On Classical, Real-Time, and Time-Bounded Verification

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Oxford University Computing Laboratory

(Joint work with Alex Rabinovich and James Worrell)

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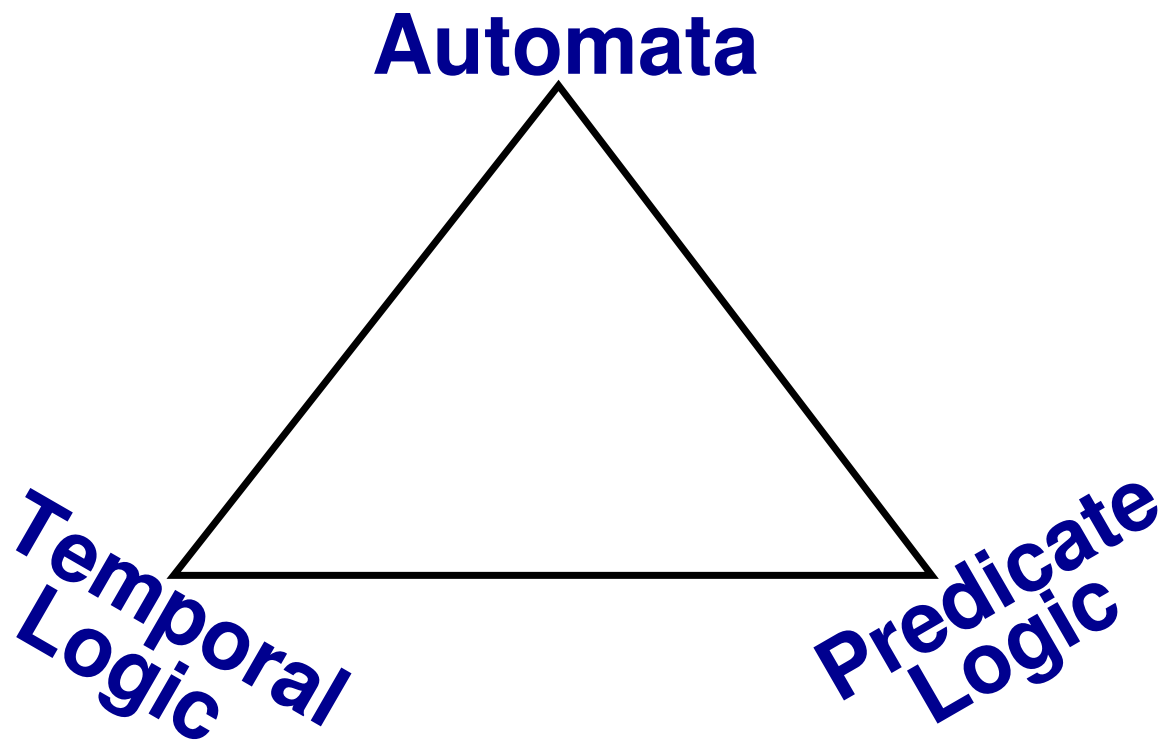
The Classical Theory of Verification



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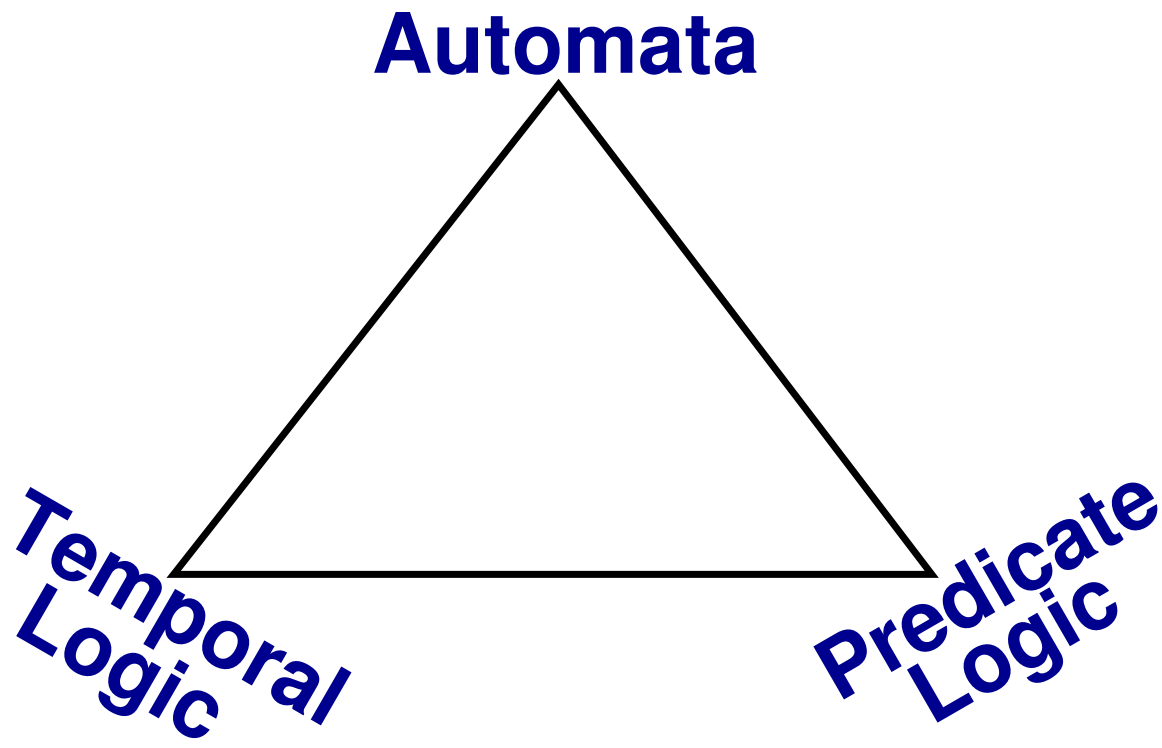

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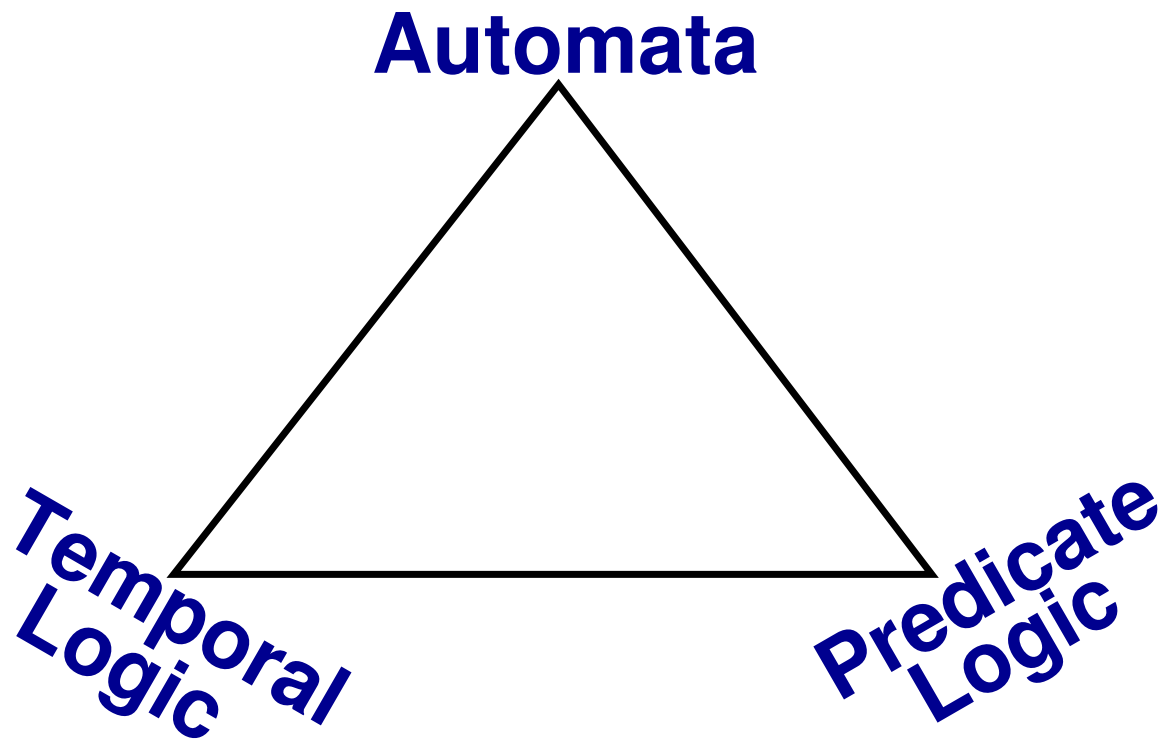

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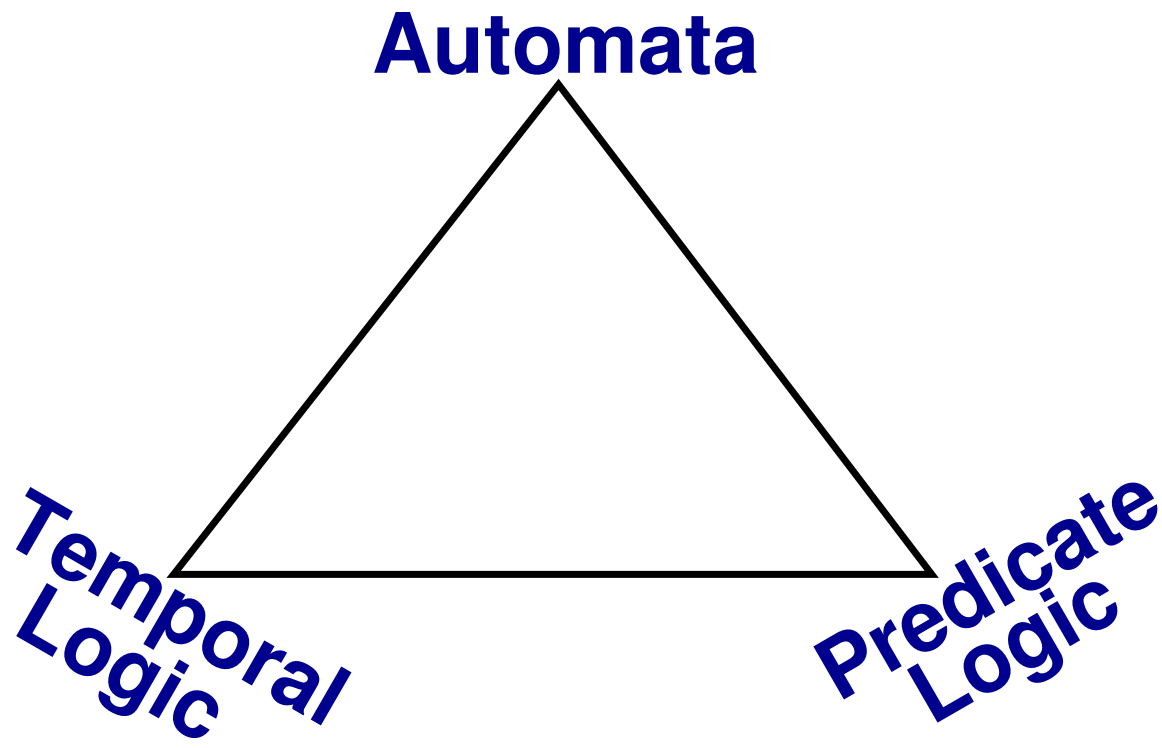

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- Note: focus on linear time (as opposed to branching time).

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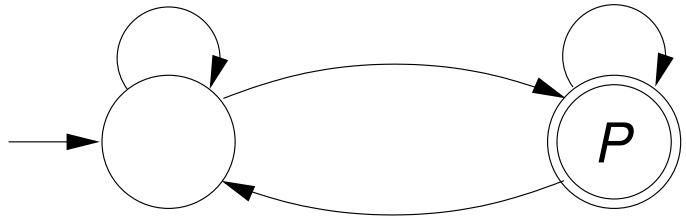

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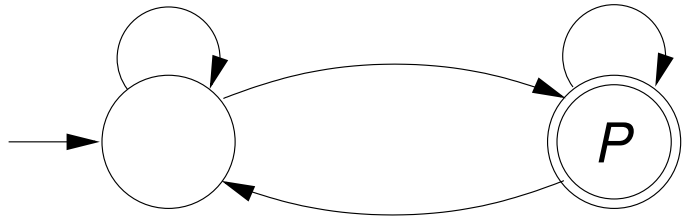

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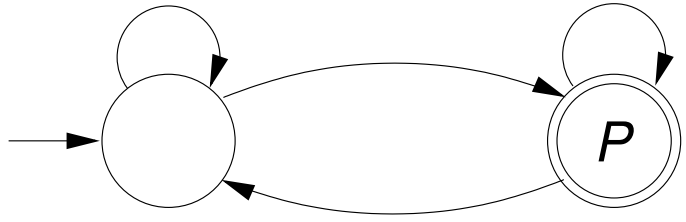

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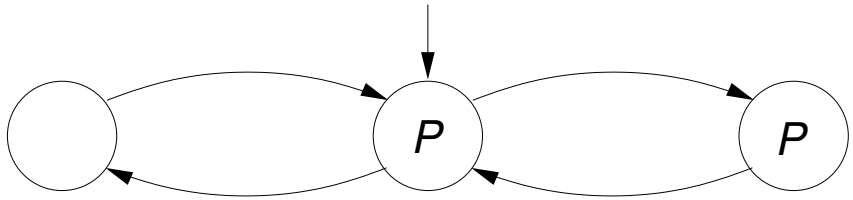

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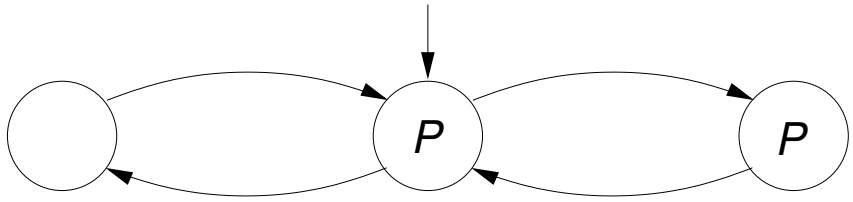

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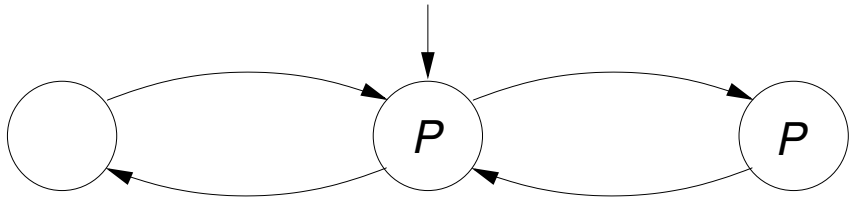

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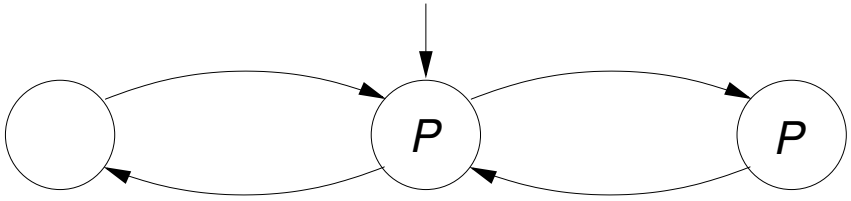

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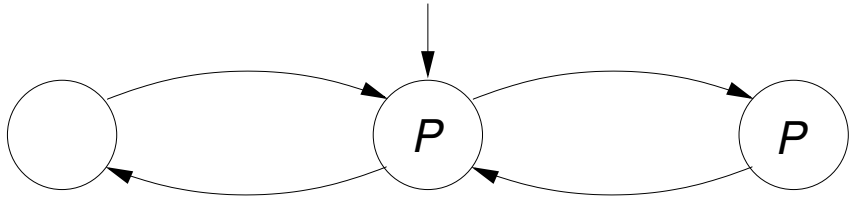

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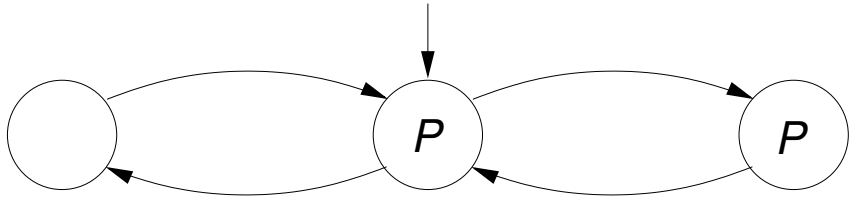

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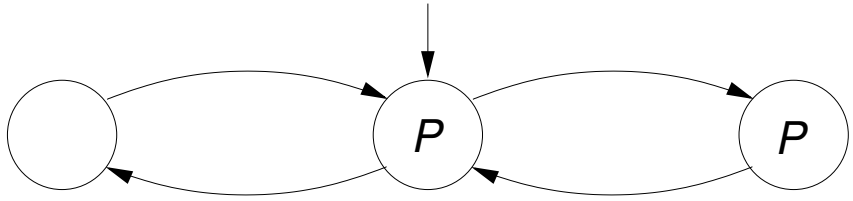

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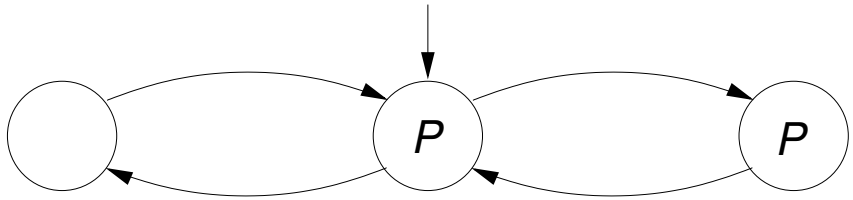

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Corollary (Church 1960)

The model-checking problem for automata against MSO($<$) specifications is decidable:

$$M \models \varphi \quad \textit{iff} \quad L(M) \cap L(A_{\neg\varphi}) = \emptyset$$

Algorithmic Complexity

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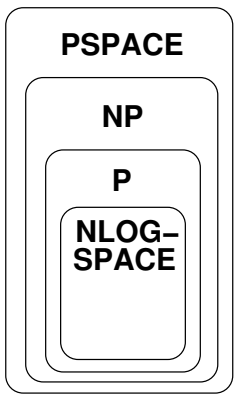

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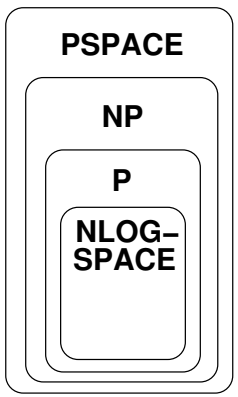

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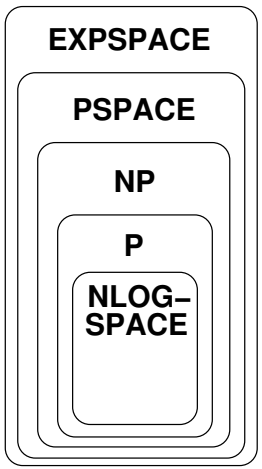

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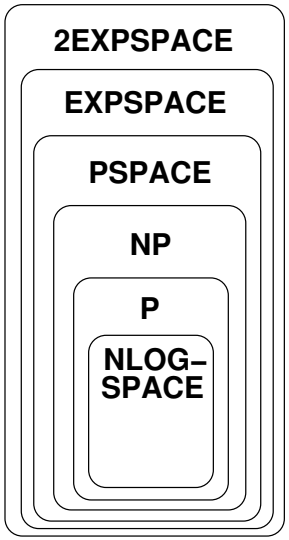

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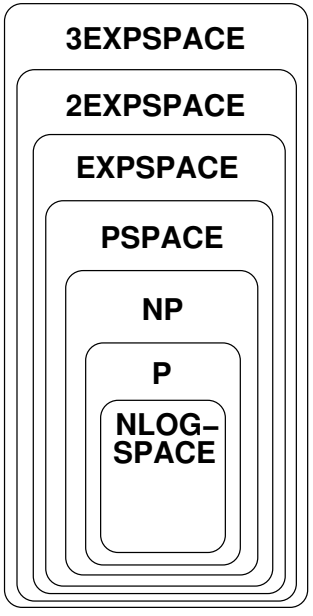

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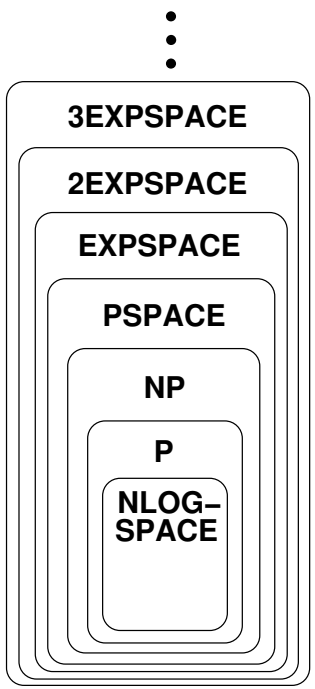

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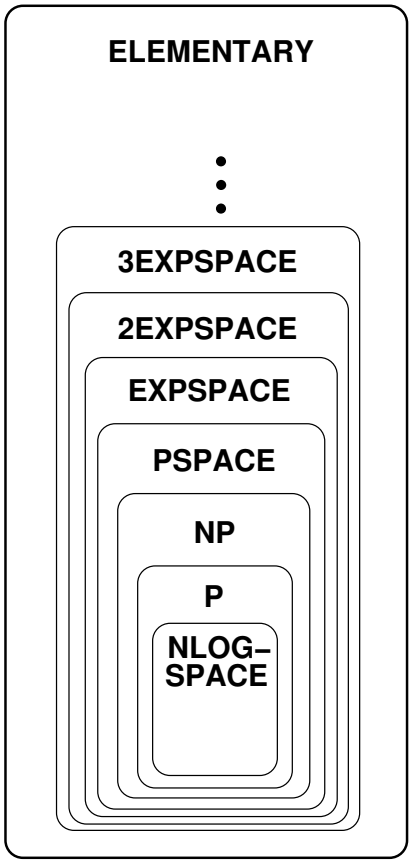

