Reachabilty in Succinct and Parametric One-Counter Automata

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ACTS
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Parameters Everywhere

Boltzman's constant	k
Planck's constant	$\hbar$
Speed of light	c
Gravitational constant	G

...

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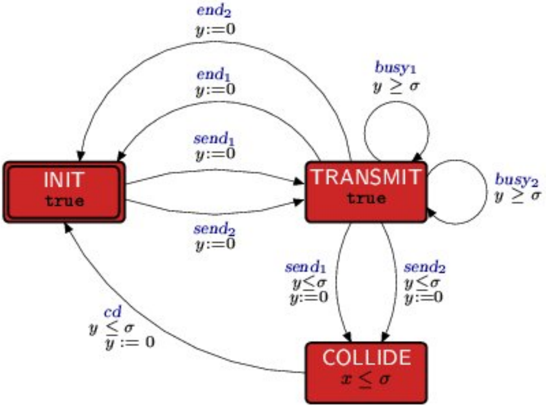
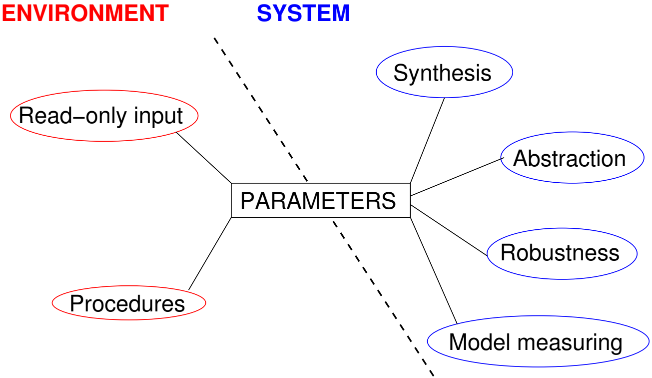
A More Tractable Example

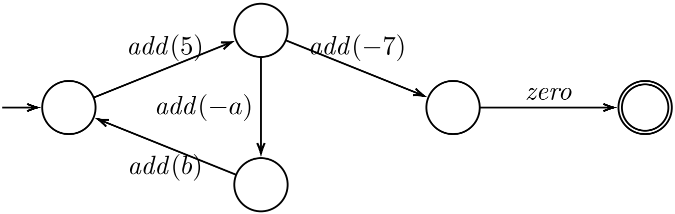



ENVIRONMENT
SYSTEM
Synthesis
Read–only input
Abstraction
PARAMETERS
Robustness
Procedures
Model measuring

Parametric State Machines

- Flat counter machines with parameters
  (Bozga, Iosif, Lakhnech 06)
- Reversal-bounded counter machines with read-only input
  (Dang, Ibarra 93 ; . . .)
- Timed automata with parametric guards
  (Alur, Henzinger, Vardi 93 ; André, Encrenaz, Fribourg 09)
- Counter machines with weights/costs
  (Xie, Dang, Ibarra 03)

Parametric One-Counter Automata

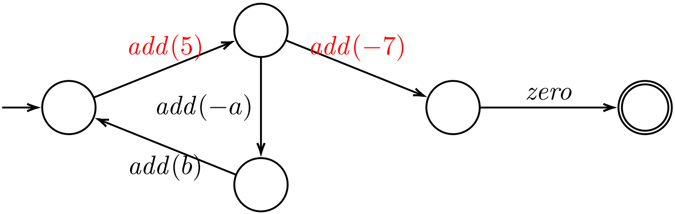


One-counter automata: NFA with one counter over $\mathbb{N}$

Succinct: Numbers encoded in binary

Parametric: Increment and decrement counter by parametric values

Parametric One-Counter Automata



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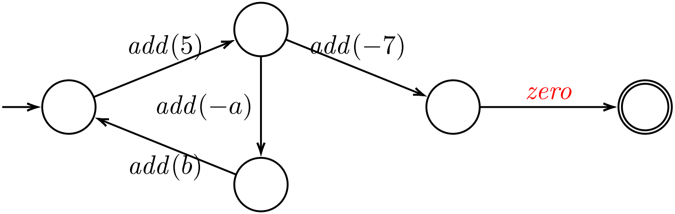

