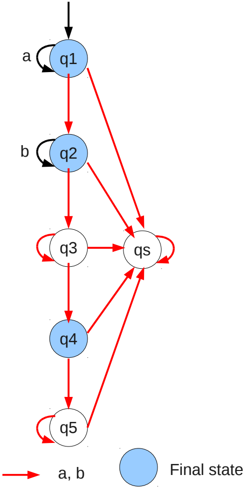

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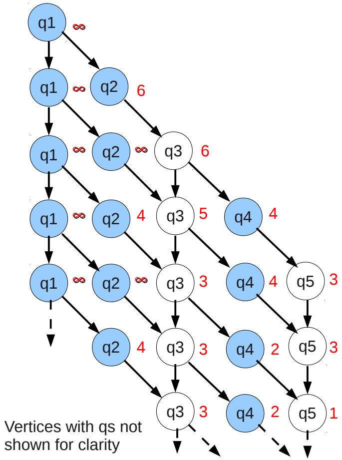
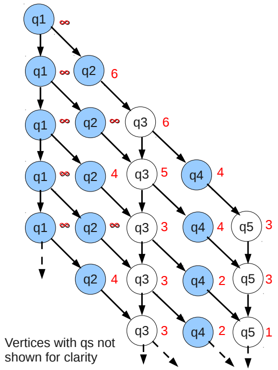


Minimal full rankings

Given run DAG G, full ranking r^* of G is **minimal** iff for all full rankings r of G, $r^*(v) \leq r(v)$ for all vertices v in G.

Non-minimal full ranking



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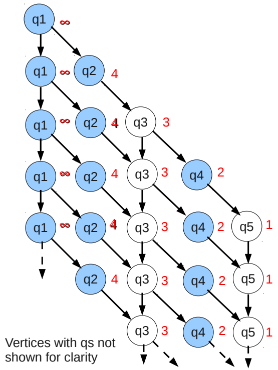

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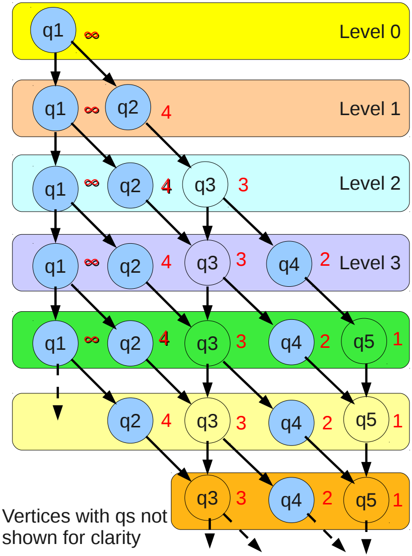
- Every ∞ ranked vertex has at least one descendant with the largest non-infinity rank in range of the ranking function.

Intuition of disambiguation construction



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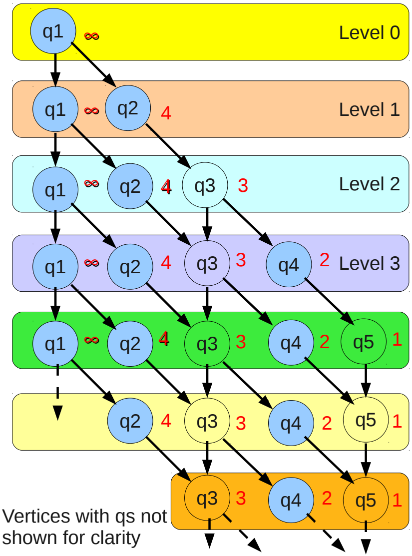

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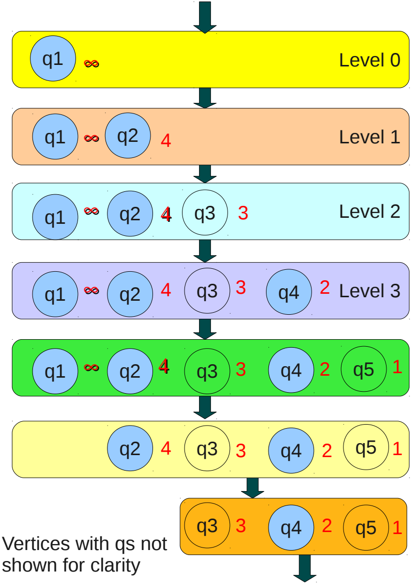

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  - Ensure that each finite segment satisfies relevant property checkable over finite steps
- Acceptance condition simply ensures that every finite segment of an infinite run satisfies relevant properties and root vertex is ranked ∞

State representation

State of resulting automaton:

$$(S, O, X, f, i), \text{ where}$$

- S : subset of states of NBW in current level
- f : ranking function at current level
- i : rank of vertices for which (decomposed) global properties are currently being checked
- $O \subseteq S$: subset of states with rank i for which global properties yet to be checked
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Total count of states is $O(n^2.(0.76n)^n)$

- Uses a modification of a counting argument used by Schewe (2009) for NBW complementation

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- Any two accepting runs must differ in the ranking of at least one level
- Since minimal ranking is unique, only one accepting run possible

Conclusion

- Using a variant of KV-ranking (similar to that used by Carton and Michel), we obtain a UNBW (not SUNBW) with better bound than reported in the literature
- We conjecture that this matches the lower bound for disambiguation
- Shows potential close connection between disambiguation and complementation