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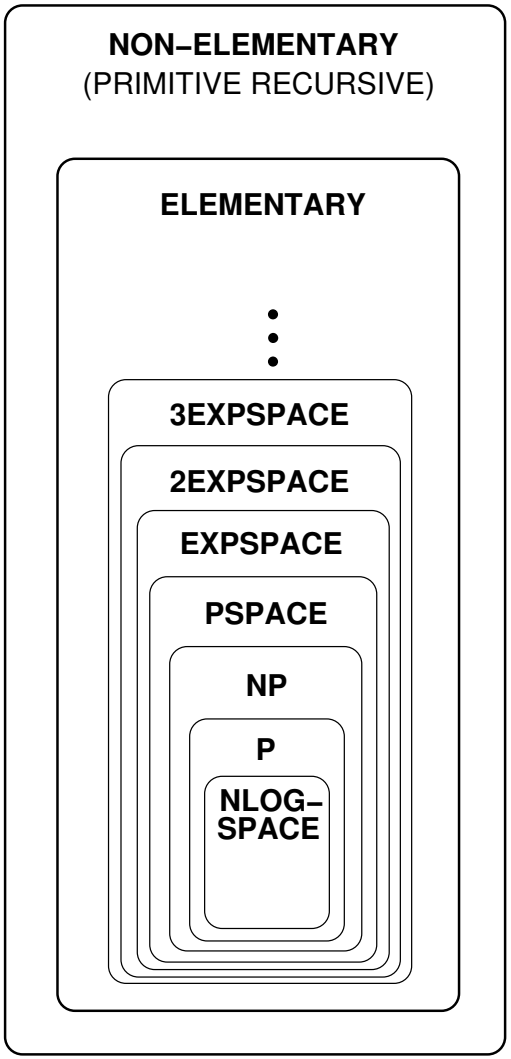

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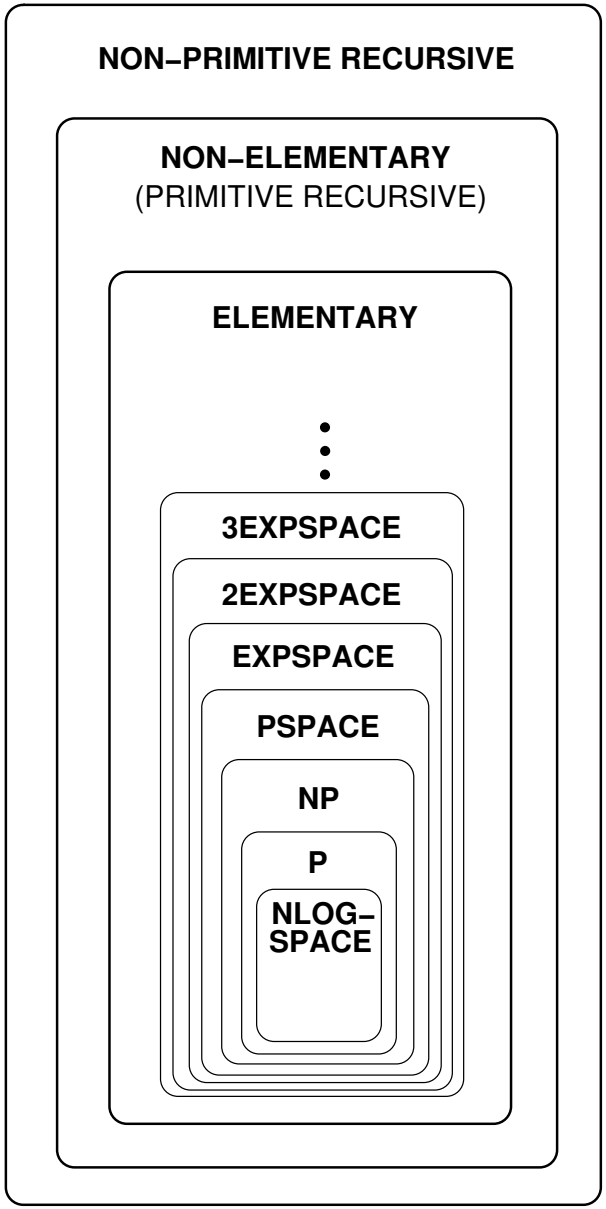

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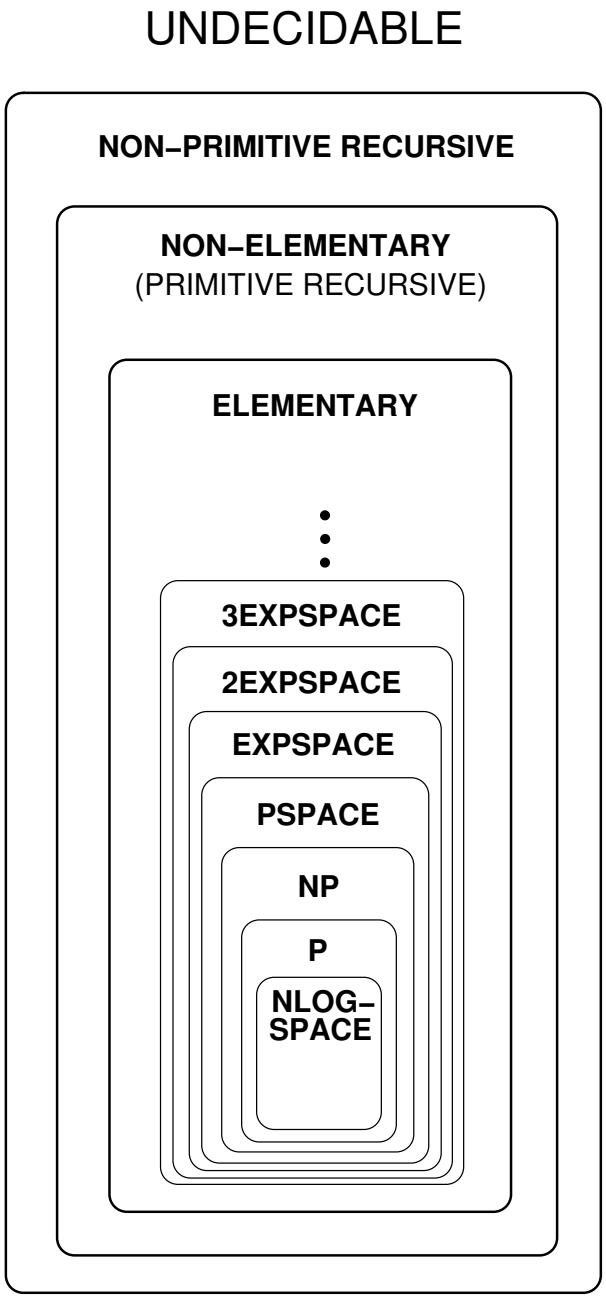

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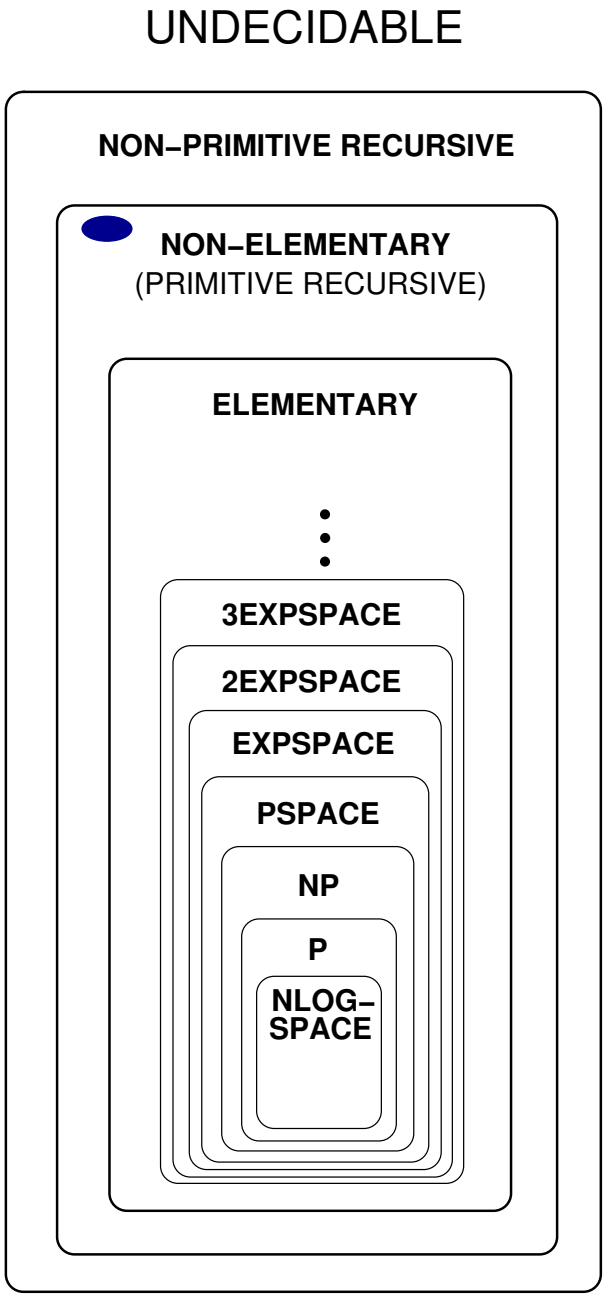

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But amazingly:

Theorem (Sistla & Clarke 1982)

LTL satisfiability and model checking are PSPACE-complete.

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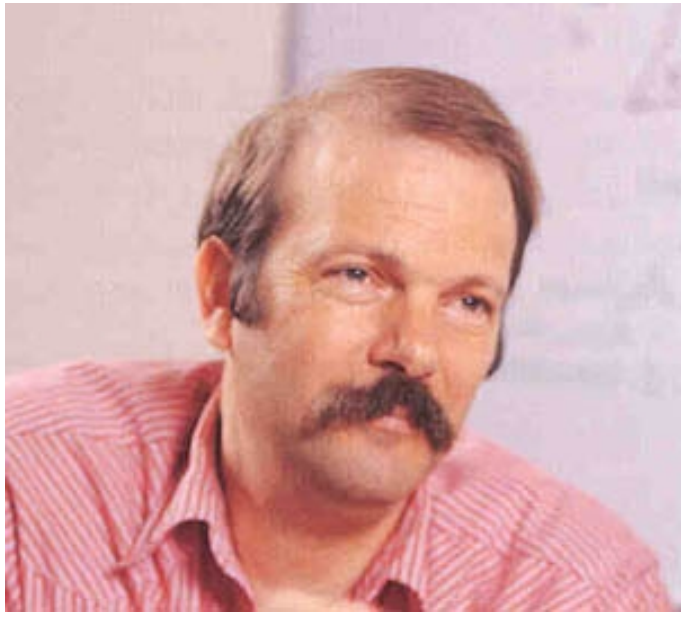

Moshe Vardi

Logics and Automata

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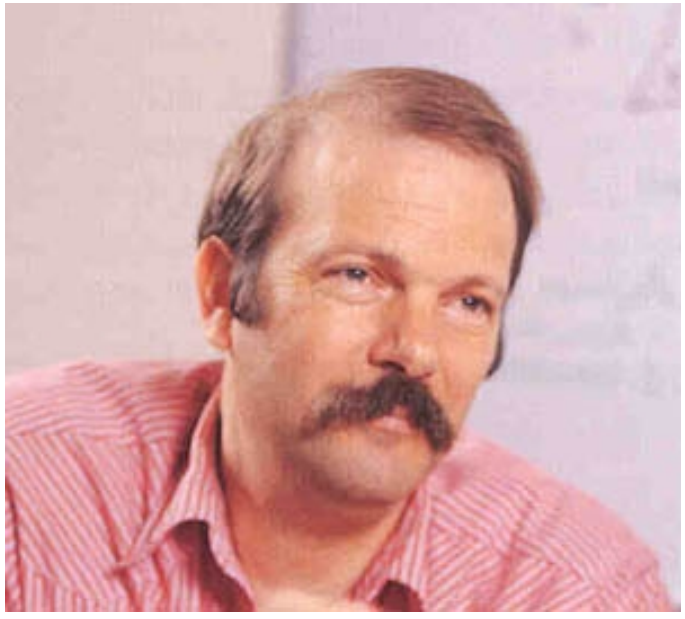
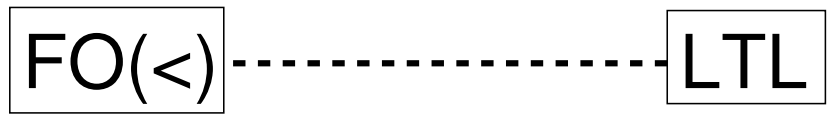


Moshe Vardi

Theorem

Automata are closed under all Boolean operations. Moreover, the language inclusion problem ($L(A) \subseteq L(B)$?) is PSPACE-complete.

The Classical Theory: Expressiveness



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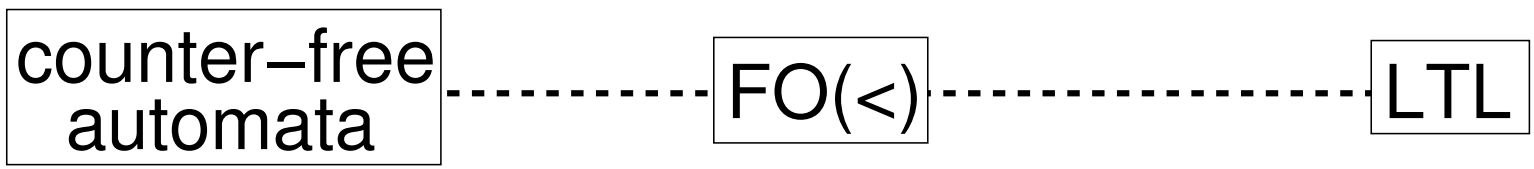

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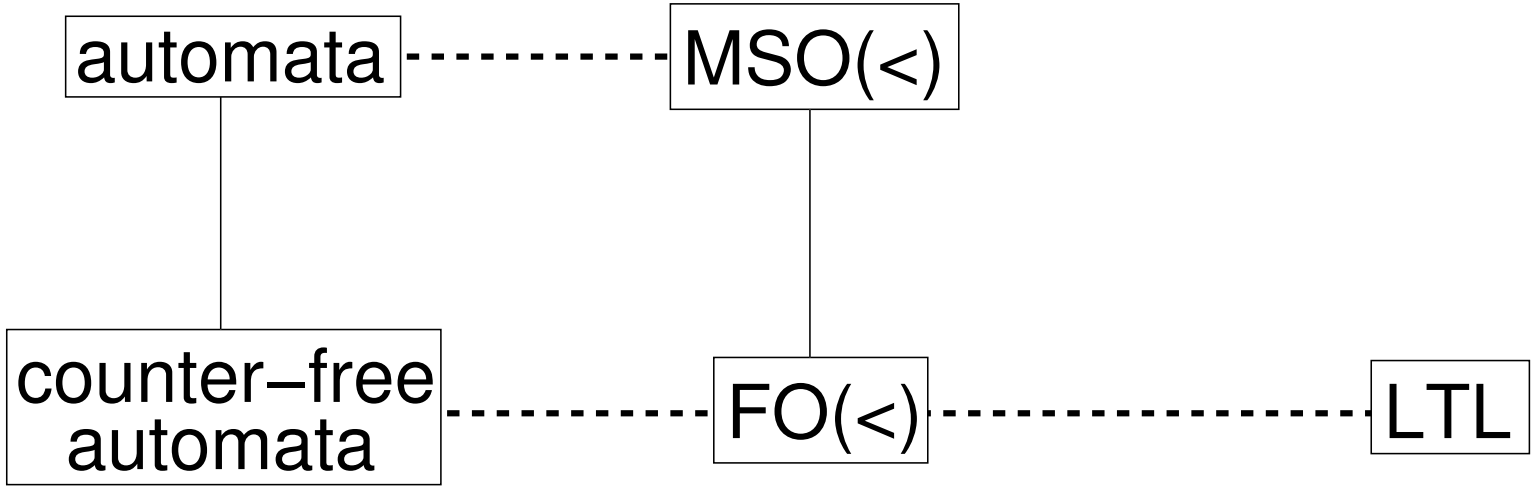

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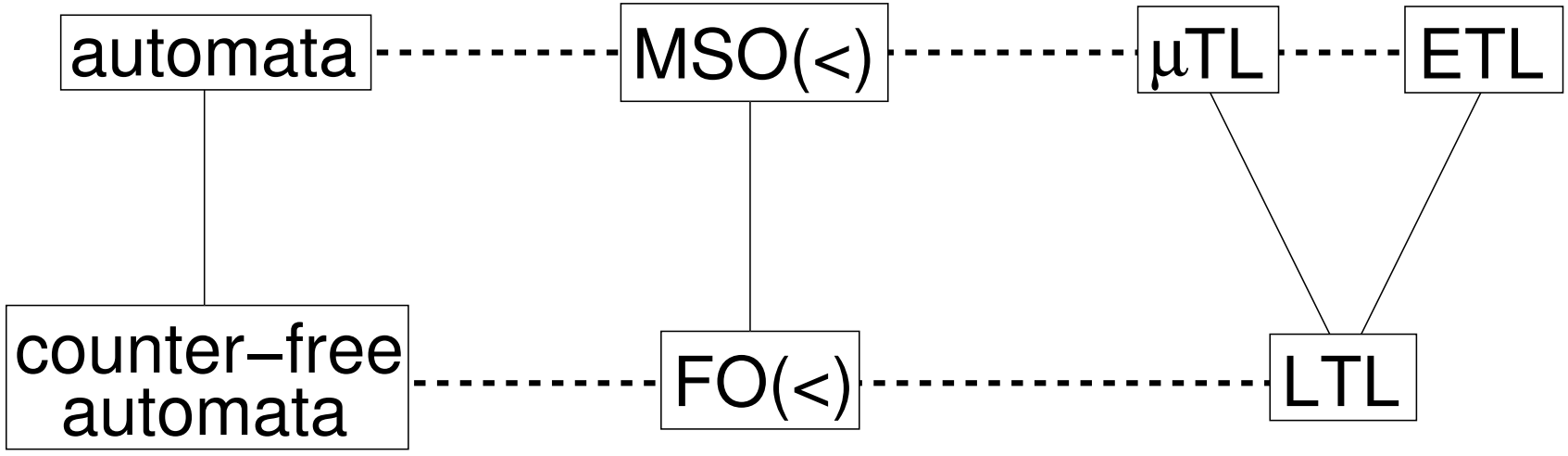
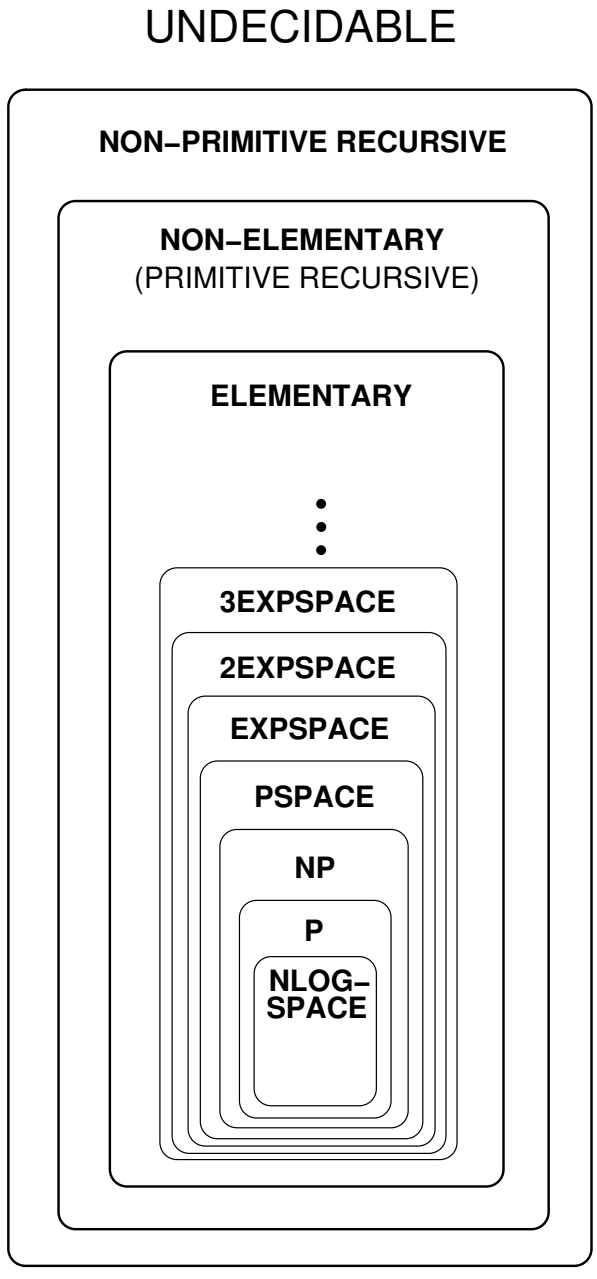


The Classical Theory: Complexity



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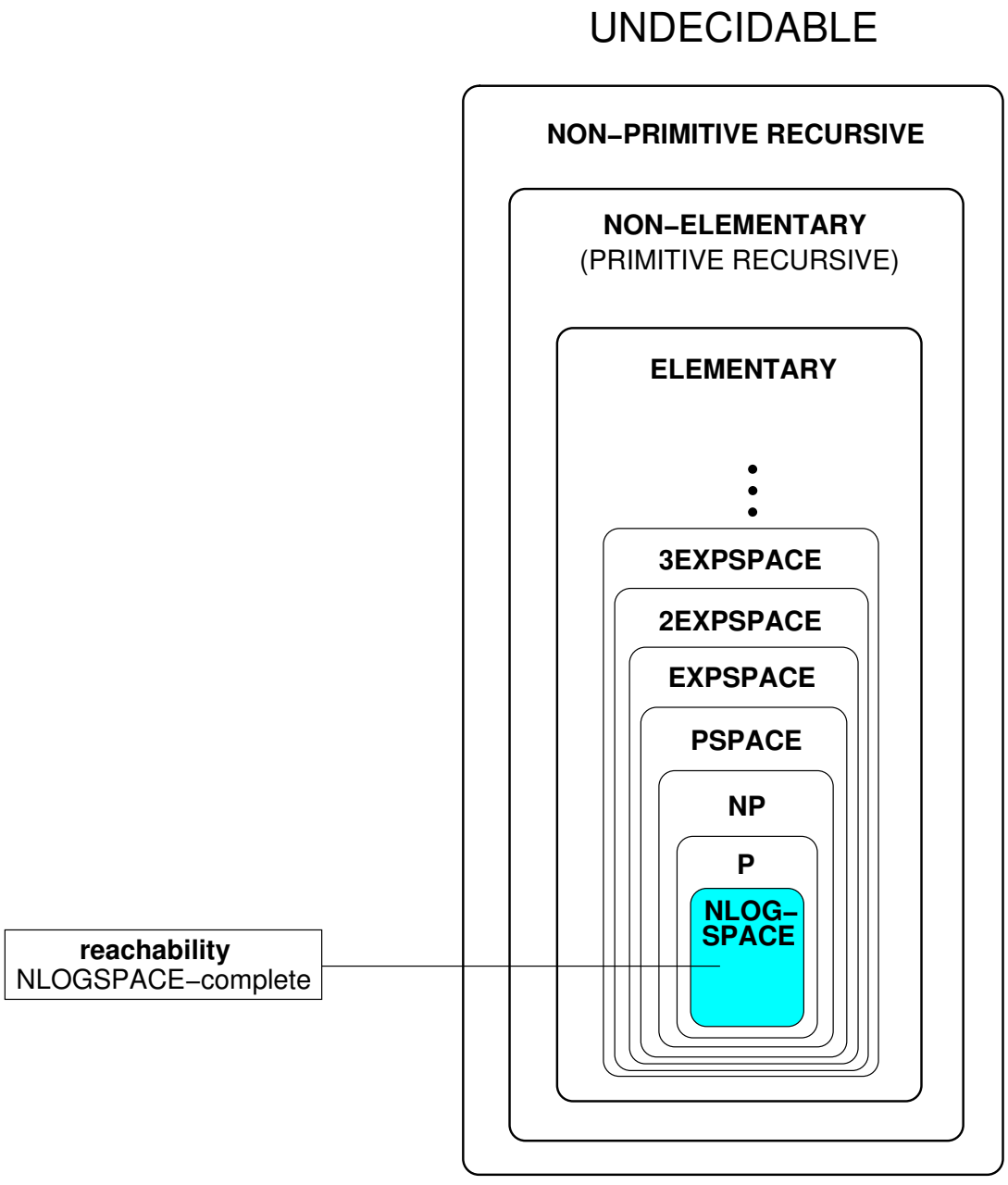