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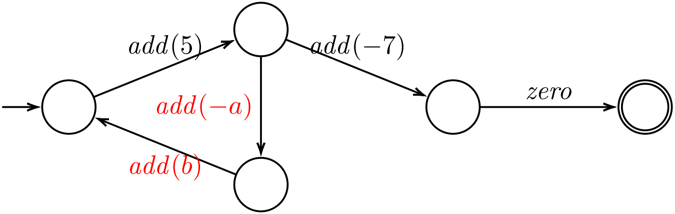

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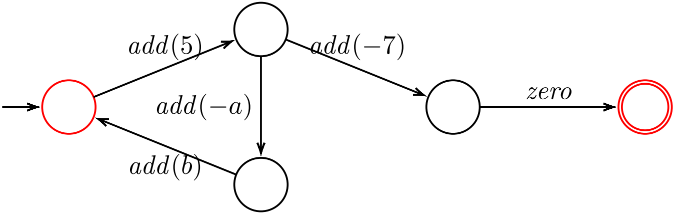


Are there values for the parameters such that a final configuration is reachable from an initial configuration?

Main result

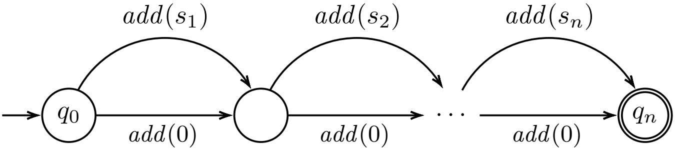
Theorem

The reachability problem for parametric one-counter automata is NP*-complete.*

NP-Hardness

Reduction from SUBSETSUM:

Instance: $S = \{s_1, s_2 \ldots, s_n\} \subseteq \mathbb{N}$ and target $t \in \mathbb{N}$
Question: Is there $S' \subseteq S$ such that $\sum_{s \in S'} s = t$?



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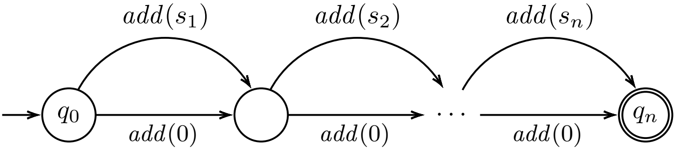

Problem becomes NLOGSPACE-complete when numbers are encoded in unary (Lafourcade *et al.*, 2004)

Presburger Arithmetic

- First-order theory of the natural numbers with addition is decidable (Presburger 29)
- Adding multiplication or divisibility leads to undecidability of satisfiability (Gödel 31, Robinson 49)
- Existential fragment of PA with divisibility is decidable (Lipshitz 78)
  - Terms: linear polynomials $A(\vec{x}) = a_0 + a_1 x_1 + \ldots + a_n x_n$
  - Atomic formulas: $A(\vec{x}) \leq B(\vec{x})$ and $A(\vec{x}) | B(\vec{x})$
  - Formulas: $\exists x_1 \cdots \exists x_n : \varphi(\vec{x})$

Presburger+Divisibility –> Reachability

Idea. Given $\varphi(\vec{x})$, construct counter machine $\mathcal{C}_\varphi$ with parameters $\vec{x}$ such that $\varphi(\vec{x})$ iff $(q_s, 0) \rightsquigarrow (q_t, 0)$:

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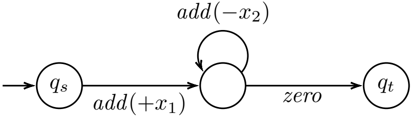
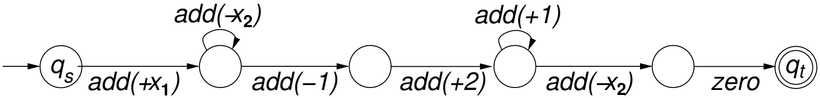
- $x_1 \mid x_2$



Presburger + Divisibility –> Reachability

Idea. Given formula $\varphi(\vec{x})$, construct counter machine $\mathcal{C}_\varphi$ such that $\varphi(\vec{x})$ holds iff $(q_s, 0) \rightsquigarrow (q_t, 0)$ in $\mathcal{C}_\varphi$.

- $\varphi_1 \wedge \varphi_2$: sequential composition of $\mathcal{C}_{\varphi_1}$ and $\mathcal{C}_{\varphi_2}$
- $\varphi_1 \vee \varphi_2$: parallel composition of $\mathcal{C}_{\varphi_1}$ and $\mathcal{C}_{\varphi_2}$
- $x_2 \nmid x_1$



NP-Hardness Again

- **Theorem** (Manders, Adelman 76). The following problem is NP-complete:

  Given integers α, β, γ does there exist $x \leq \gamma$ such that

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- Easily encoded into Presburger arithmetic with divisibility
- Reachability is NP-hard on counter machines even if we fix the underlying graph of states and transitions.

Words of Wisdom

Words of Wisdom

"If you can't solve a problem, there is an easier problem you can't solve." - ***George Pólya***

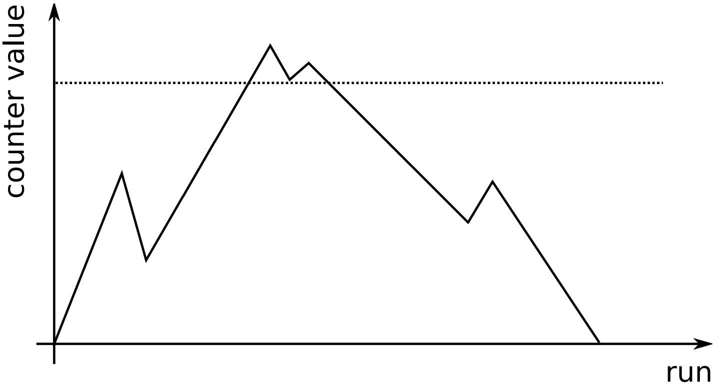
The non-parametric case

NP-Membership of Reachability

Three stages to show membership in NP:

1. Establish a bound on the length of a run
2. Find certificate of a run of polynomial size
3. Ensure certificate can be verified in non-deterministic polynomial time

Truncating Runs (Lafourcade *et al.*, 2004)



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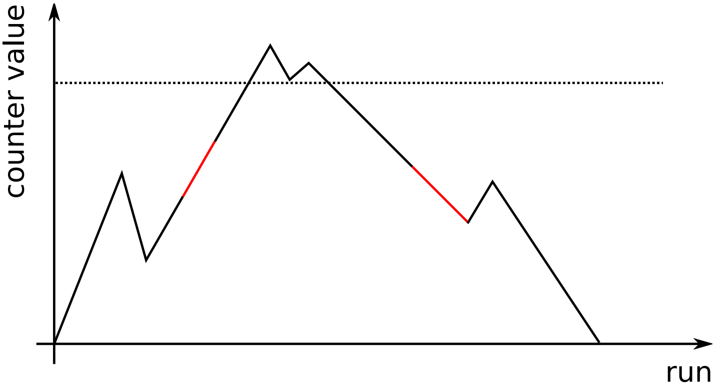