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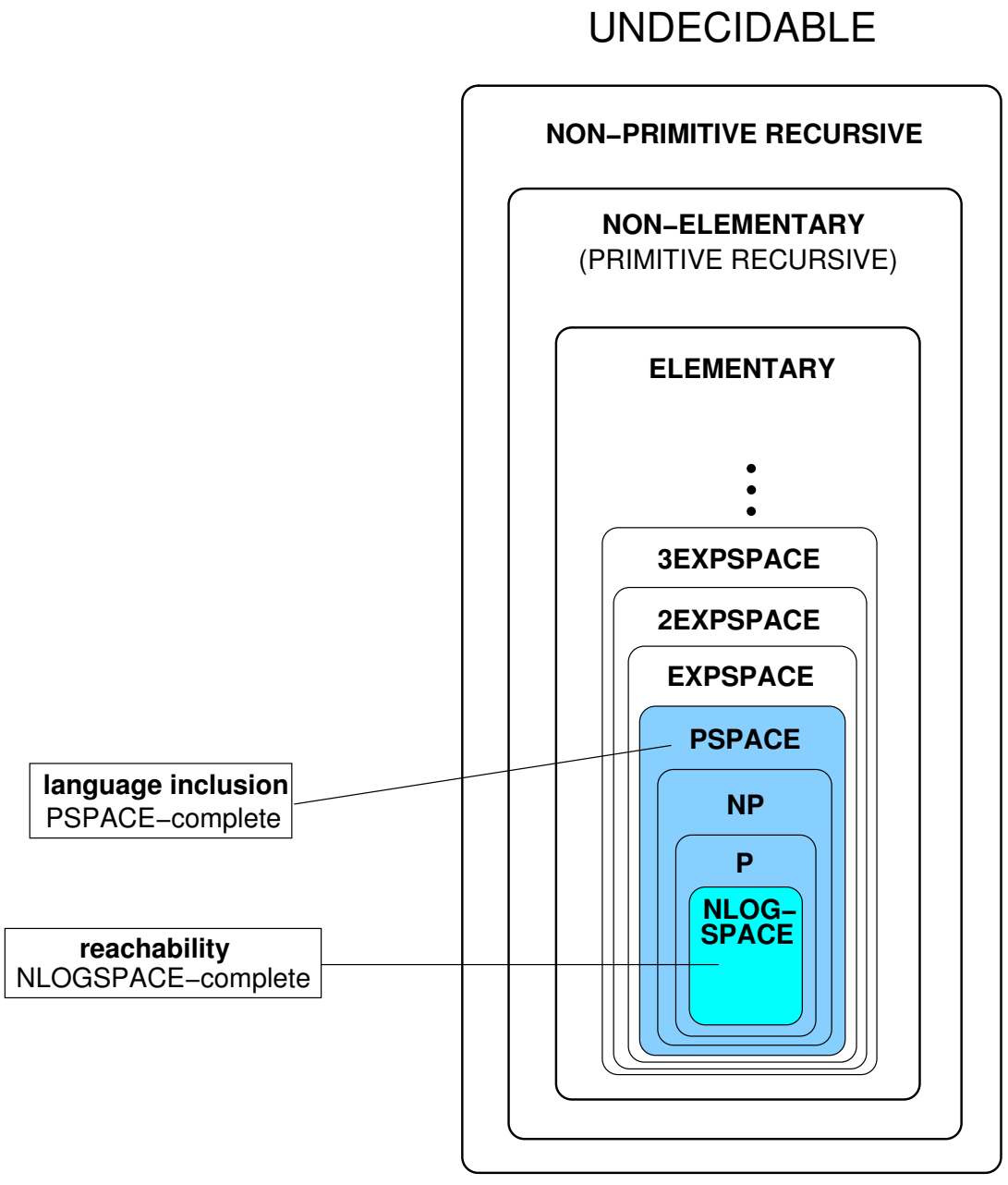

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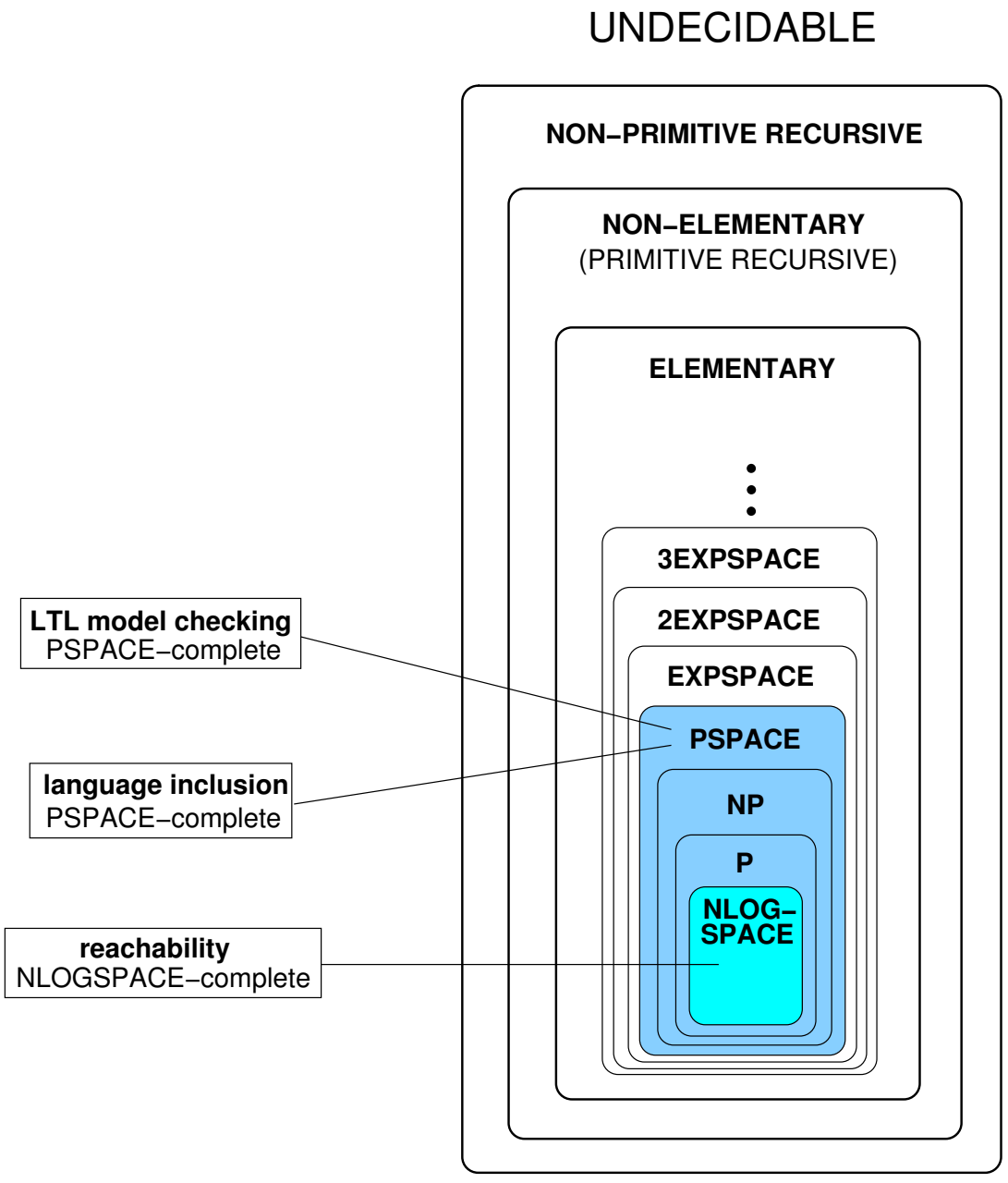

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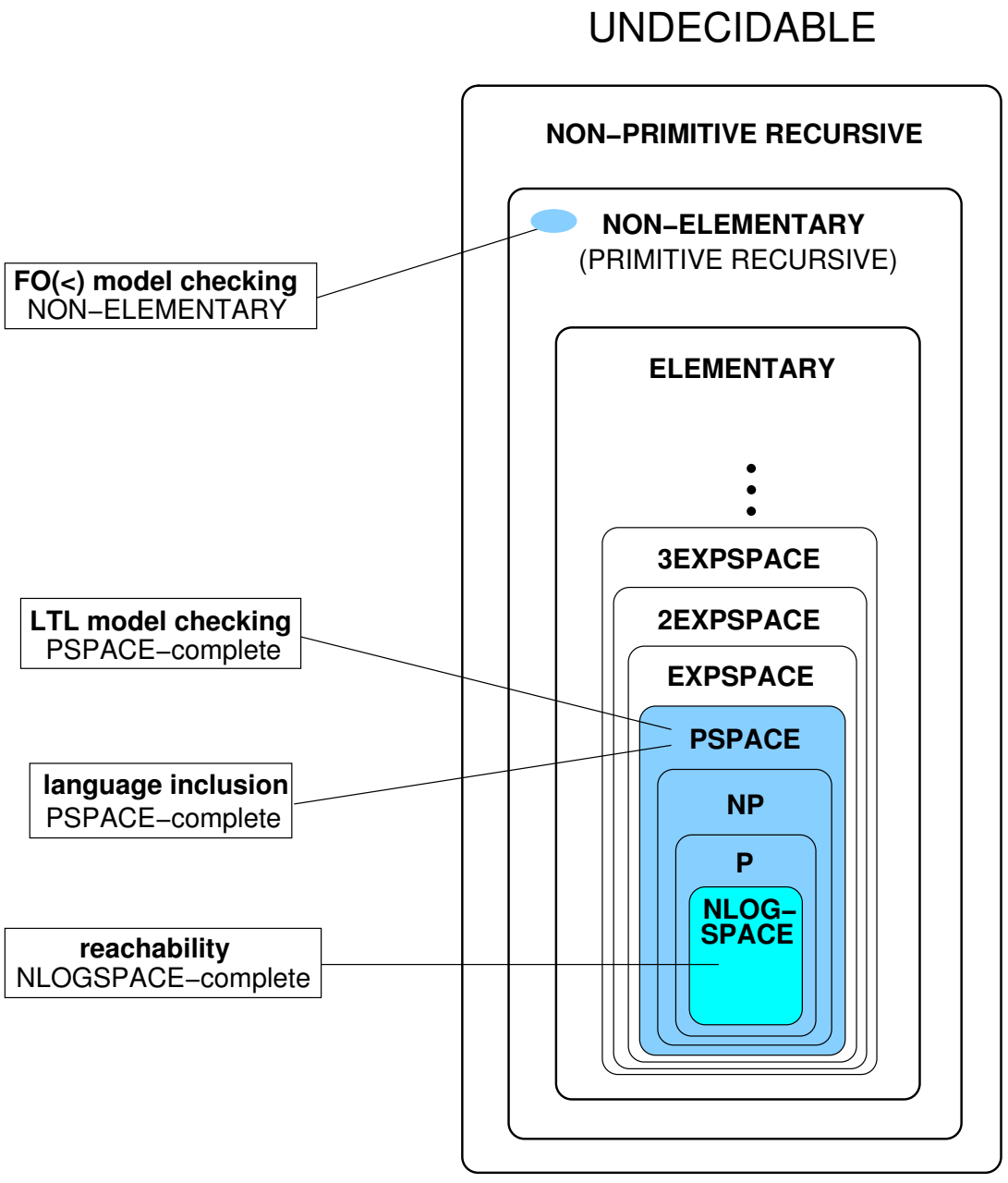

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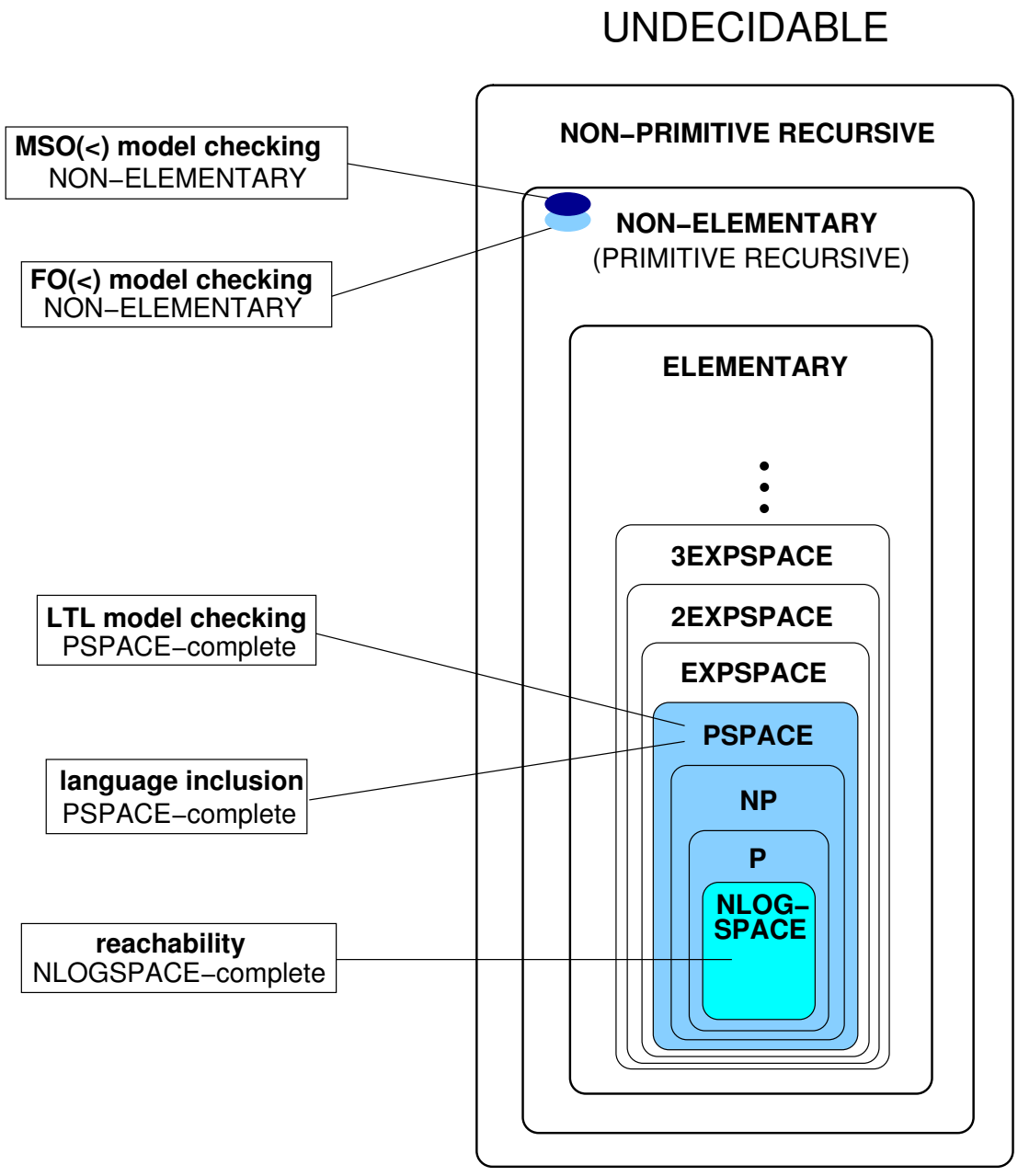

A Login Protocol

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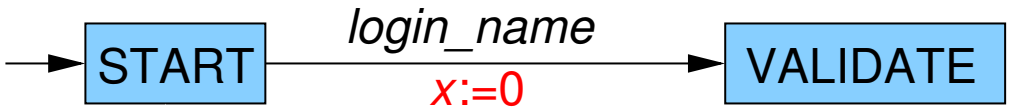

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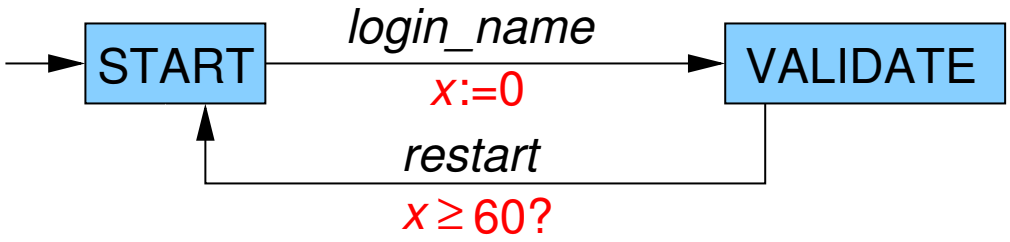

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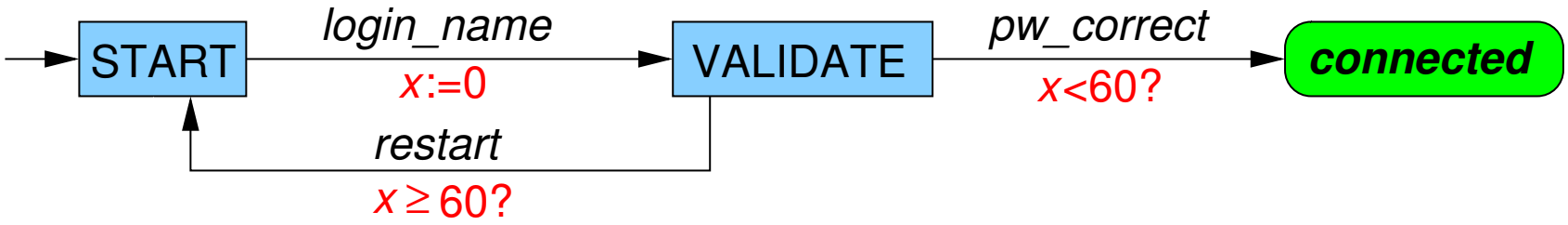

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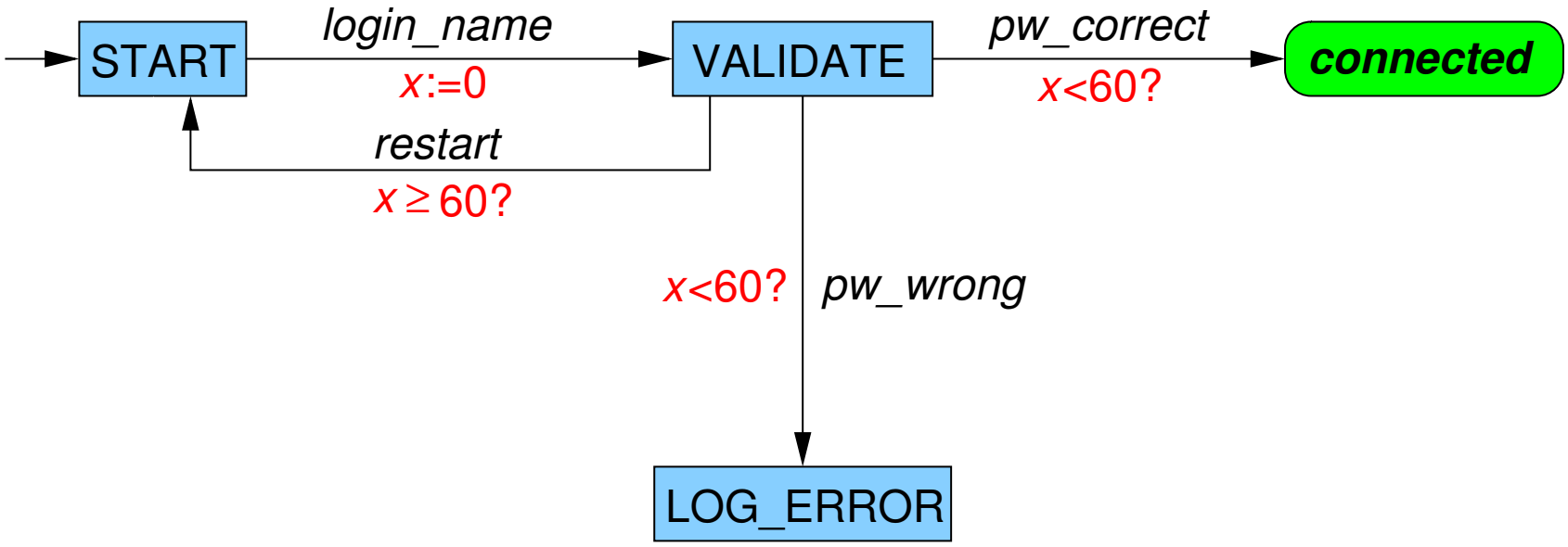