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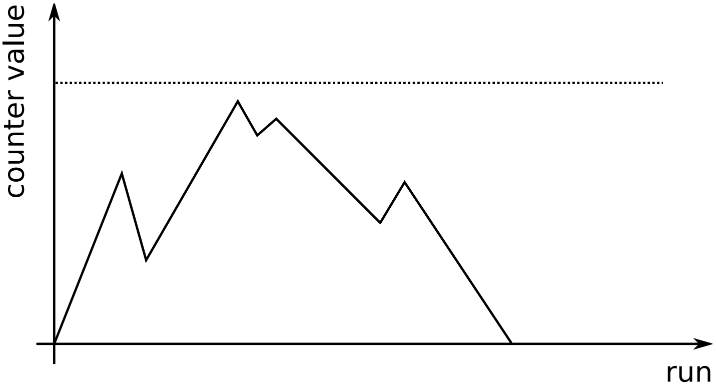

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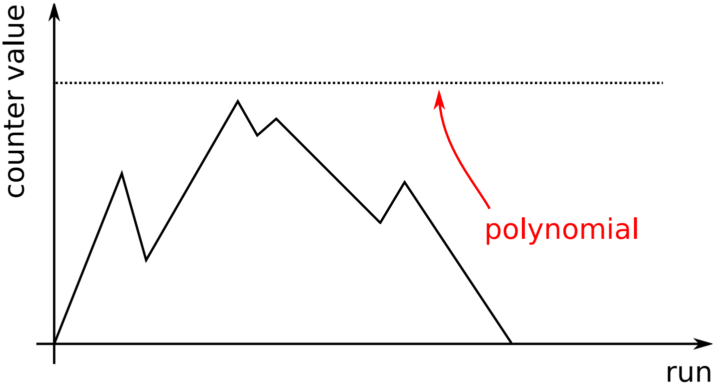

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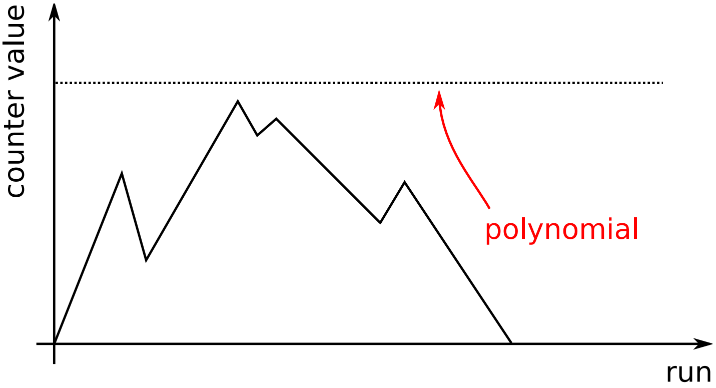

$\rightsquigarrow$ PSPACE upper bound for reachability

NP-Membership of Reachability

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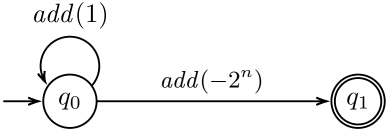
Runs of Exponential Length



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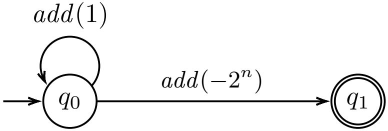
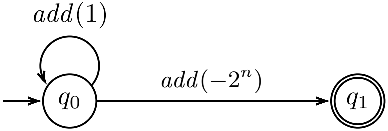


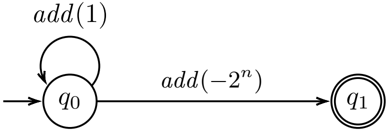
$(q_0, 0)$

Runs of Exponential Length



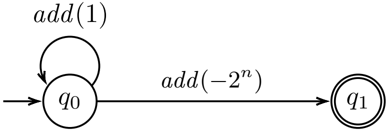
$(q_0, 0) \rightarrow (q_0, 1)$

Runs of Exponential Length



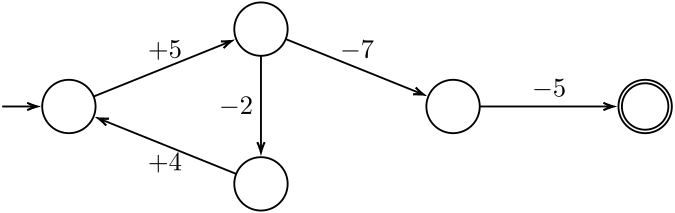
$$(q_0, 0) \rightarrow (q_0, 1) \rightarrow (q_0, 2)$$

Runs of Exponential Length



$$(q_0, 0) \rightarrow (q_0, 1) \rightarrow (q_0, 2) \rightarrow \cdots \rightarrow (q_1, 2^n) \rightarrow (q_1, 0)$$

Flow Networks



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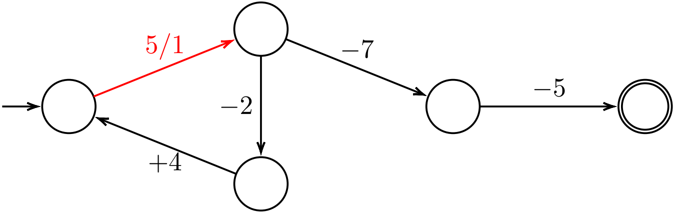

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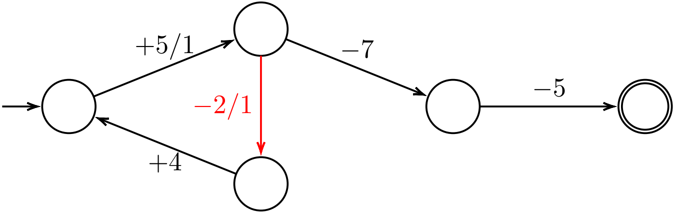

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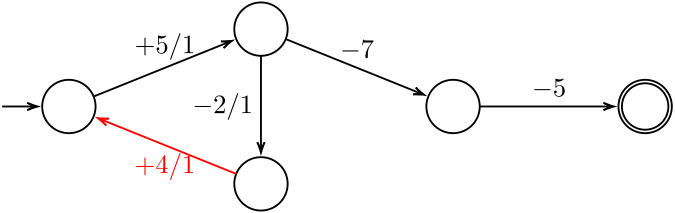

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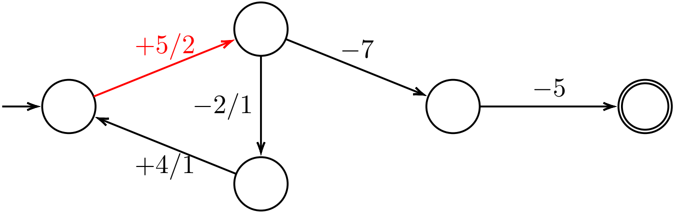

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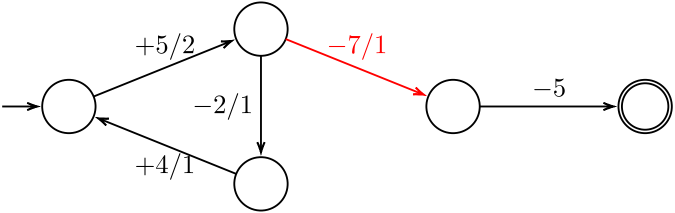

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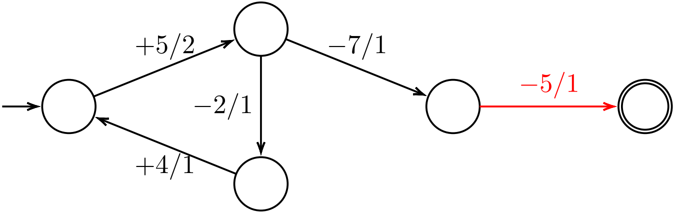