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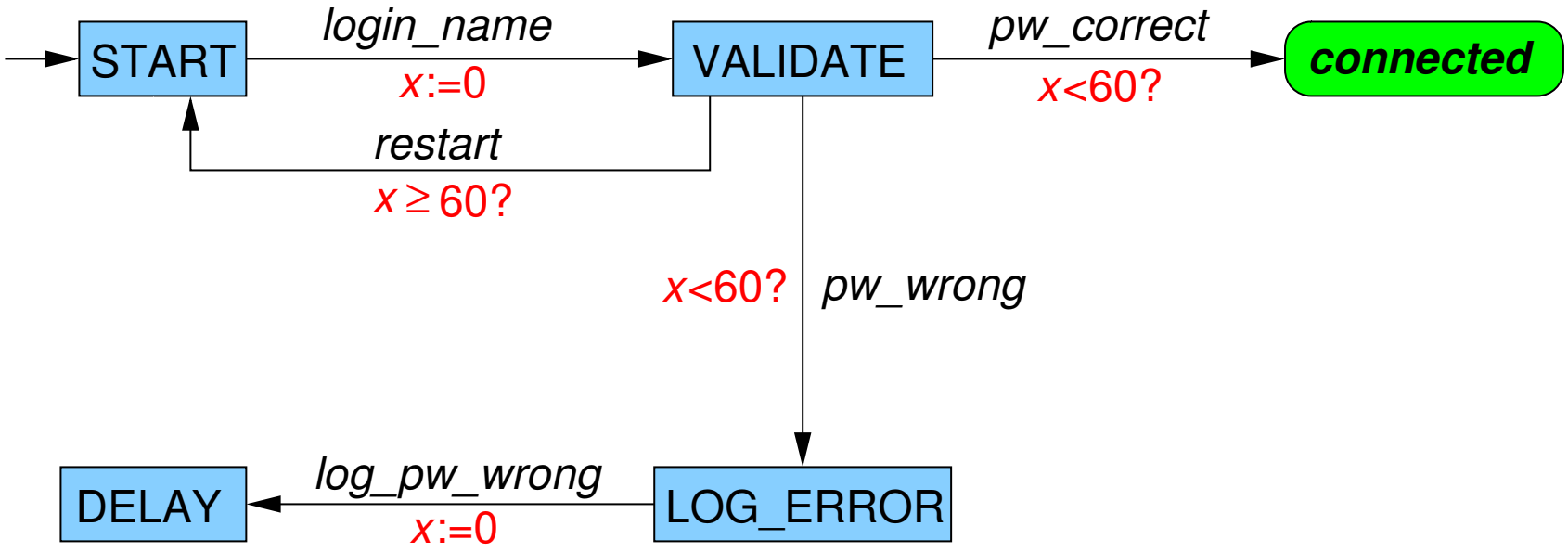

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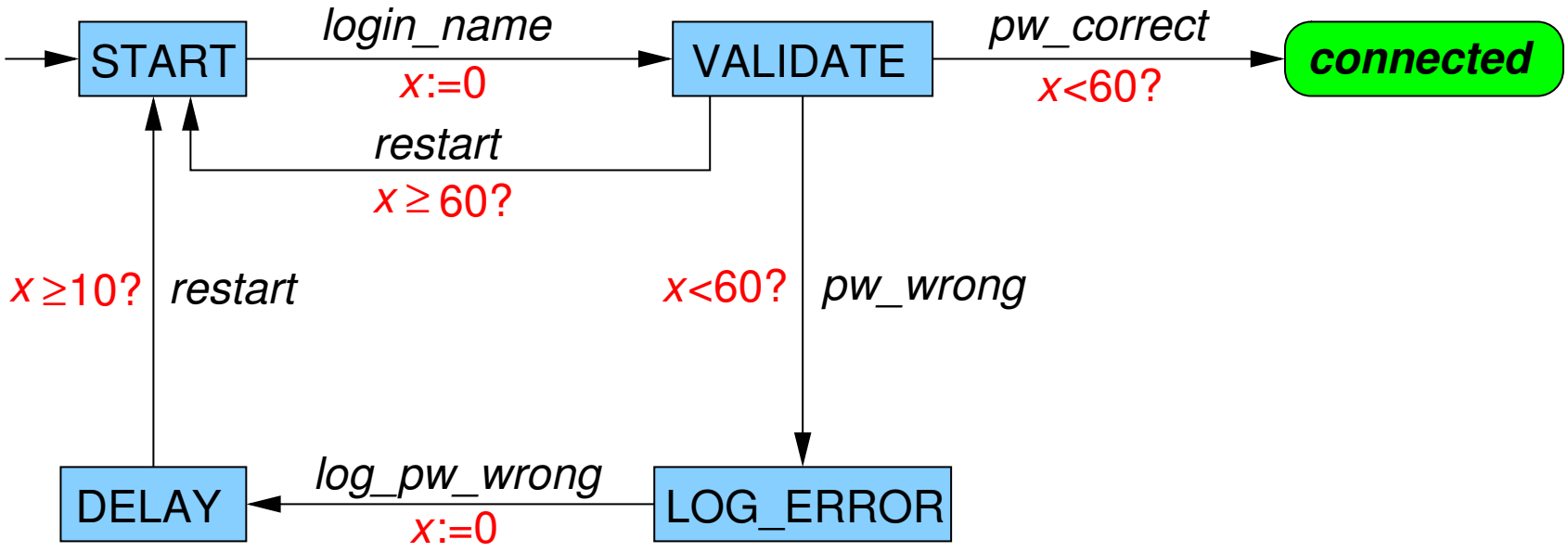

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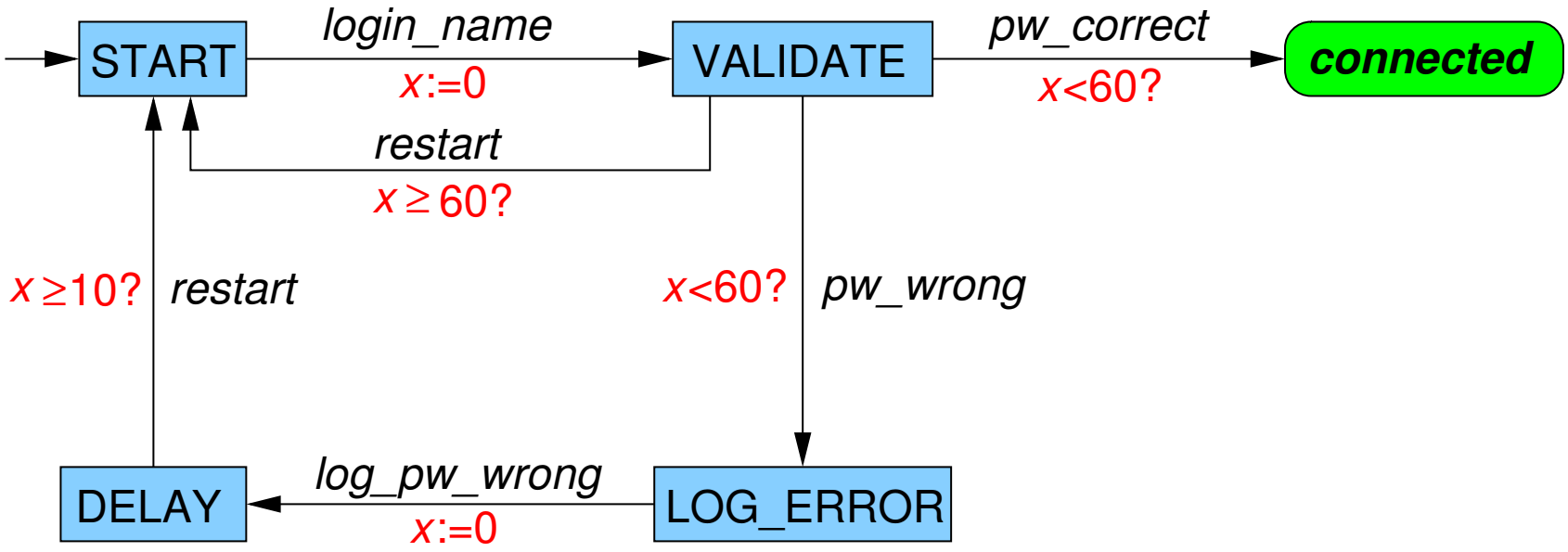
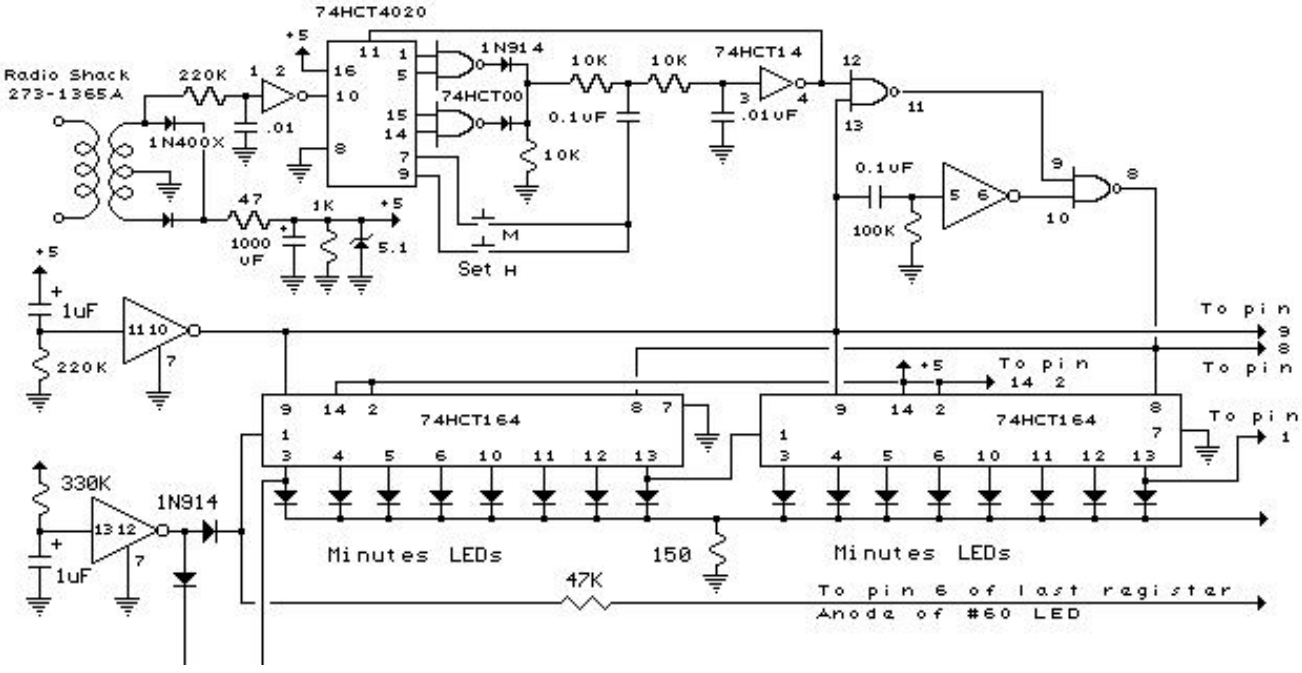
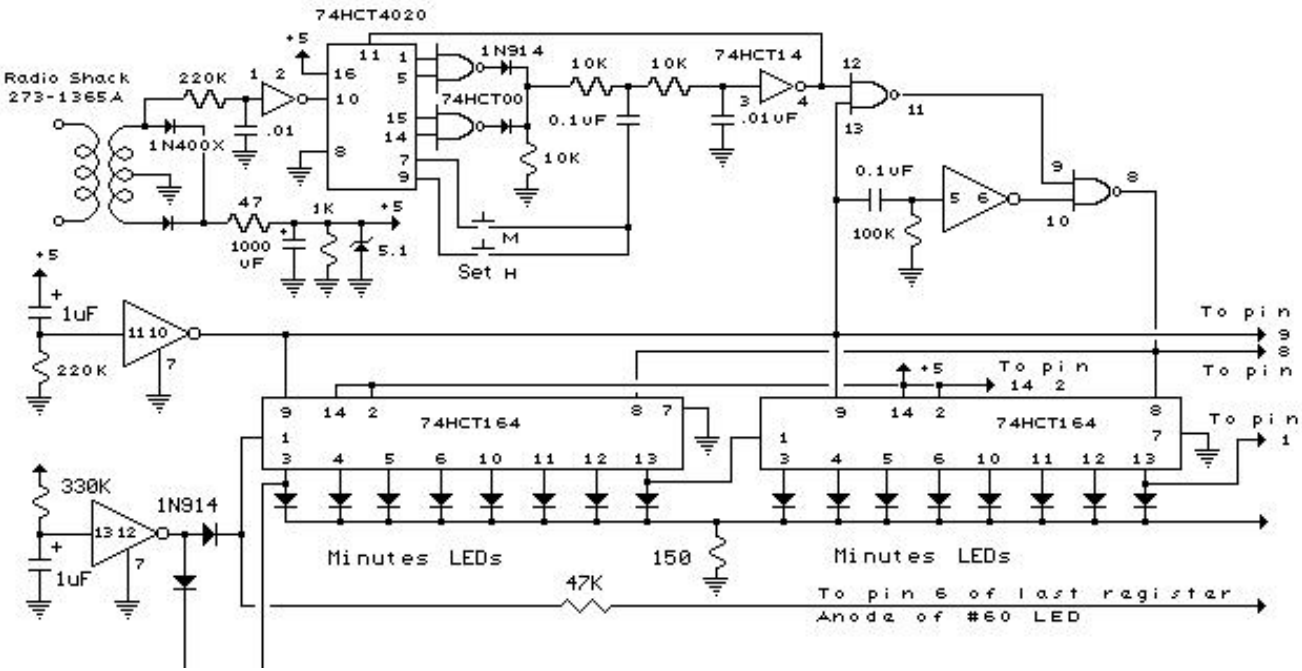


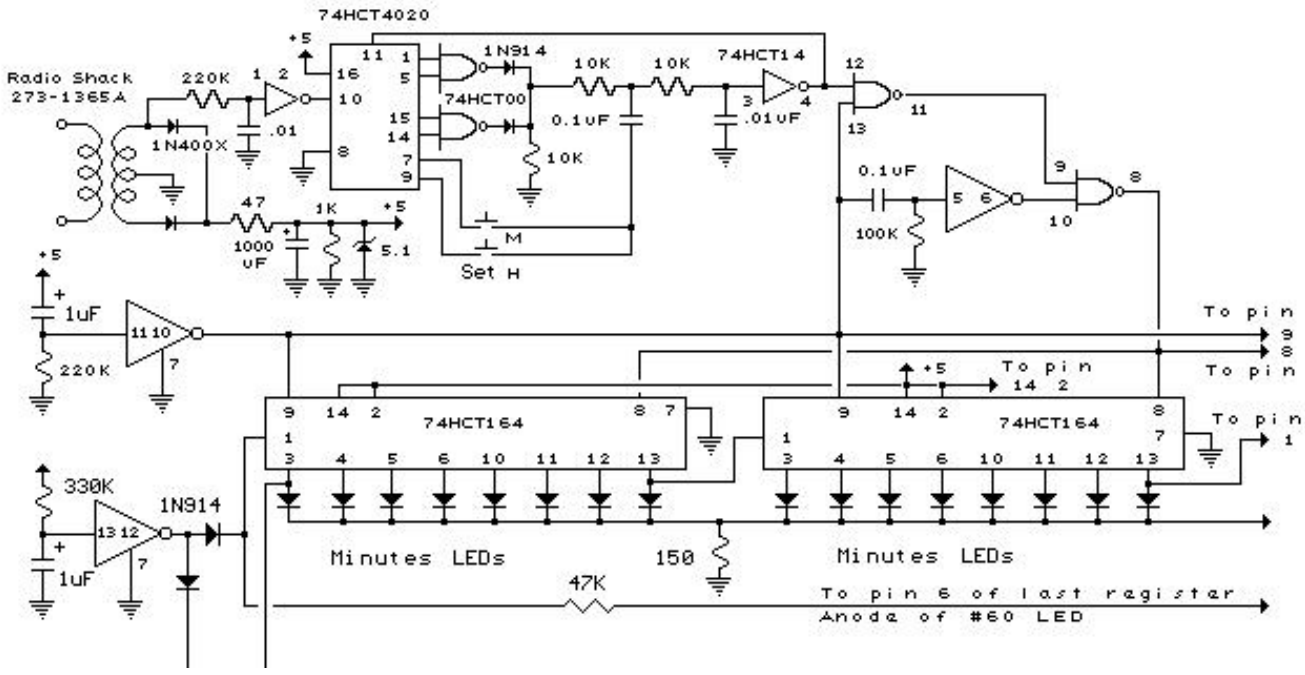
SPECIFICATION: $\Box(pw\_wrong \longrightarrow \Box_{[0,10)} \neg restart)$


74HCT4020
Radio Shack
273-1365A
220K
1N400X
.01
47
1000
uF
1K
5.1
+5
1N914
74HCT00
10K
0.1uF
74HCT14
.01uF
M
Set H
100K
1uF
220K
330K
74HCT164
Minutes LEDs
150
47K
To pin 9
To pin 8
To pin 14 2
To pin 1
To pin 6 of last register
Anode of #60 LED



[. . .] When power is applied, a single ‘1’ bit is loaded into the first stage of both the minutes and hours registers. To accomplish this, a momentary low reset signal is sent to all the registers (at pin 9) and also a NAND gate to lock out any clock transitions at pin 8 of the minutes registers. At the same time, a high level is applied to the data input lines of both minutes and hours registers at pin 1. A single positive going clock pulse is generated at the end of the reset signal which loads a high level into the first stage of the minutes register. The rising edge of first stage output at pin 3 advances the hours and a single bit is also loaded into the hours register. Power should remain off for 3 seconds before being re-applied to allow the filter and timing capacitors to discharge. [. . .]

(Bill Bowden, `www.circuitdb.com/circuits/id/98`)



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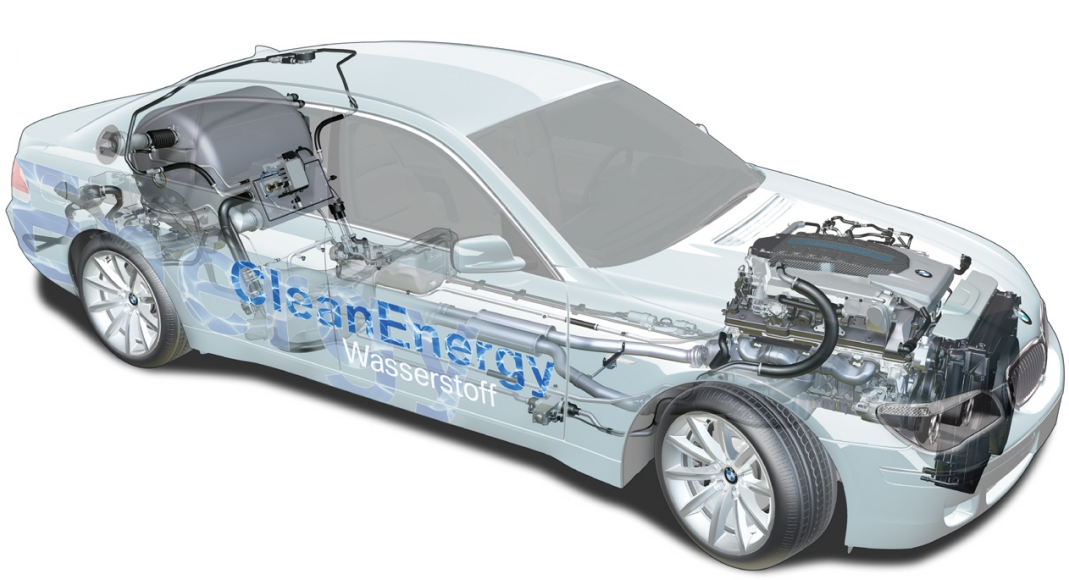
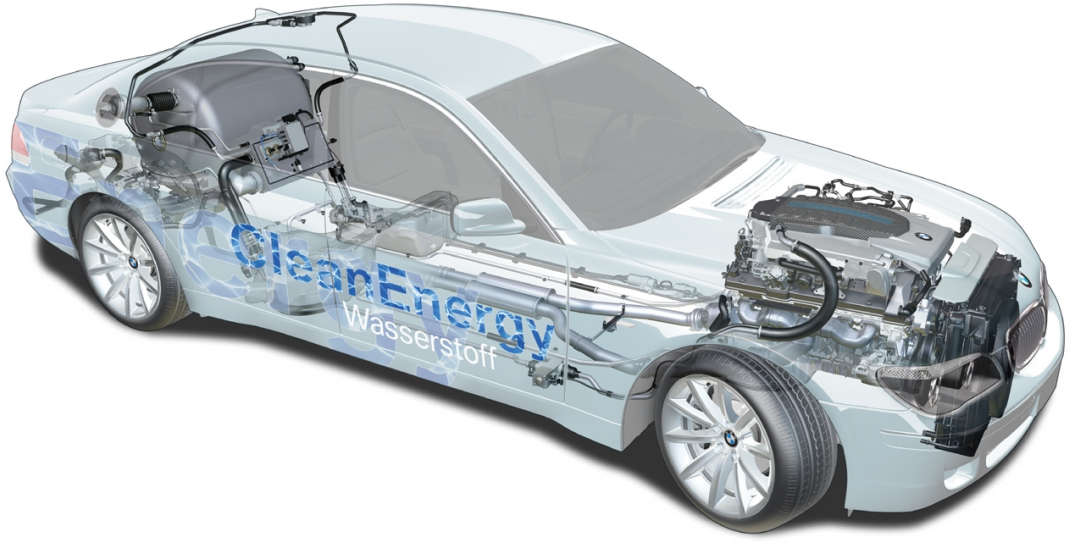
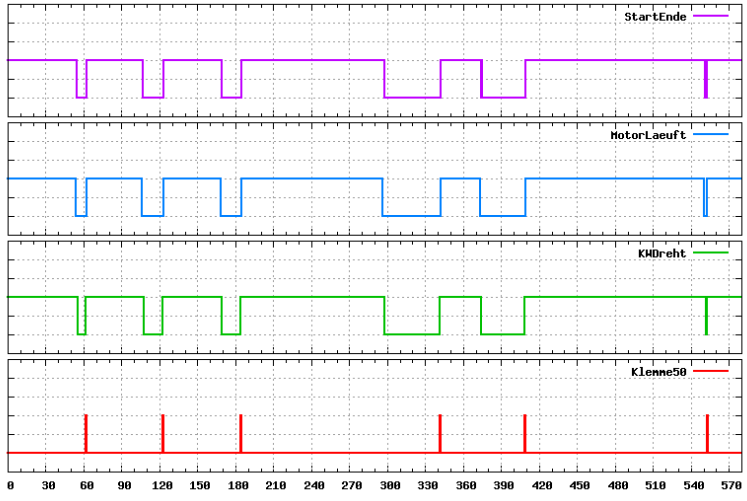
BMW Hydrogen 7



BMW Hydrogen 7





Timed Systems

Timed systems occur in:

- Hardware circuits
- Communication protocols
- Cell phones
- Plant controllers
- Aircraft navigation systems
- . . .

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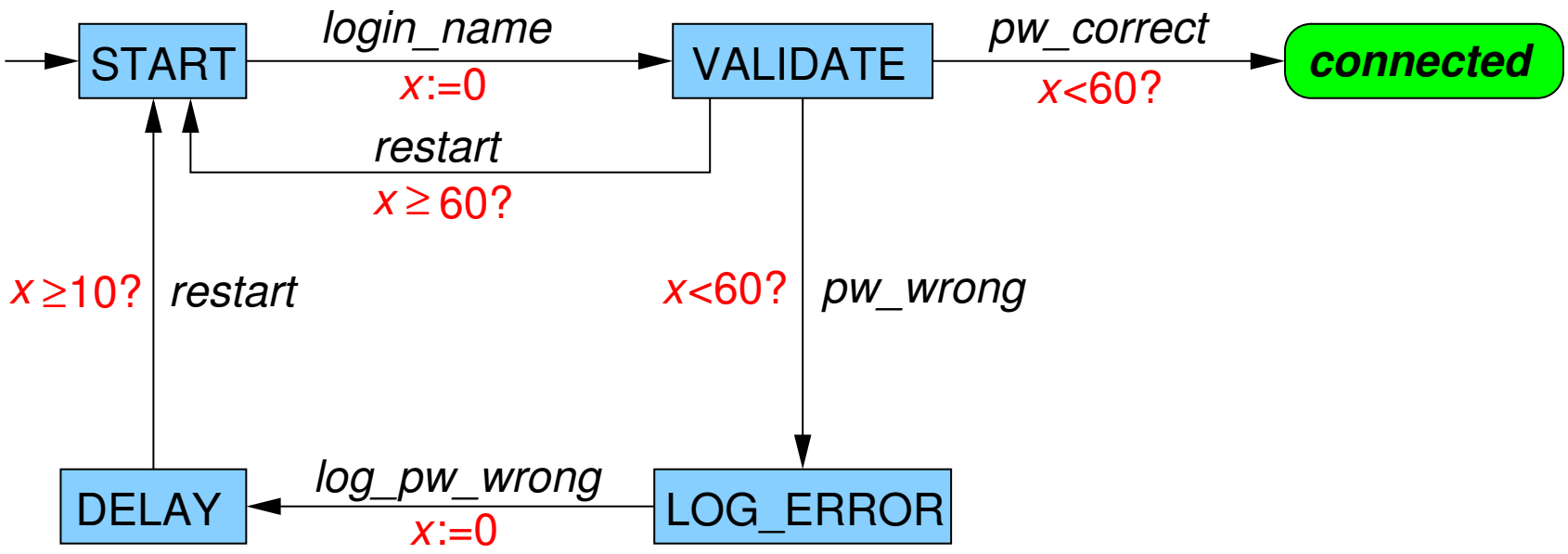
In many instances, it is **crucial** to accurately model the timed behaviour of the system.

From Qualitative to Quantitative

“Lift the classical theory to the real-time world.”

Boris Trakhtenbrot, LICS 1995

Timed Automata



Timed Automata

Timed automata were introduced by Rajeev Alur at Stanford during his PhD thesis under David Dill:

- Rajeev Alur, David L. Dill: *Automata For Modeling Real-Time Systems*. ICALP 1990: 322-335
- Rajeev Alur, David L. Dill: *A Theory of Timed Automata*. TCS 126(2): 183-235, 1994

Timed Automata

Time is modelled as the non-negative reals, $\mathbb{R}_{\geq 0}$.

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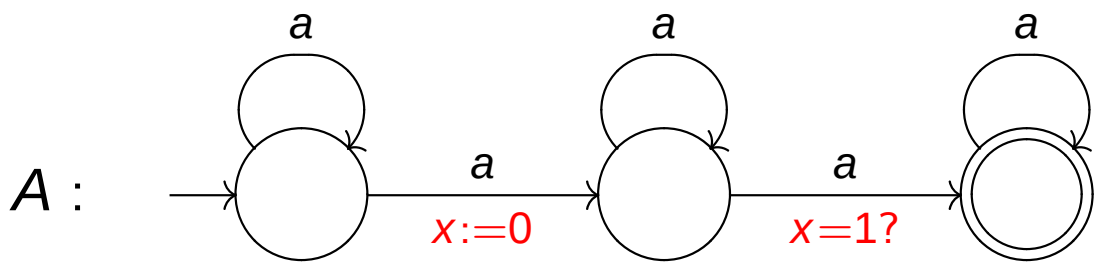
Unfortunately:

Theorem (Alur & Dill 1990)

Language inclusion is undecidable for timed automata.

An Uncomplementable Timed Automaton



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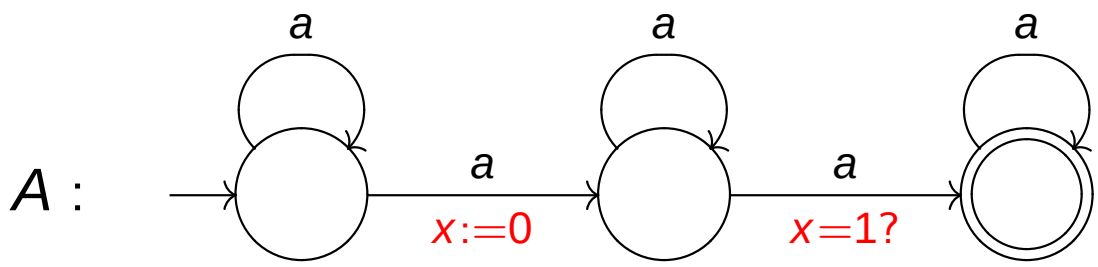

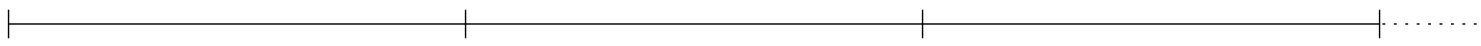

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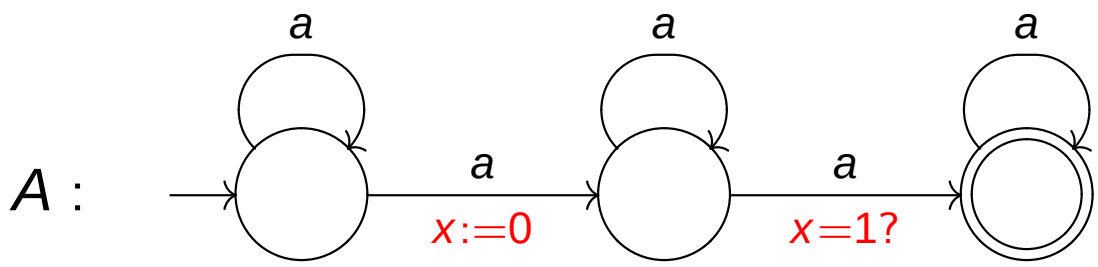

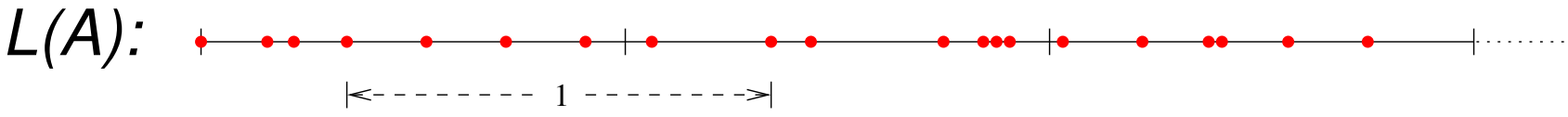

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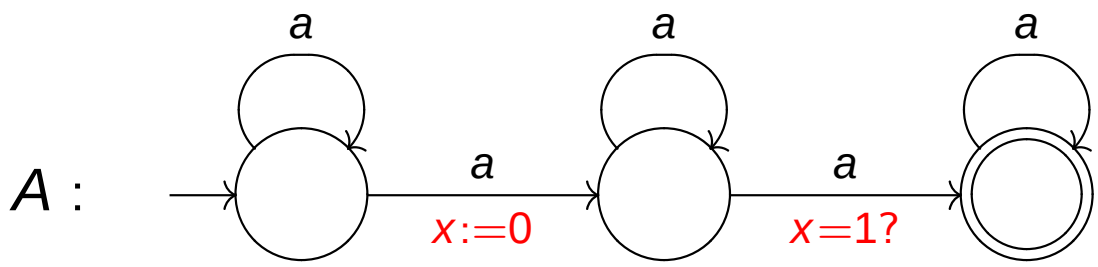

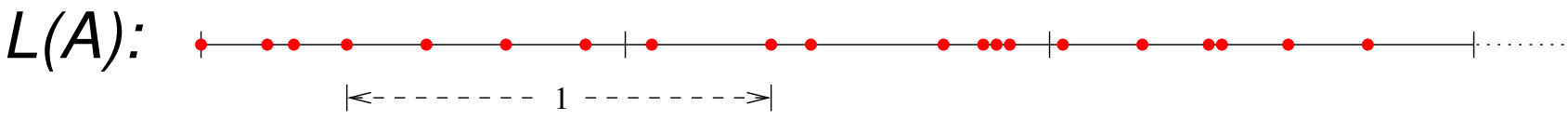

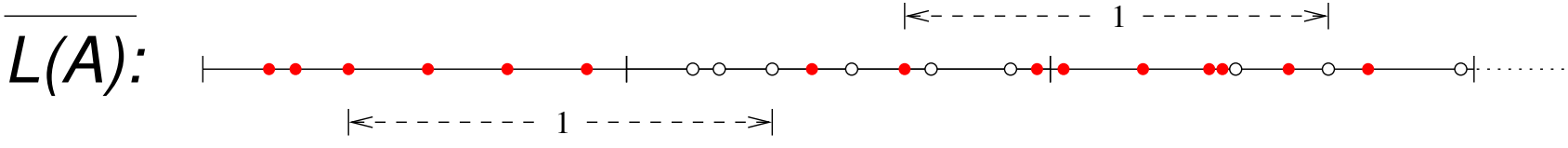

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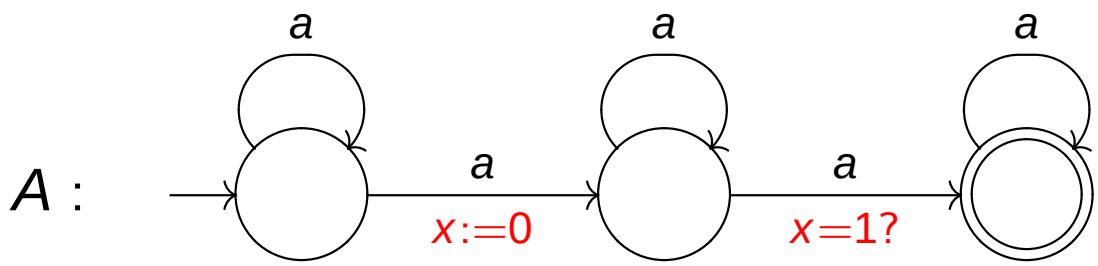

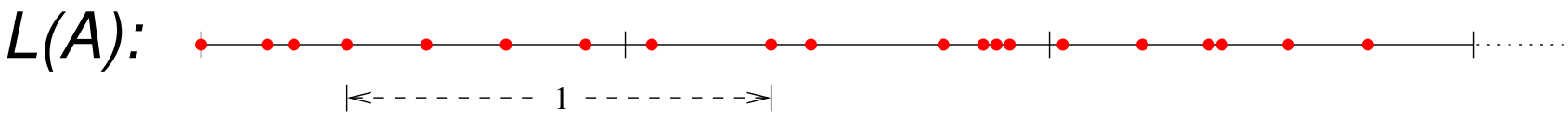

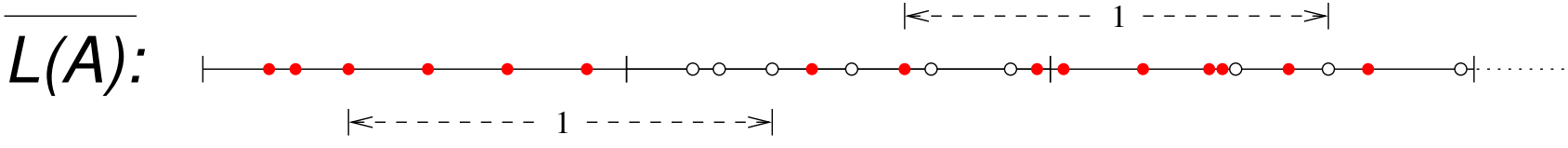

A cannot be complemented:
There is no timed automaton B with $L(B) = \overline{L(A)}$.

Metric Temporal Logic

Metric Temporal Logic (MTL) [Koymans; de Roever; Pnueli $\sim$1990] is a central quantitative specification formalism for timed systems.

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$$\forall x\,(\mathit{PEDAL}(x) \rightarrow \exists y\,(x + 5 \leq y \leq x + 10 \wedge \mathit{BRAKE}(y)))$$

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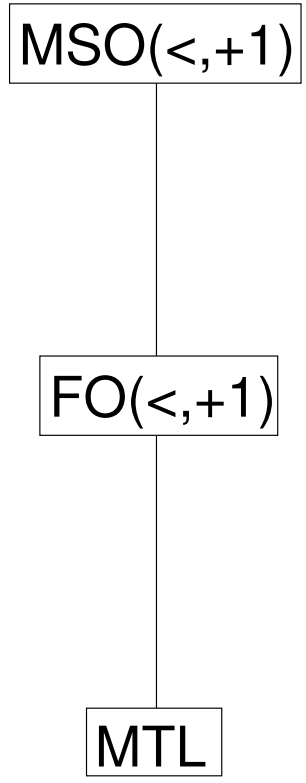
Corollary: FO($<$,$+1$) and MSO($<$,$+1$) satisfiability and model checking are undecidable over $\mathbb{R}_{\geq 0}$.

The Real-Time Theory: Expressiveness



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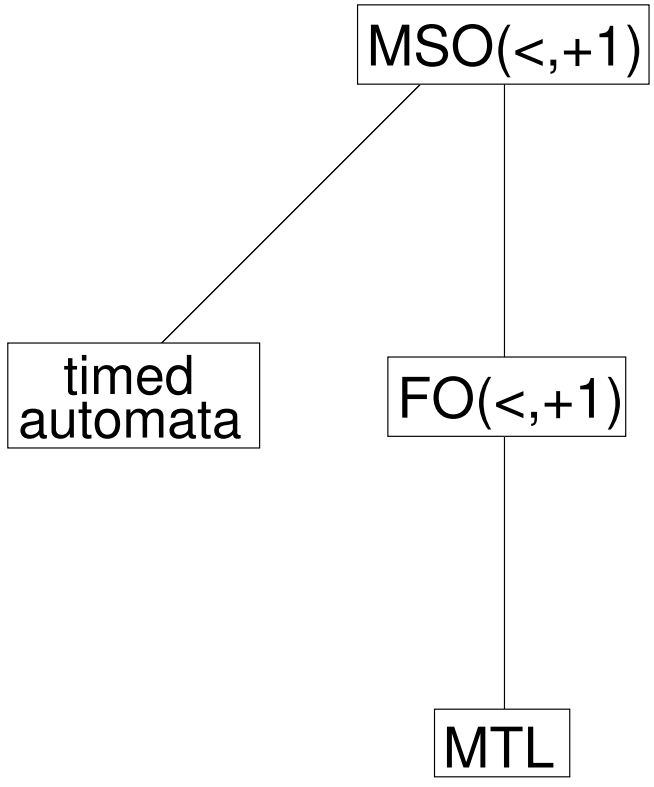
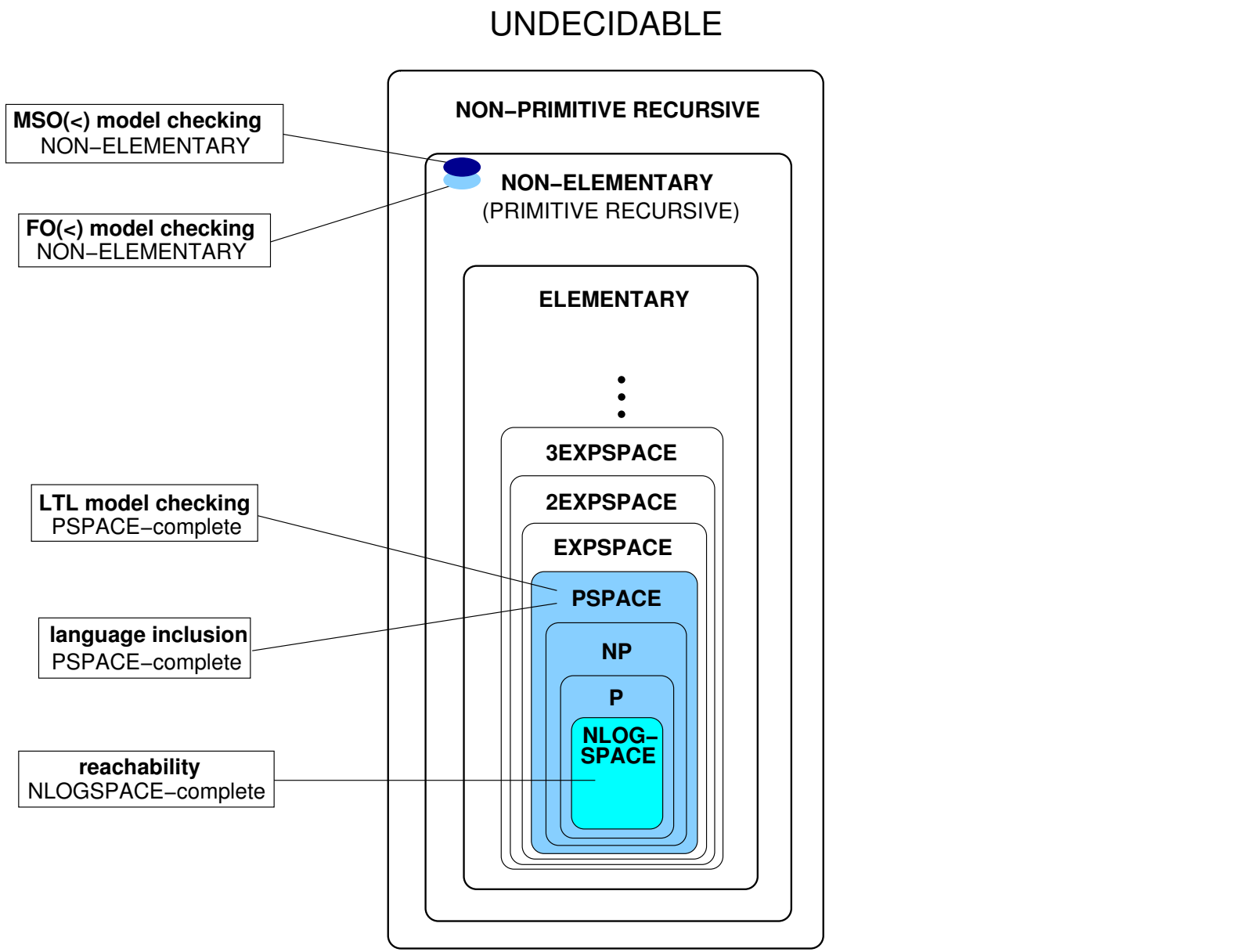


The Real-Time Theory: Complexity

Classical Theory

Real–Time Theory



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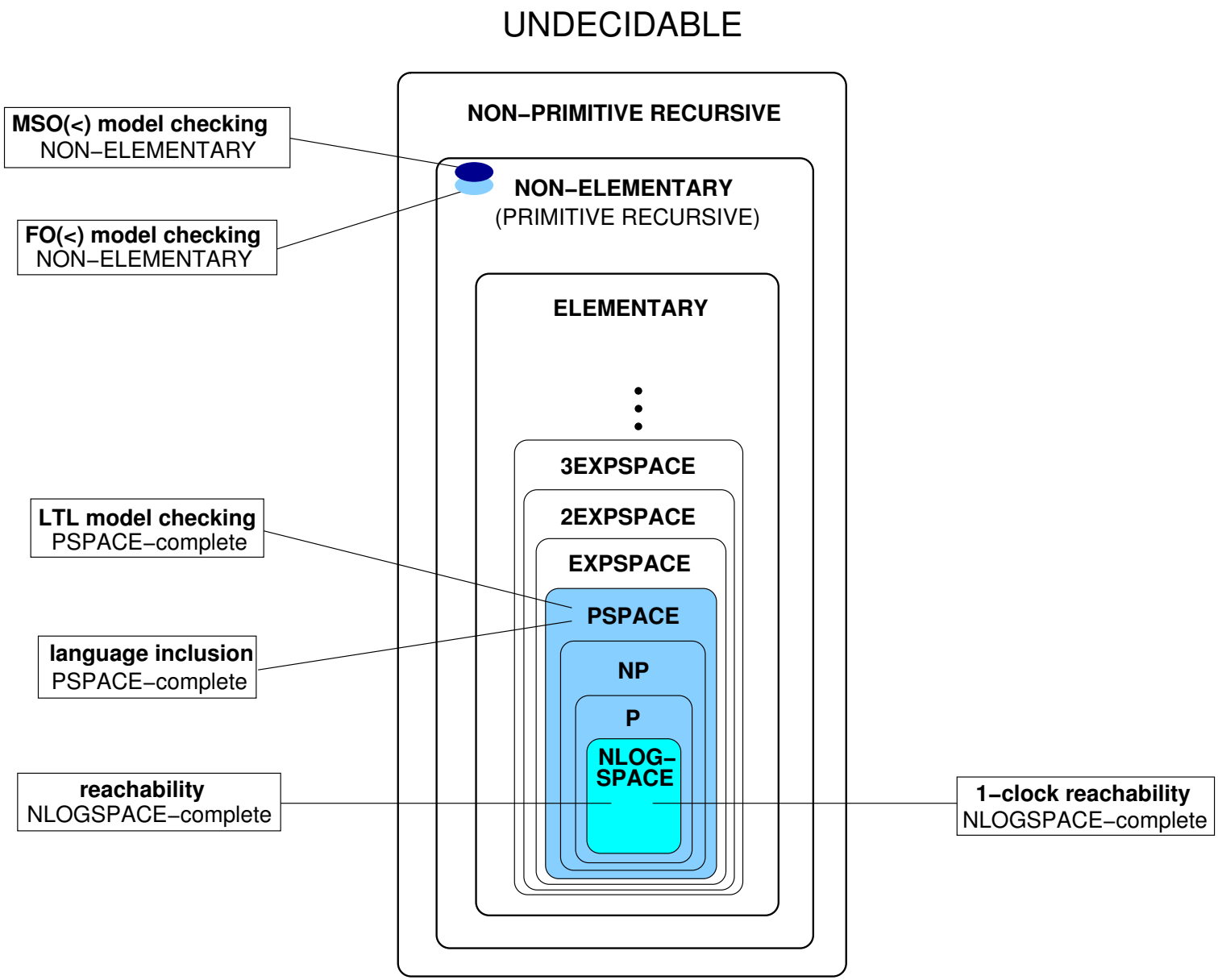
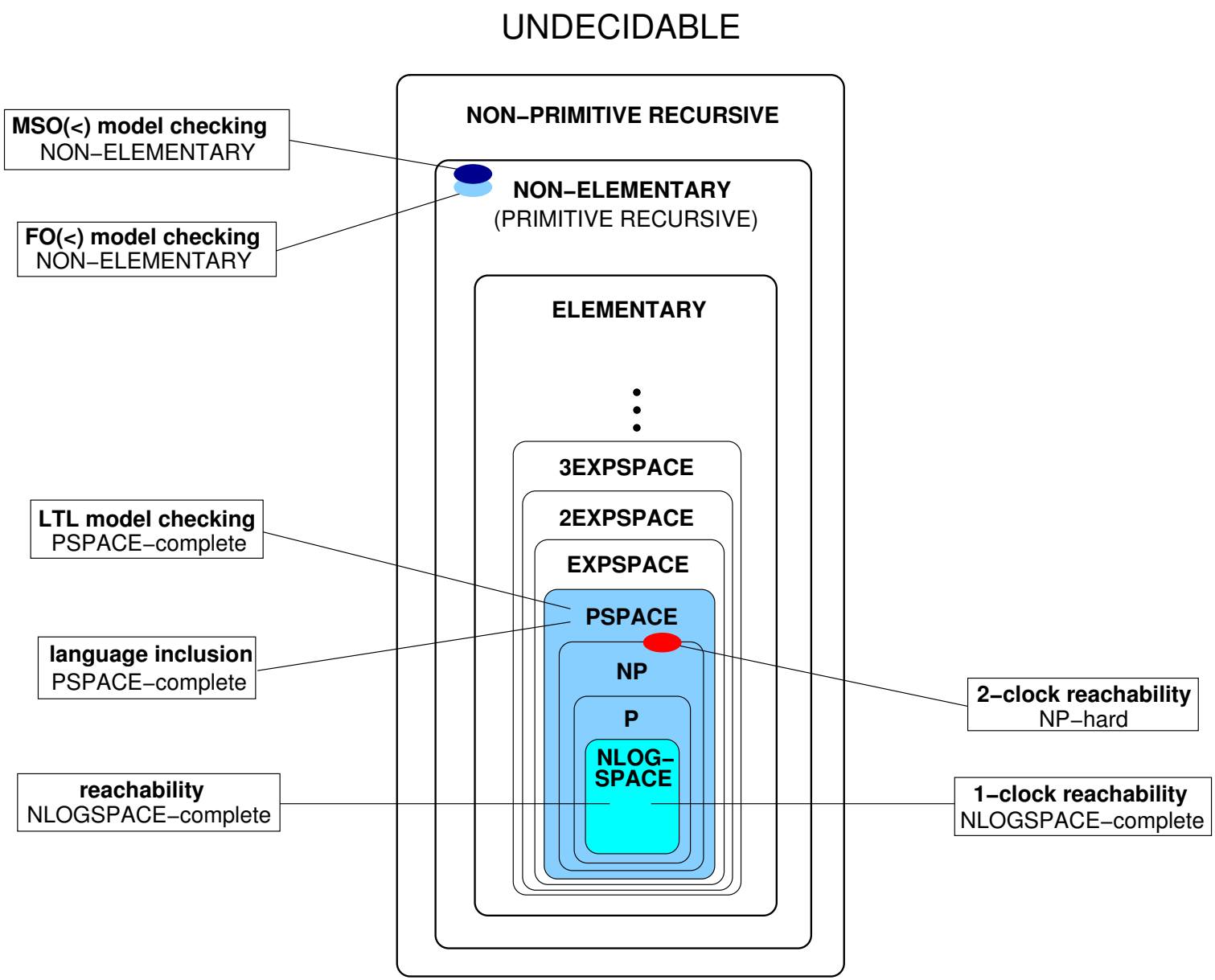
Classical Theory **Real–Time Theory**