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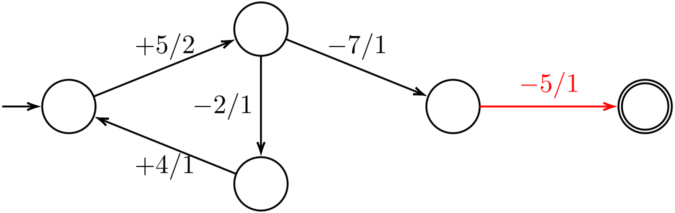
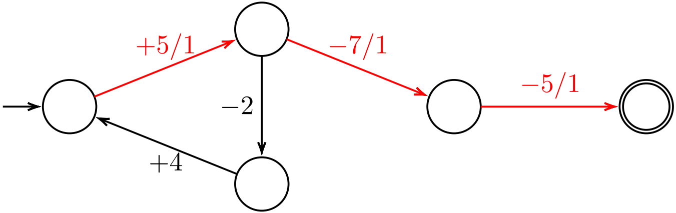


⇝ assign to each edge the number of times it is taken:

Flow Networks



but flow network does not necessarily correspond to a run

NP-Membership of Reachability

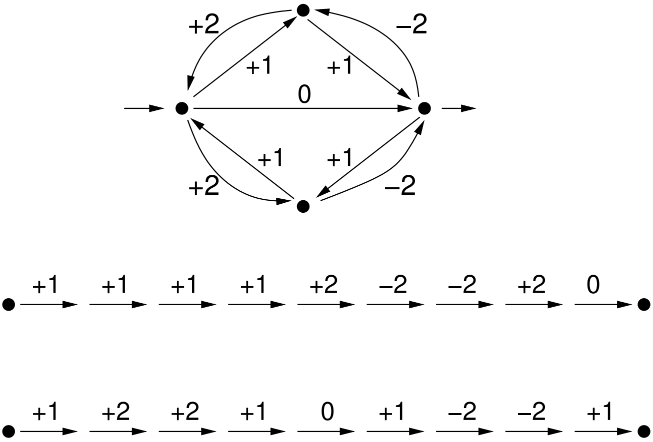
Three stages to show membership in NP:

1. Establish a bound on the length of a run
2. Find certificate of polynomial size of a run
3. Ensure certificate can be verified in non-deterministic polynomial time

Three Simple Cases

1. Flow network begins with a positive cycle and ends with a negative cycle
2. Flow network has no positive cycles
3. Flow network has no negative cycles

Positive Cycles and Positive Cycles



Positive Cycle and Negative Cycle

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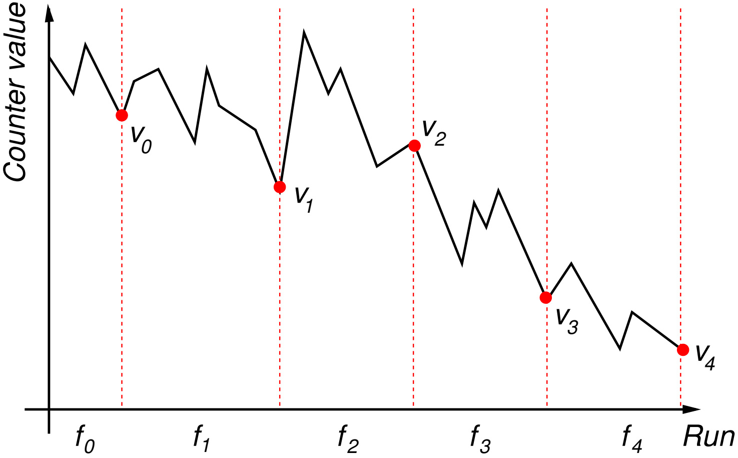
No Positive Cycles



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- Guess **elimination order** on vertices
  - $v_0, v_1, \ldots, v_4$

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- Corresponding flow decomposition
  - $f = f_0 + f_1 + \cdots + f_4$

No Positive Cycles

- Guess **elimination order** on vertices
  - $v_0, v_1, \ldots, v_4$
- Corresponding flow decomposition
  - $f = f_0 + f_1 + \cdots + f_4$
- Counter never goes negative:
  - $\mathit{value}(f_0) \geq 0$
  - $\mathit{value}(f_0 + f_1) \geq 0$