The Real-Time Theory: Complexity

Classical Theory

Real–Time Theory



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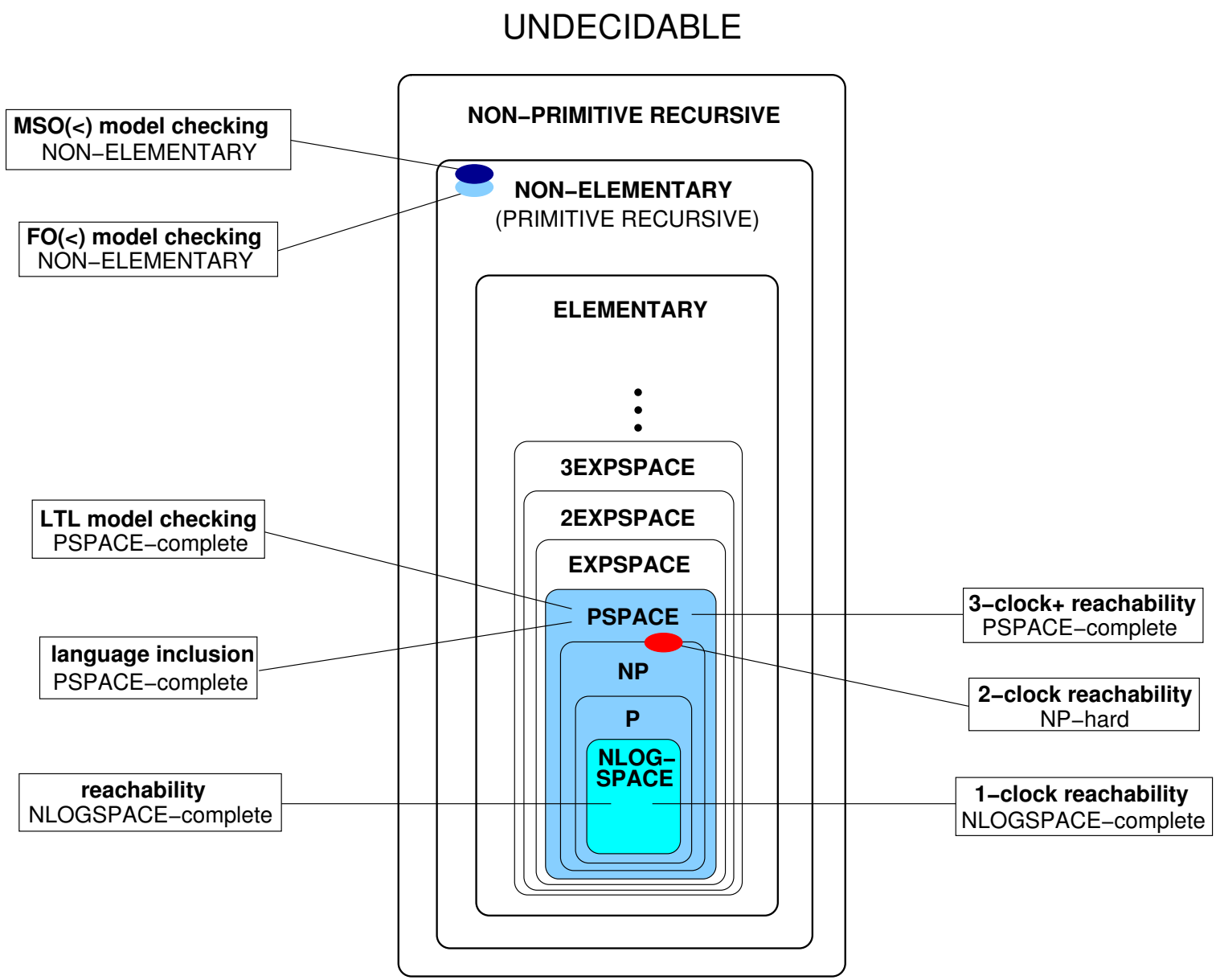
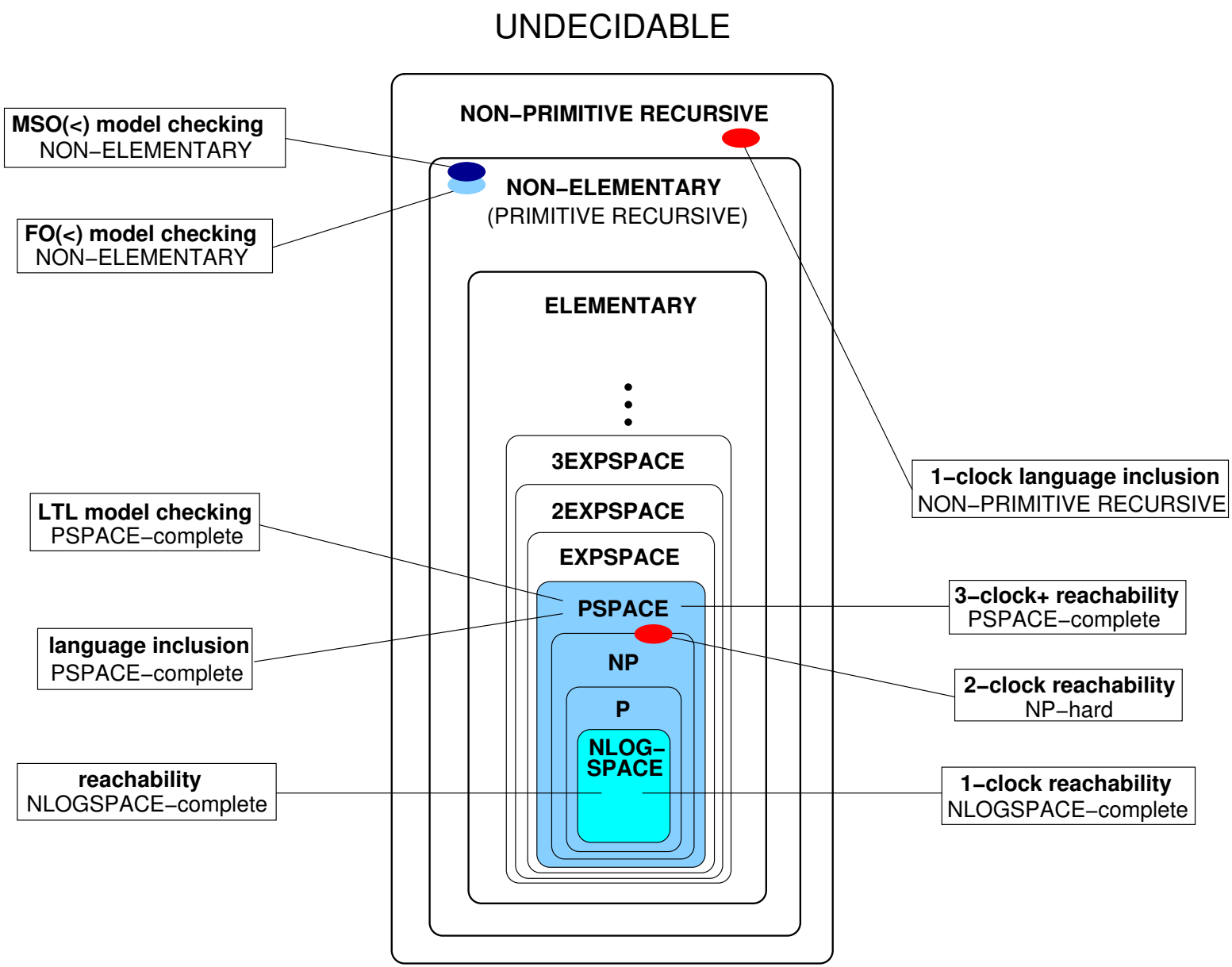


The Real-Time Theory: Complexity

Classical Theory

Real–Time Theory



The Real-Time Theory: Complexity

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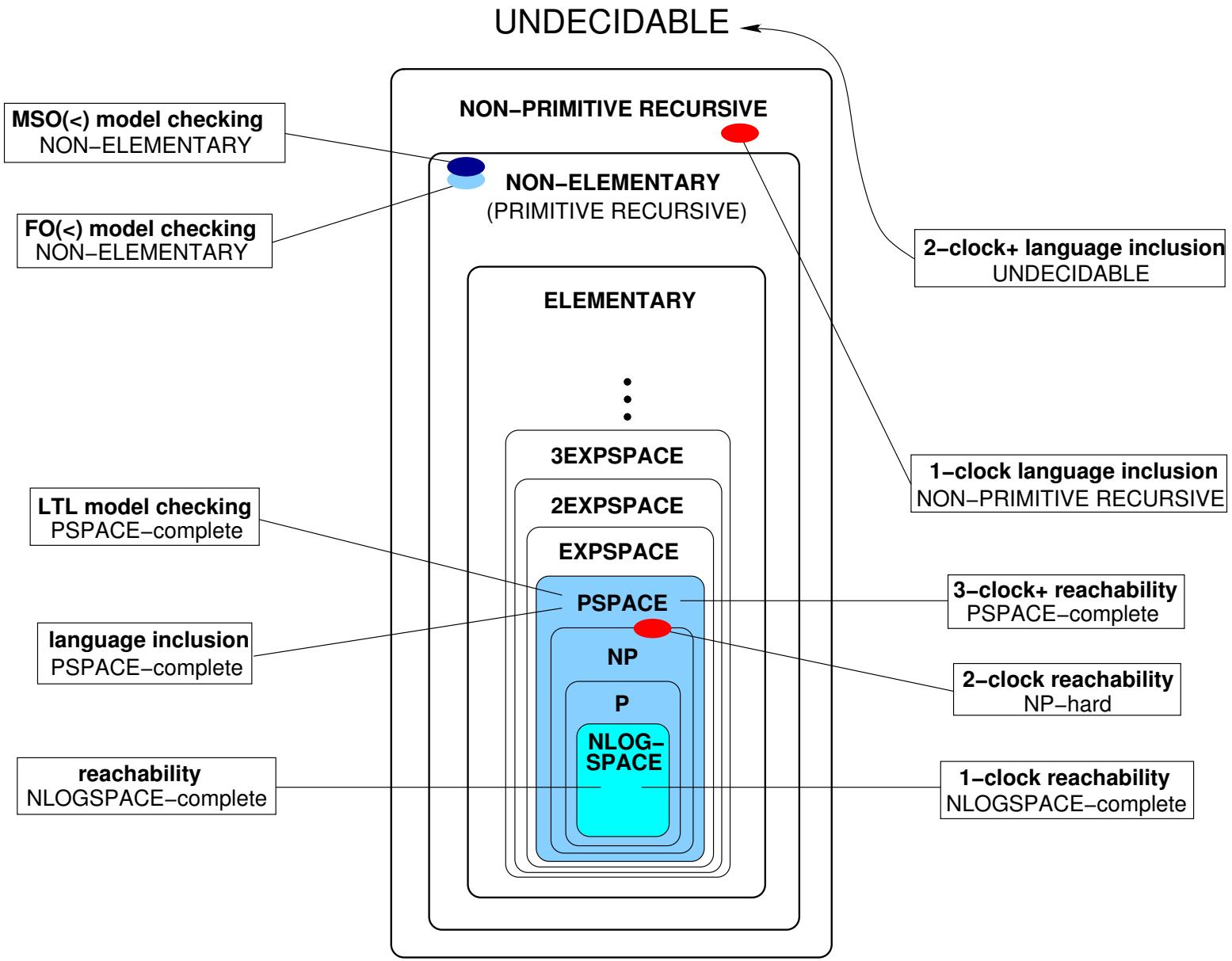

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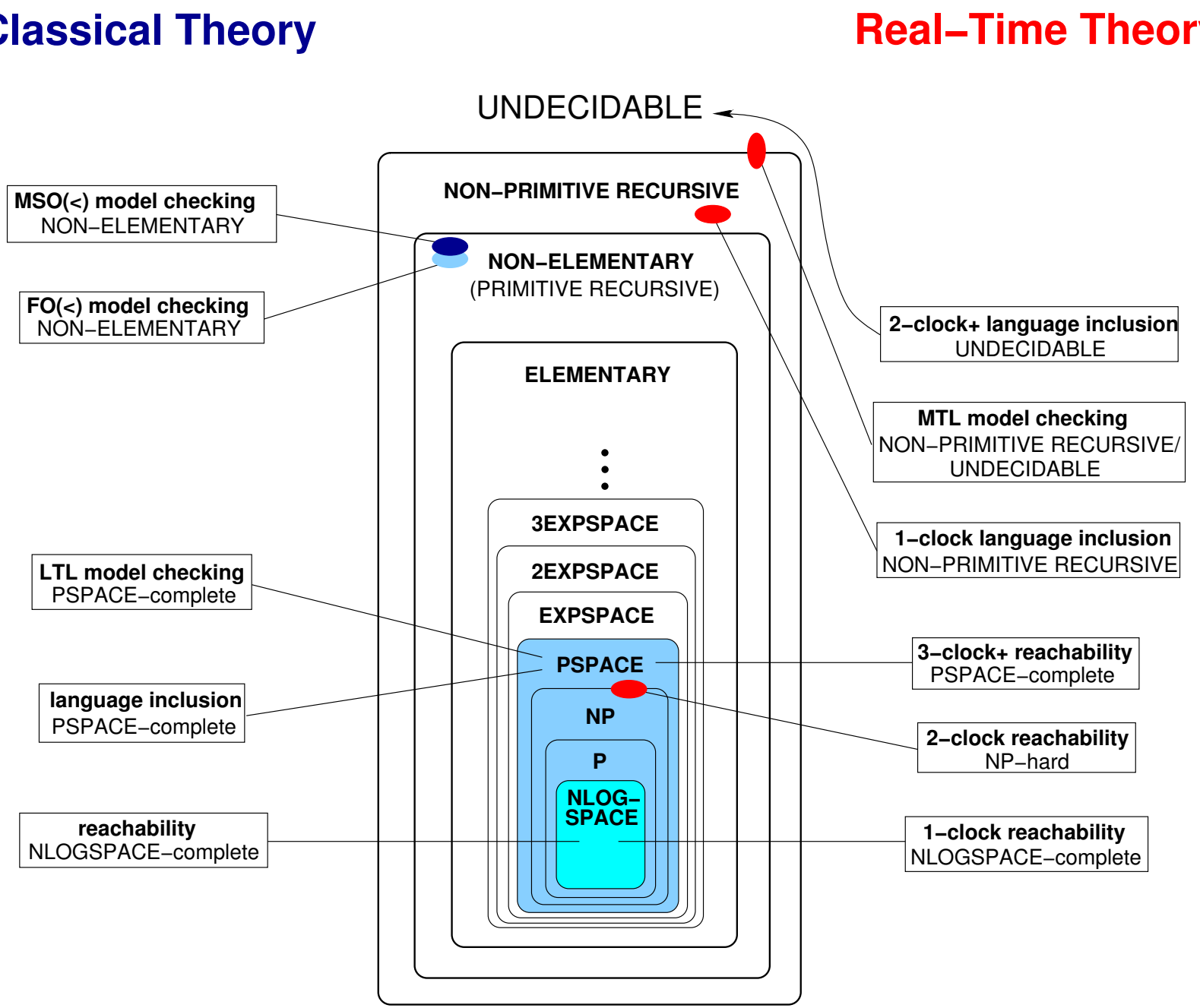

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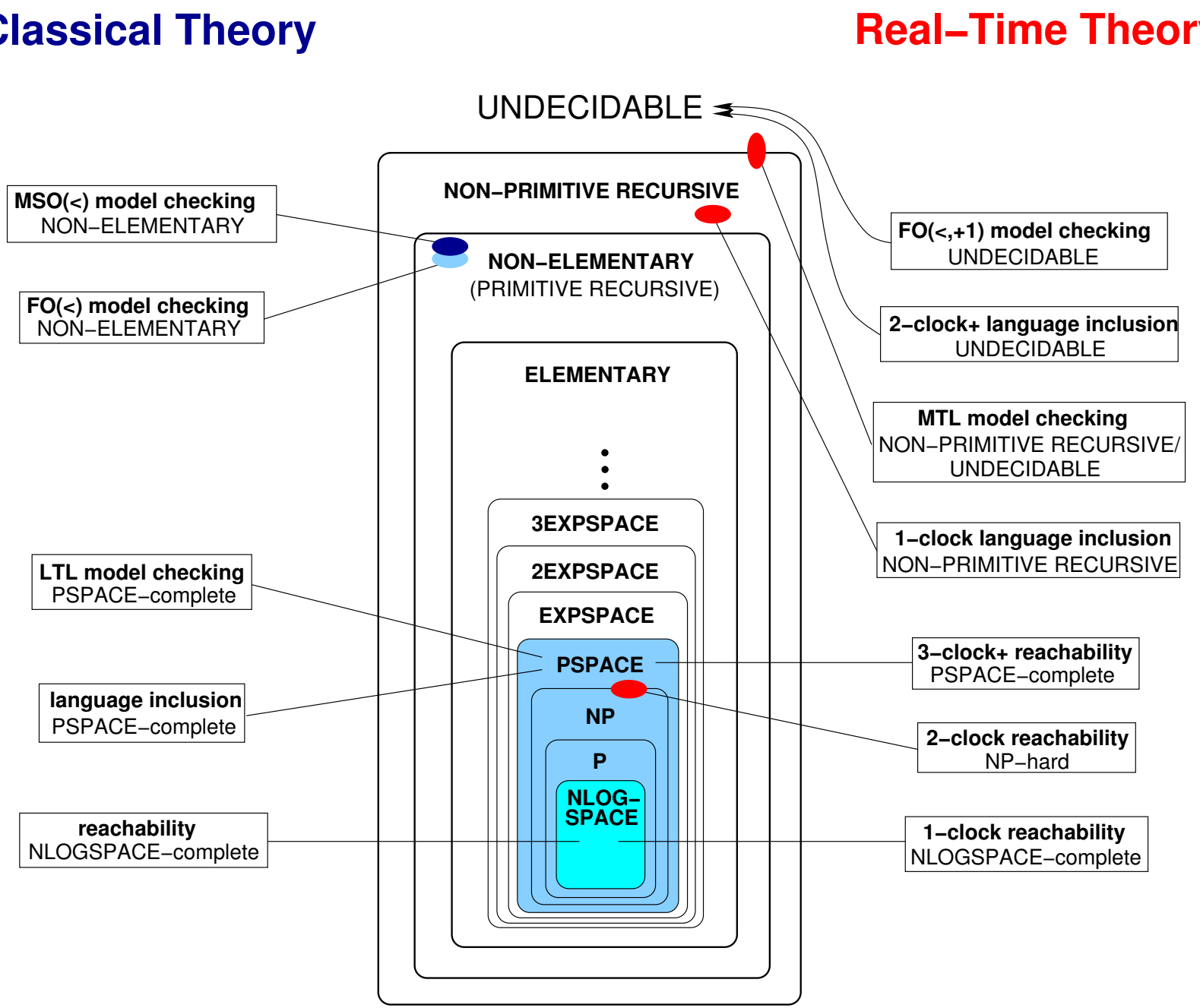

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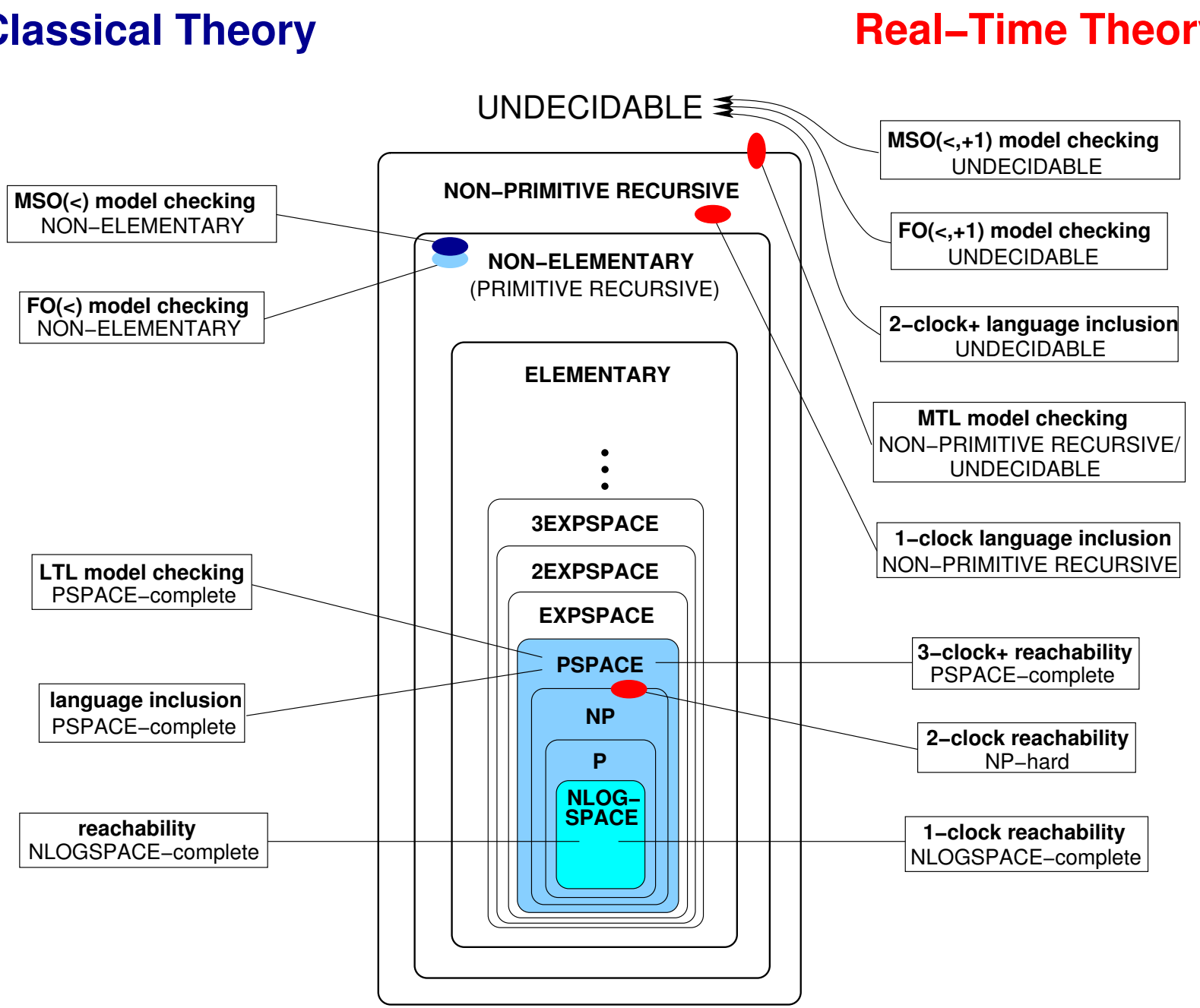

Key Stumbling Block

Theorem (Alur & Dill 1990)

Language inclusion is undecidable for timed automata.

Timed Language Inclusion: Some Related Work

- Topological restrictions and digitization techniques:
  [Henzinger, Manna, Pnueli 1992], [Bošnački 1999],
  [Ouaknine & Worrell 2003]
- Fuzzy semantics / noise-based techniques:
  [Maass & Orponen 1996],
  [Gupta, Henzinger, Jagadeesan 1997],
  [Fränzle 1999], [Henzinger & Raskin 2000], [Puri 2000],
  [Asarin & Bouajjani 2001], [Ouaknine & Worrell 2003],
  [Alur, La Torre, Madhusudan 2005]
- Determinisable subclasses of timed automata:
  [Alur & Henzinger 1992], [Alur, Fix, Henzinger 1994],
  [Wilke 1996], [Raskin 1999]
- Timed simulation relations and homomorphisms:
  [Lynch *et al.* 1992], [Taşiran *et al.* 1996],
  [Kaynar, Lynch, Segala, Vaandrager 2003]
- Restrictions on the number of clocks:
  [Ouaknine & Worrell 2004], [Emmi & Majumdar 2006]

A Long Time Ago, circa 2003. . .

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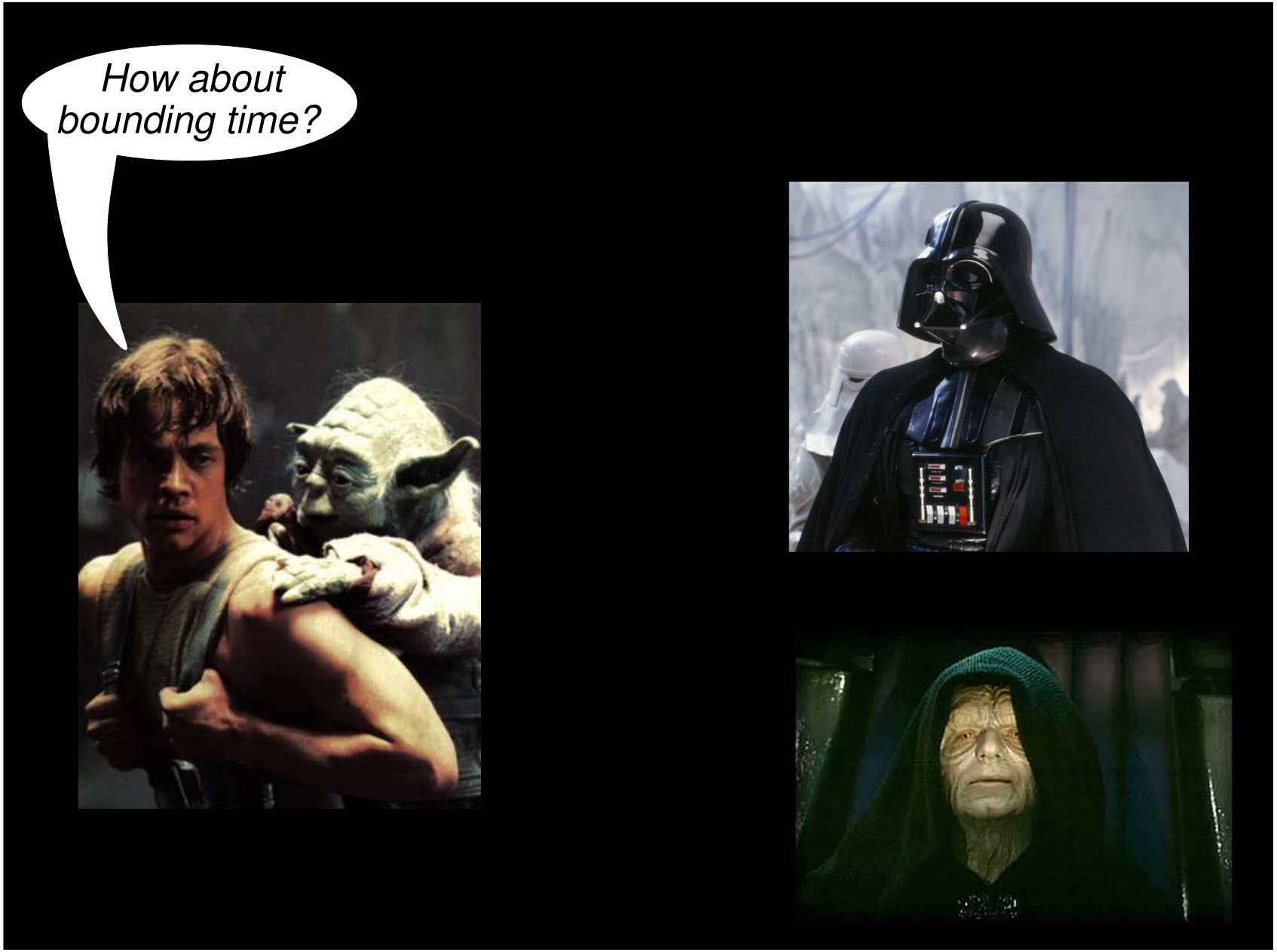

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How about bounding time?

Timed automata still cannot be complemented!

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Time-Bounded Language Inclusion

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Instance: Timed automata A, B, and time bound $T \in \mathbb{N}$

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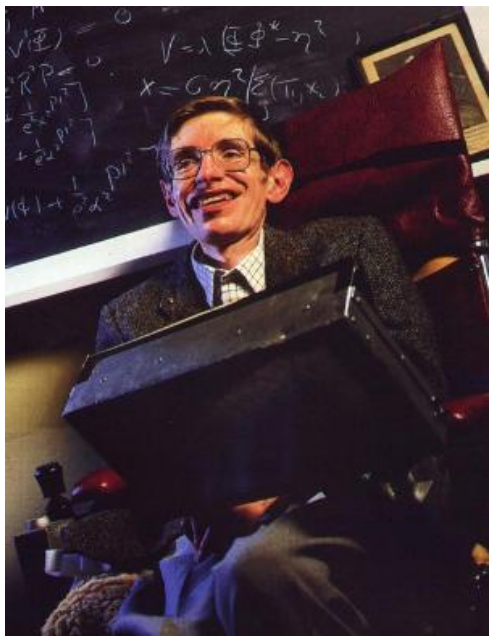
- Inspired by Bounded Model Checking.
- Timed systems often have time bounds (e.g. timeouts), even if total number of actions is potentially unbounded.
- Universe's lifetime is believed to be bounded anyway...



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- This reverses Vardi's 'automata-theoretic approach to verification' paradigm!

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By contrast,

Theorem

- *MSO($<$) is decidable over* $\mathbb{N}$ *[Büchi 1960]*
- *MSO($<$) is decidable over* $\mathbb{Q}$*, via [Rabin 1969]*

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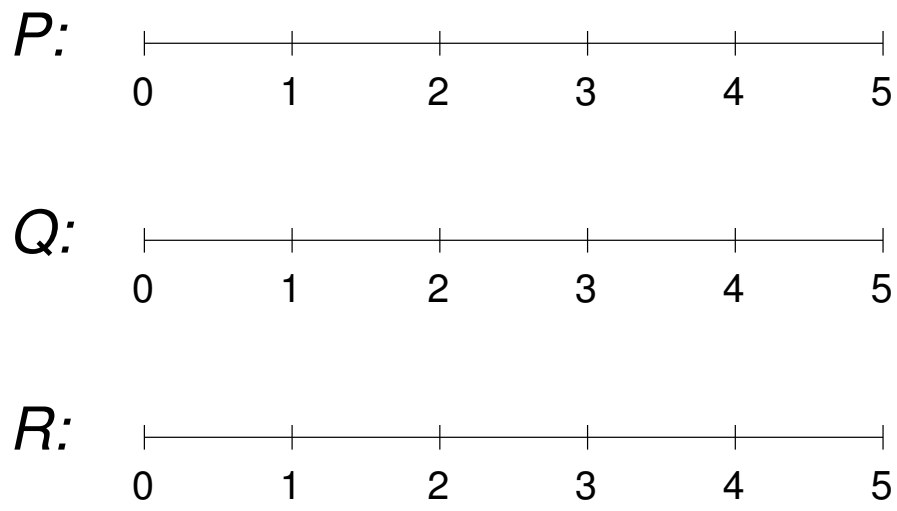

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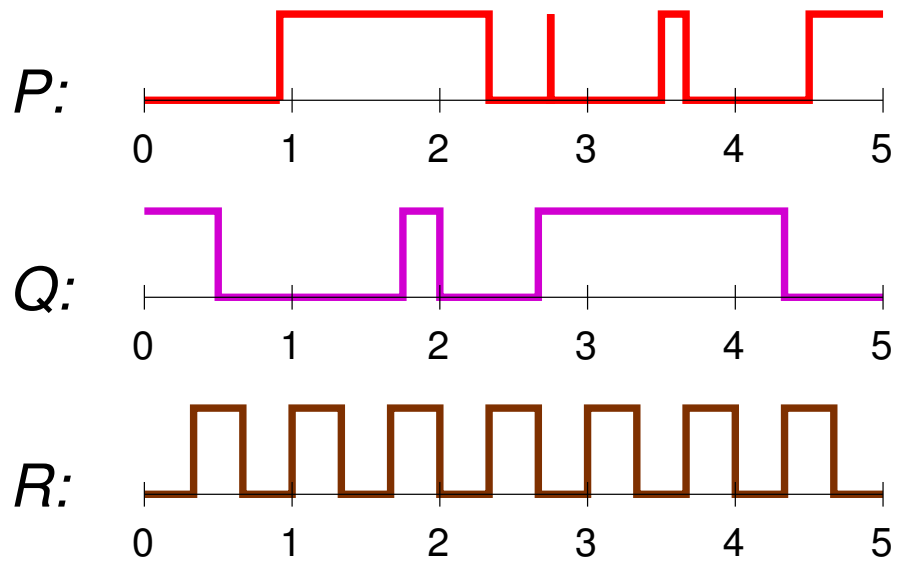

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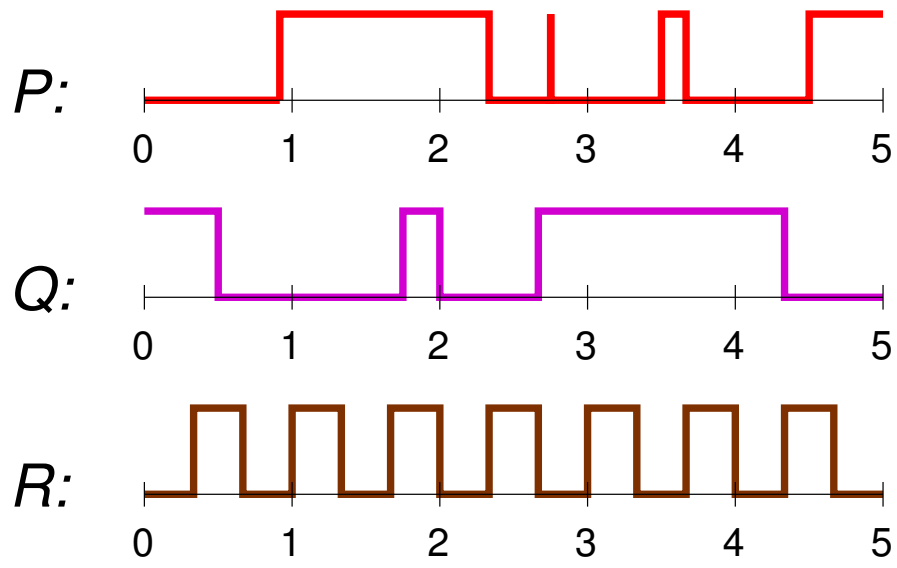

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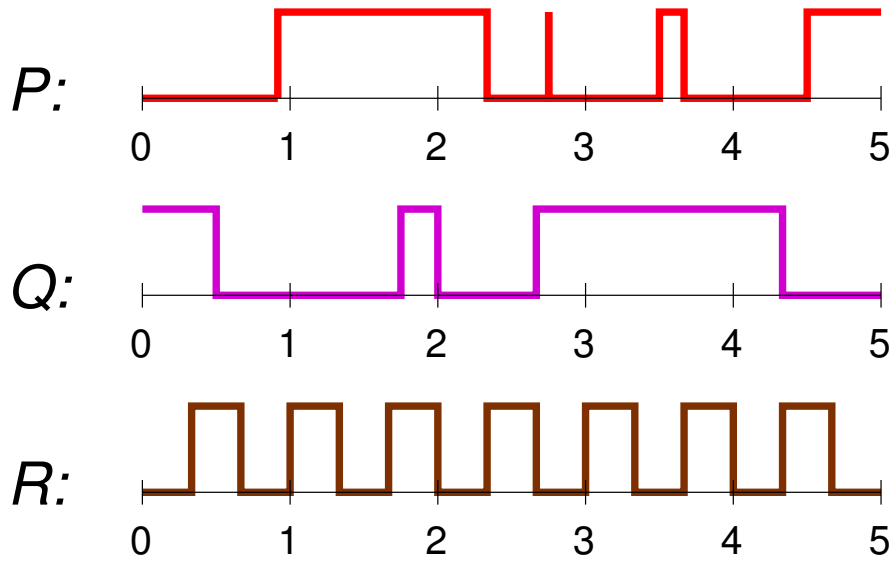

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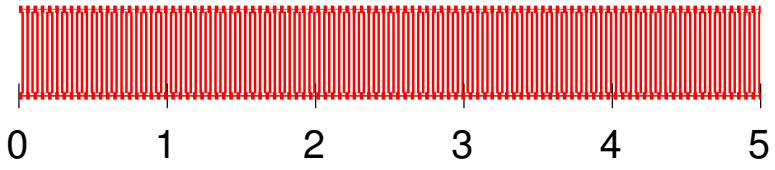

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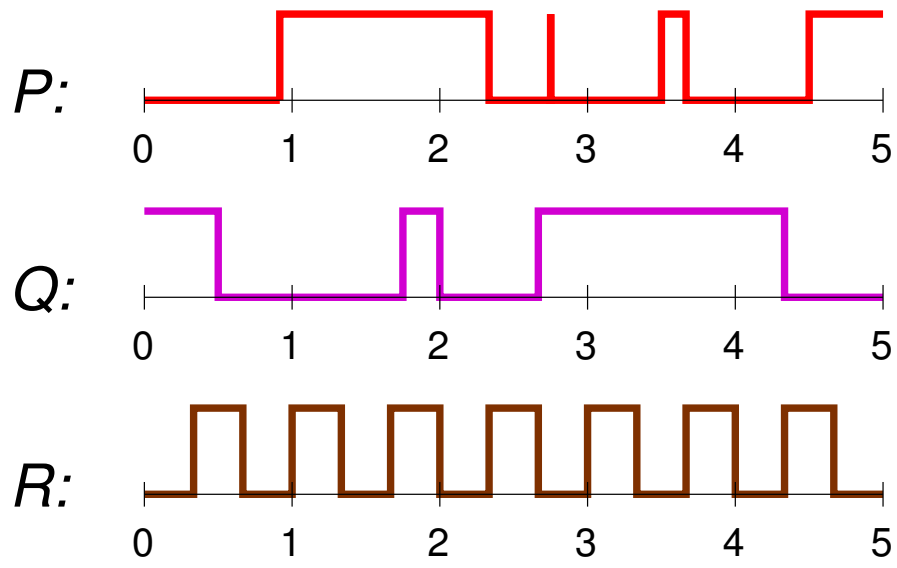

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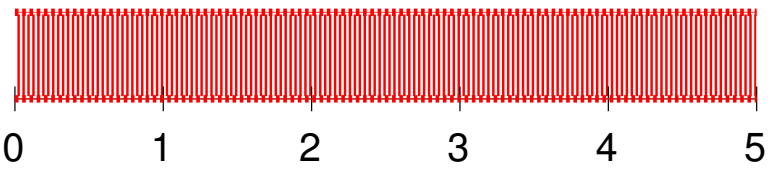

Then:

Theorem (Rabinovich 2002)

MSO(<) satisfiability over finitely-variable flows is decidable.

The Time-Bounded Theory of Verification

Theorem

For any fixed bounded time domain $[0, T)$*, the satisfiability and model-checking problems for MSO(*$<$*,*$+1$*), FO(*$<$*,*$+1$*), and MTL are all decidable, with the following complexities:*

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Theorem

Given timed automata A, B, and time bound $T \in \mathbb{N}$*, the language inclusion problem* $L_T(A) \subseteq L_T(B)$ *is decidable and 2EXPSPACE-complete.*

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- This can be decided in 2EXPSPACE.

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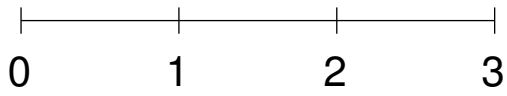
- Conclude by invoking decidability of MSO($<$).