  $\ldots$

Three Simple Cases

1. Flow network begins with a positive cycle and ends with a negative cycle

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Decomposition Lemma

Lemma

If there is a path from the initial state to the final state, then there is a path that can be written as the sum of three flow networks $f^- + f^* + f^+$, *where*

- f^- *contains no positive cycle*
- f^+ *contains no negative cycle*
- f^* *has a positive cycle at the "beginning" and a negative cycle at the "end"*

Kirchhoff Certificates

Kirchhoff certificate guessed and verified in NP:

- Flows f^-, f^+ and f^* guessed in polynomial time
- Bellman-Ford algorithm checks in polynomial time non-existence of positive cycles in f^- and negative cycles in f^+
- Elimination orderings for f^+ and f^- guessed in polynomial time

$\leadsto$ NP-algorithm

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$\rightsquigarrow$ reachability for succinct one-counter automata is NP-complete

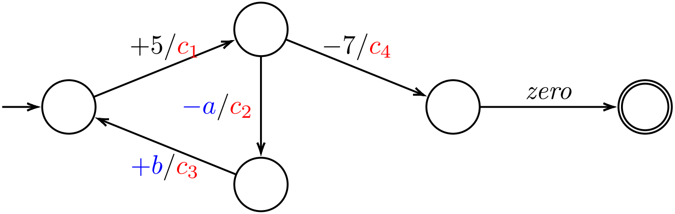
In Reality

In Reality

"It's only 10 pages in the LNCS style – we need another result!" - ***Christoph Haase***

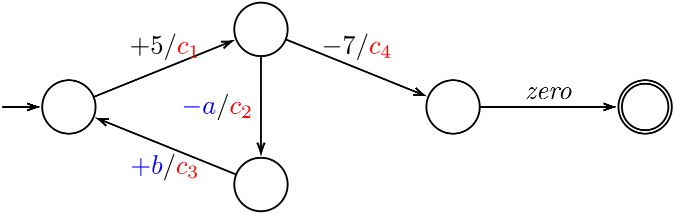
The parametric case

Symbolic Representation



- Symbolic representation of Kirchhoff certificates

Symbolic Representation



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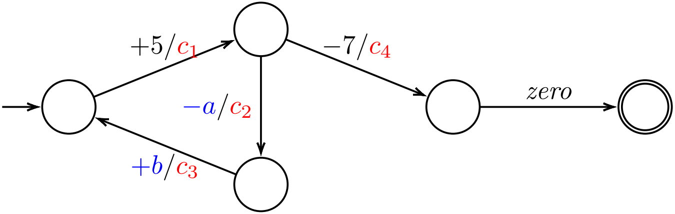
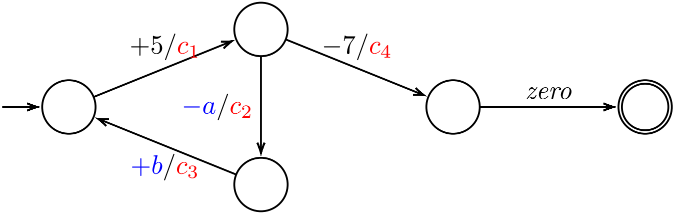
- Variables c_1, c_2, c_3, c_4 to represent flow

Symbolic Representation



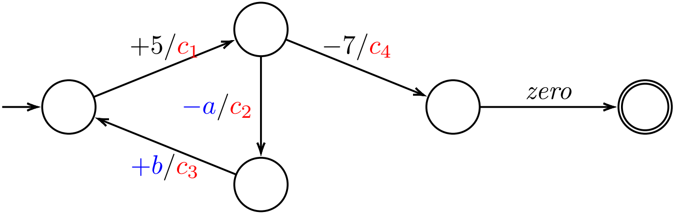
- Symbolic representation of Kirchhoff certificates
- Variables c_1, c_2, c_3, c_4 to represent flow
- Variables a, b to represent parameters

Symbolic Representation



- Flow constraints: e.g. $c_1 = c_2 + c_4$

Symbolic Representation



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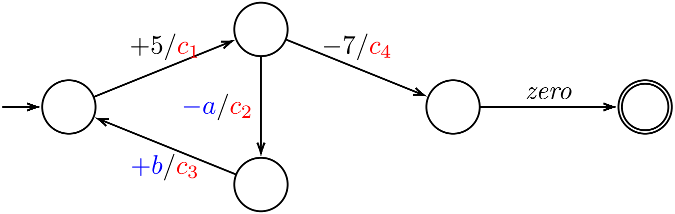
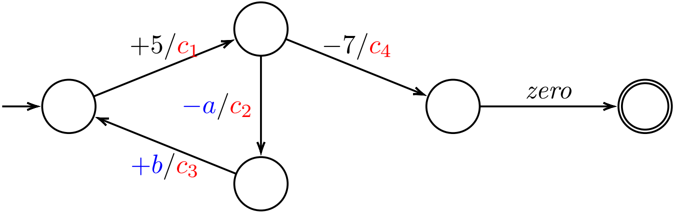
- Cycle constraints: e.g. $b - a + 5 > 0$

Symbolic Representation



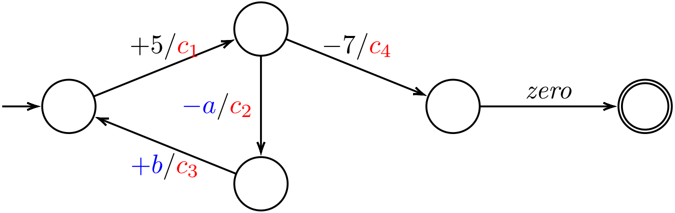
- Flow constraints: e.g. $c_1 = c_2 + c_4$
- Cycle constraints: e.g. $b - a + 5 > 0$
- Value constraints: $value(f) > 0$

Symbolic Representation



Value constraints:

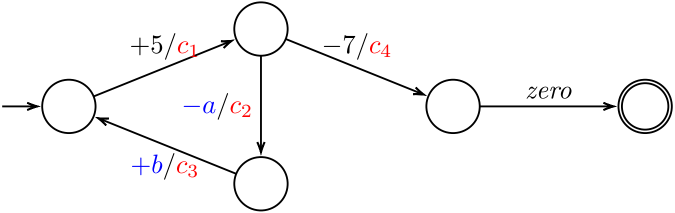
Symbolic Representation



Value constraints:

- $value(f) = 5 \cdot c_1 - a \cdot c_2 + b \cdot c_3 - 7 \cdot c_4$

Symbolic Representation



Value constraints:

- $\mathit{value}(f) = 5 \cdot c_1 - a \cdot c_2 + b \cdot c_3 - 7 \cdot c_4$
- Quadratic Diophantine equation

Flow Networks and Diophantine Equations

Some systems of quadratic Diophantine equations are decidable:

$$\begin{aligned} R_1 &= y_1 A_1(\vec{x}) + B_1(\vec{x}) \\ &\vdots \\ R_k &= y_k A_k(\vec{x}) + B_k(\vec{x}) \end{aligned}$$

Given $P \subseteq \mathbb{Z}^k$ Presburger definable, ask

$$\exists \vec{x} \exists \vec{y} P(R_1, \ldots, R_k)?$$

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$$\exists \vec{x} \exists \vec{y} P(R_1, \ldots, R_k)?$$

…translate to sentence in Presburger arithmetic with divisibility:

Summary

- Satisfiability in the existential fragment of Presburger arithmetic with divisibility is NP-complete (Lipshitz, 1976)

Summary

- Satisfiability in the existential fragment of Presburger arithmetic with divisibility is NP-complete (Lipshitz, 1976)
- All conditions of a reachability certificate can be encoded in a sentence of polynomial size in this logic

Summary

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Theorem

The reachability problem for parametric one-counter automata is NP-complete.