From MSO($<$,$+1$) to MSO($<$)



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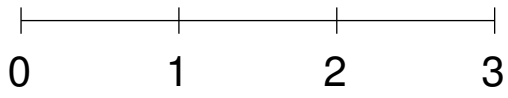

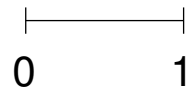

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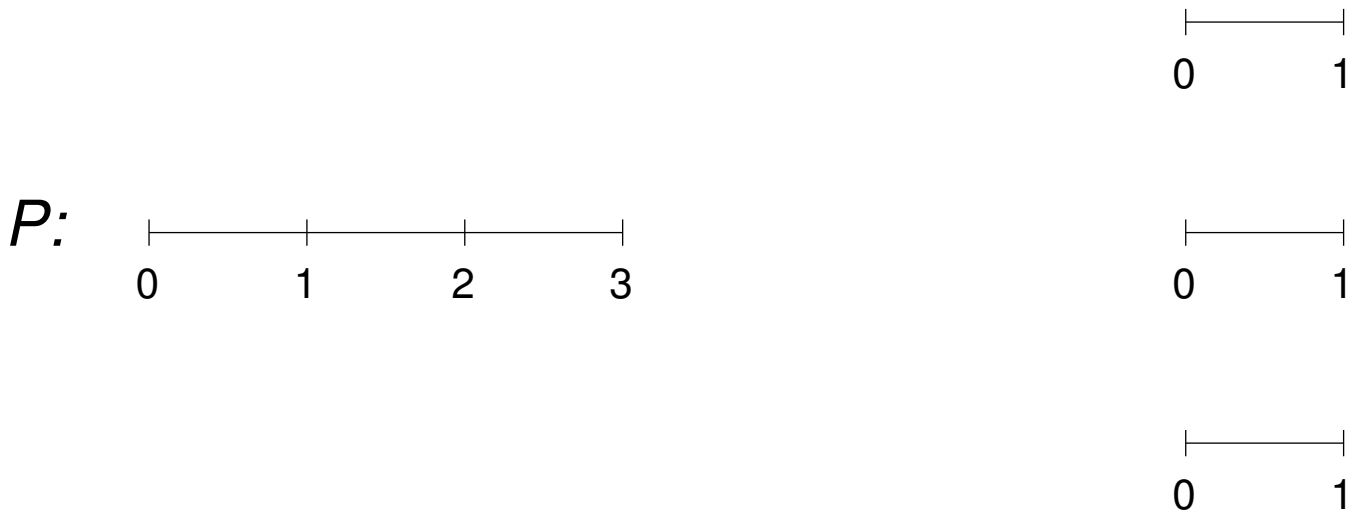

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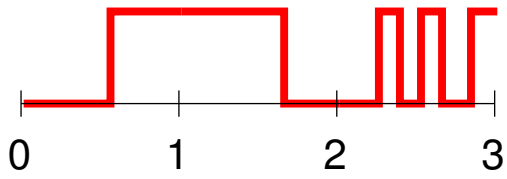

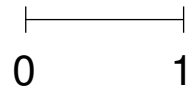

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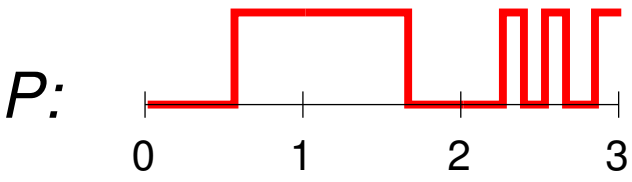

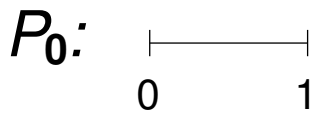

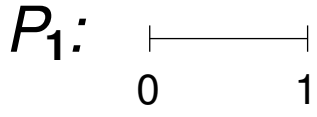

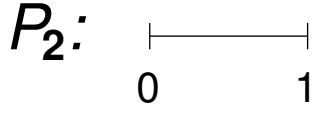

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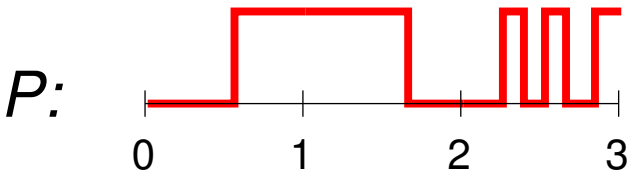

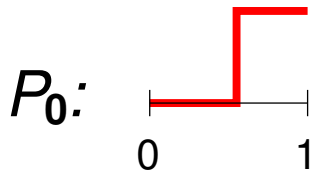

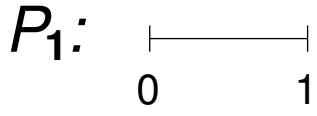

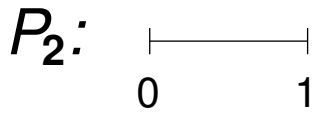

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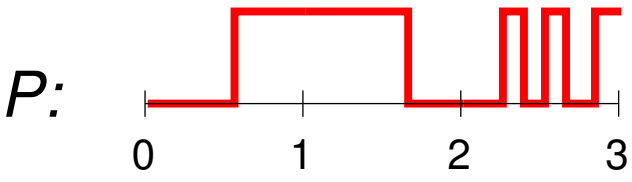

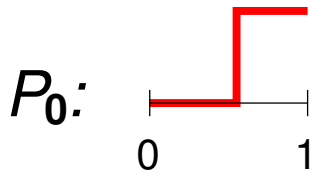

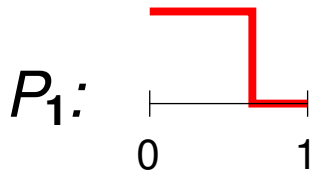

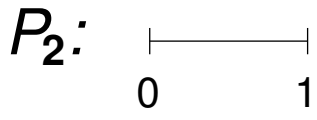

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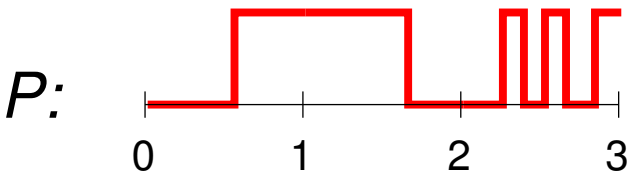

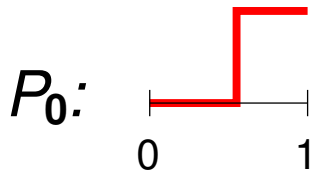

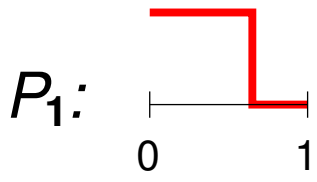

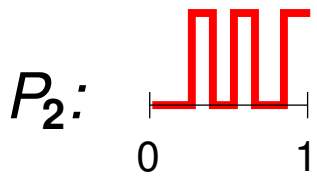

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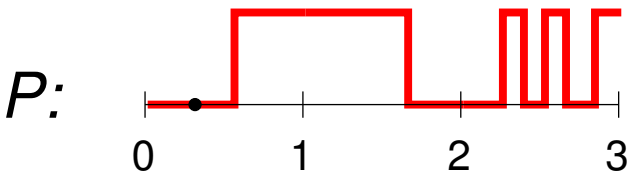

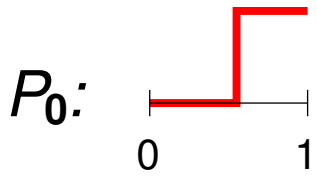

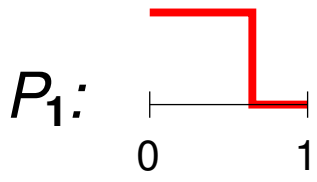

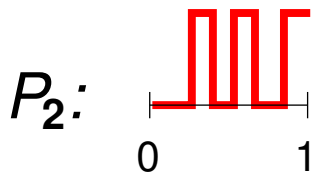

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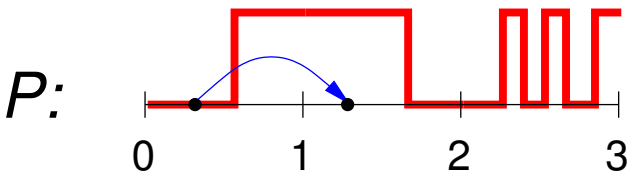

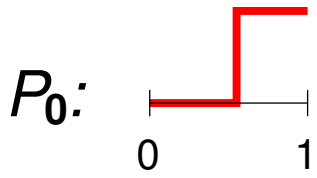

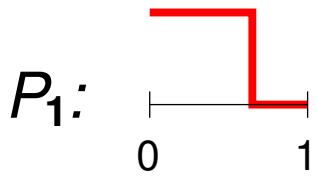

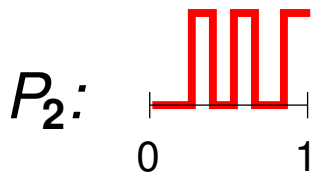

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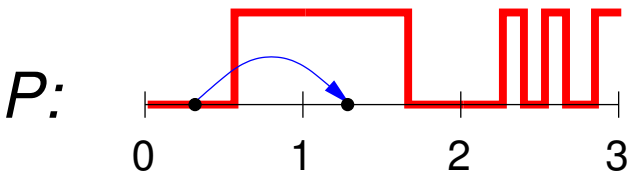

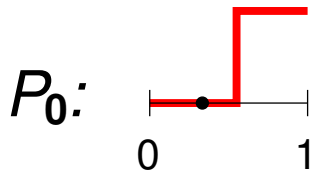

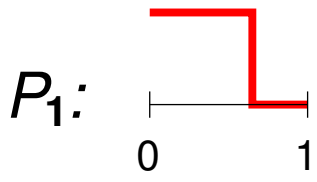

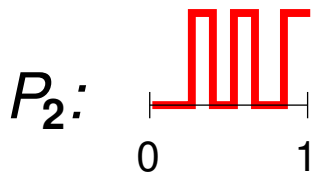

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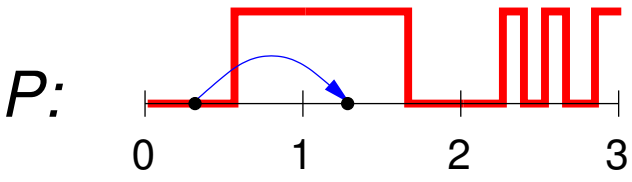

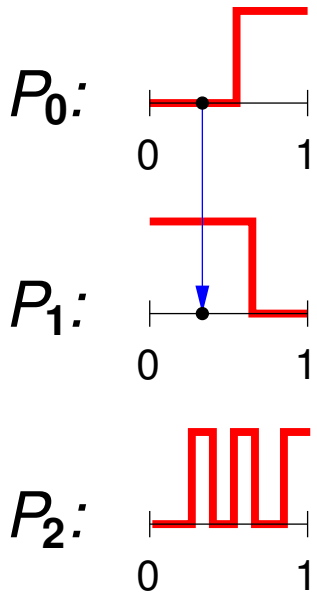

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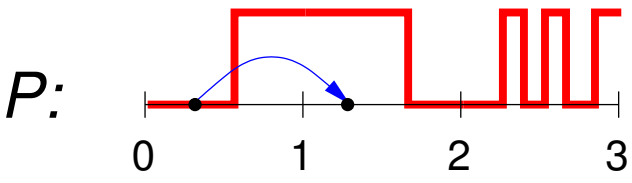

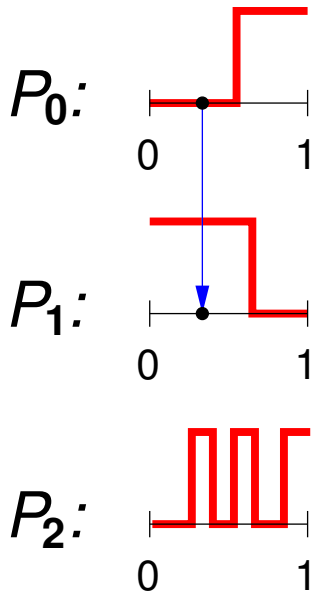

Replace every:

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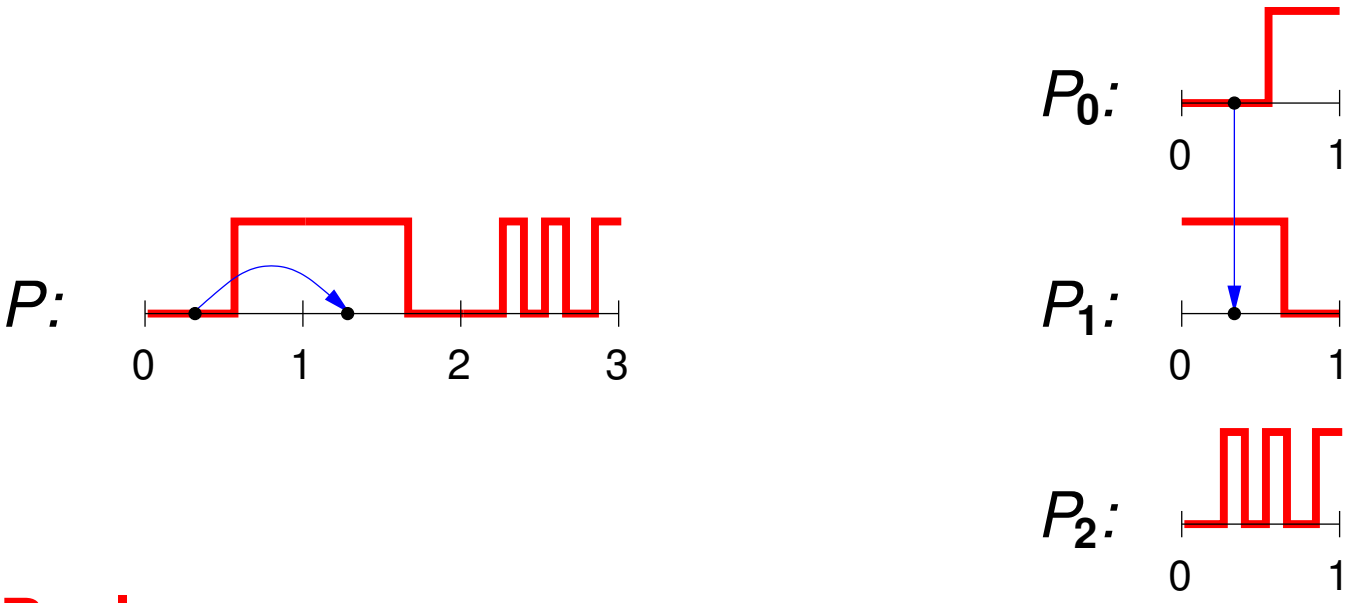

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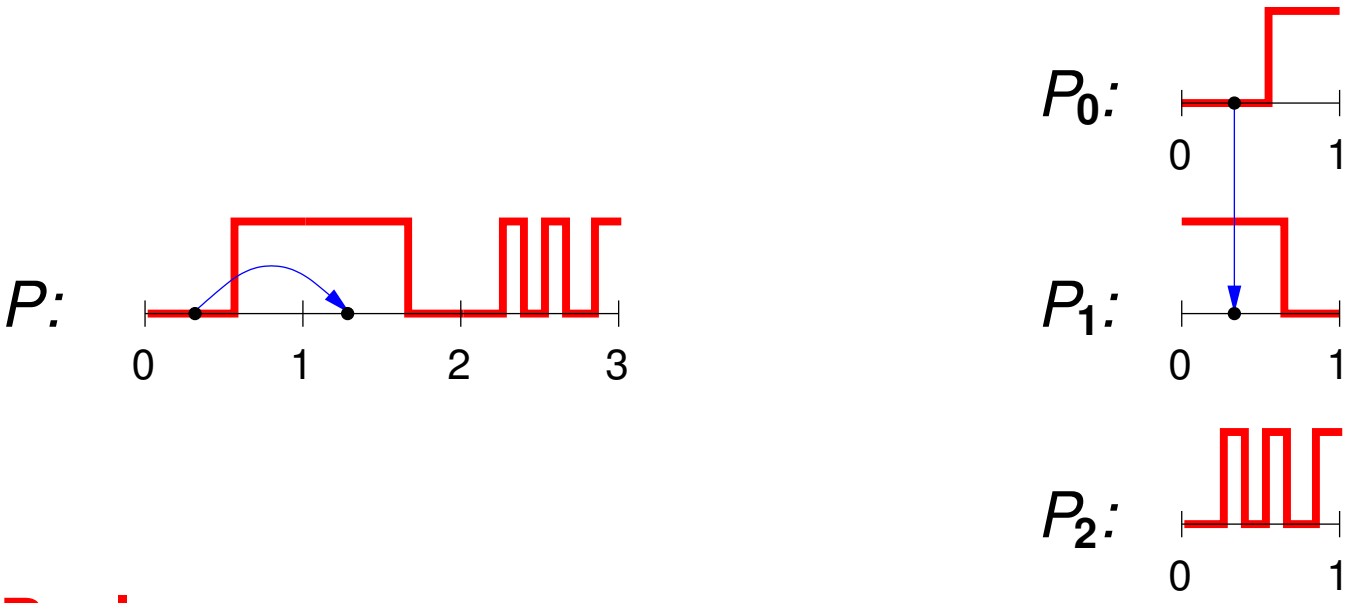

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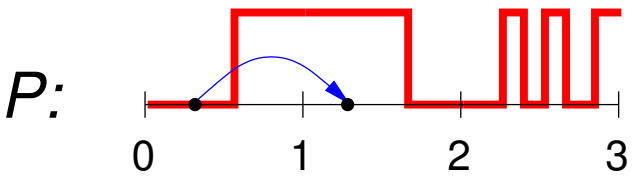

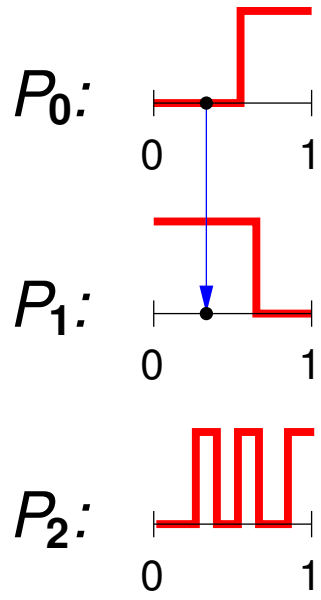

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- $x + k_1 < y + k_2$ **by**
$$\begin{cases} x < y & \text{if } k_1 = k_2 \\ \textbf{true} & \text{if } k_1 < k_2 \\ \textbf{false} & \text{if } k_1 > k_2 \end{cases}$$

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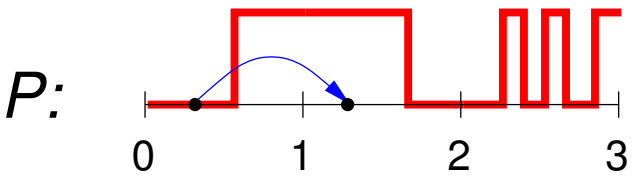

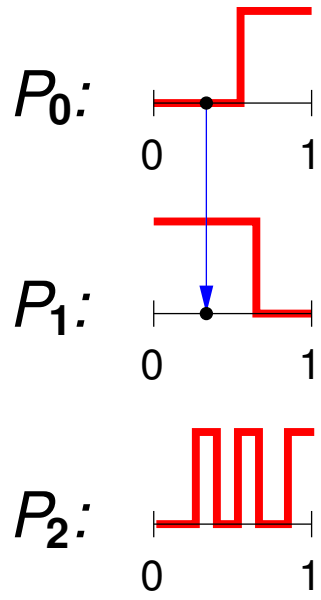

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- $P(x + k)$

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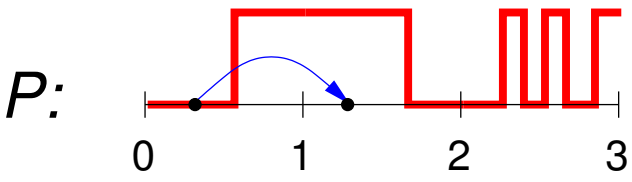

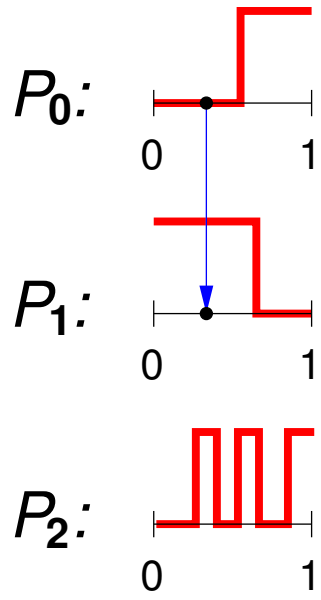

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- $P(x+k)$ **by** $P_k(x)$

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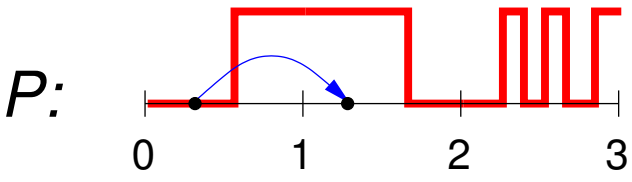

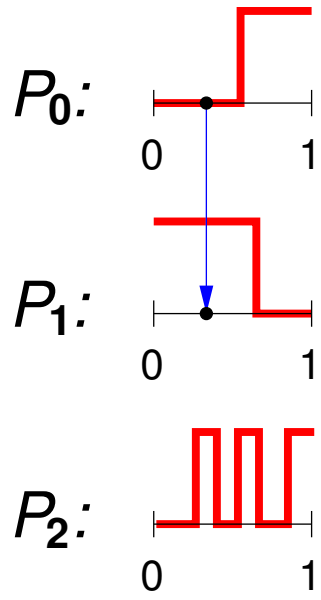

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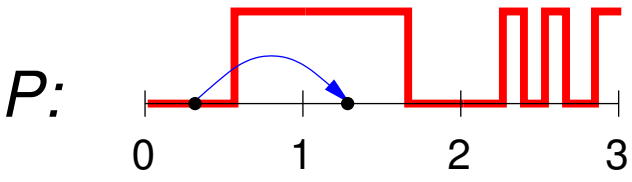

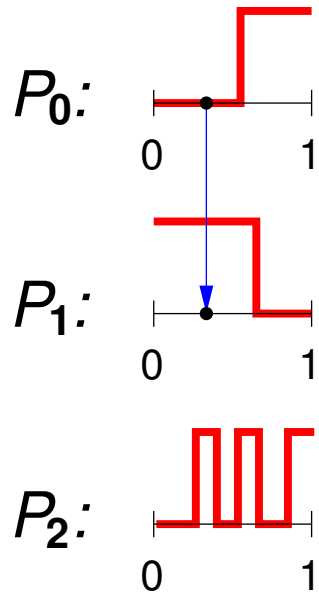

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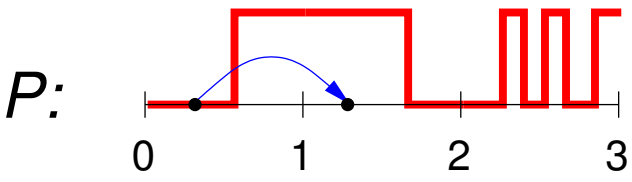

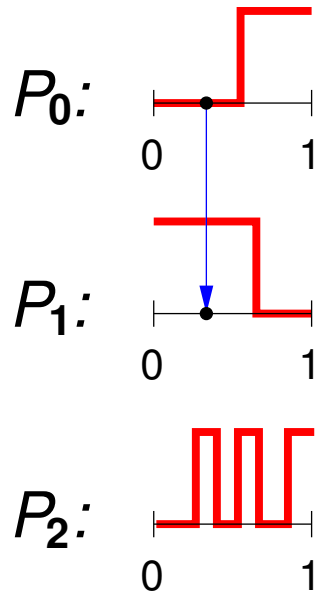

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Then φ is satisfiable over $[0, T)$ $\iff$ $\overline{\varphi}$ is satisfiable over $[0, 1)$.

The Time-Bounded Theory: Expressiveness

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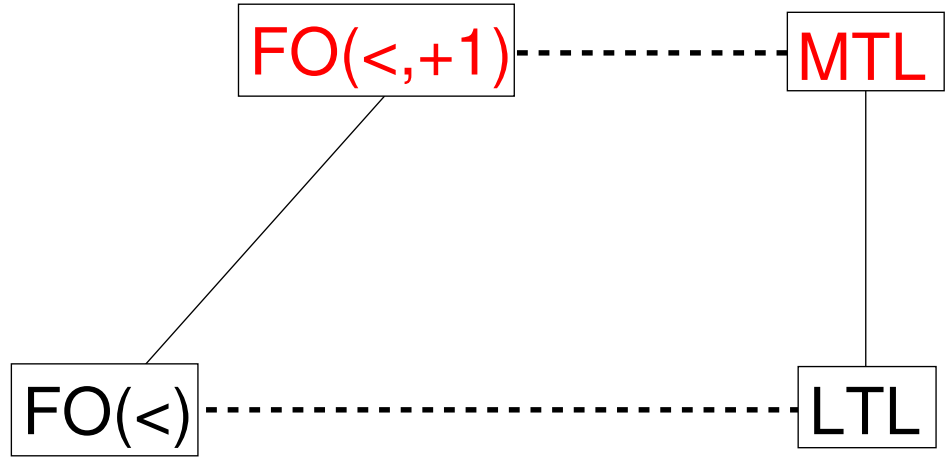

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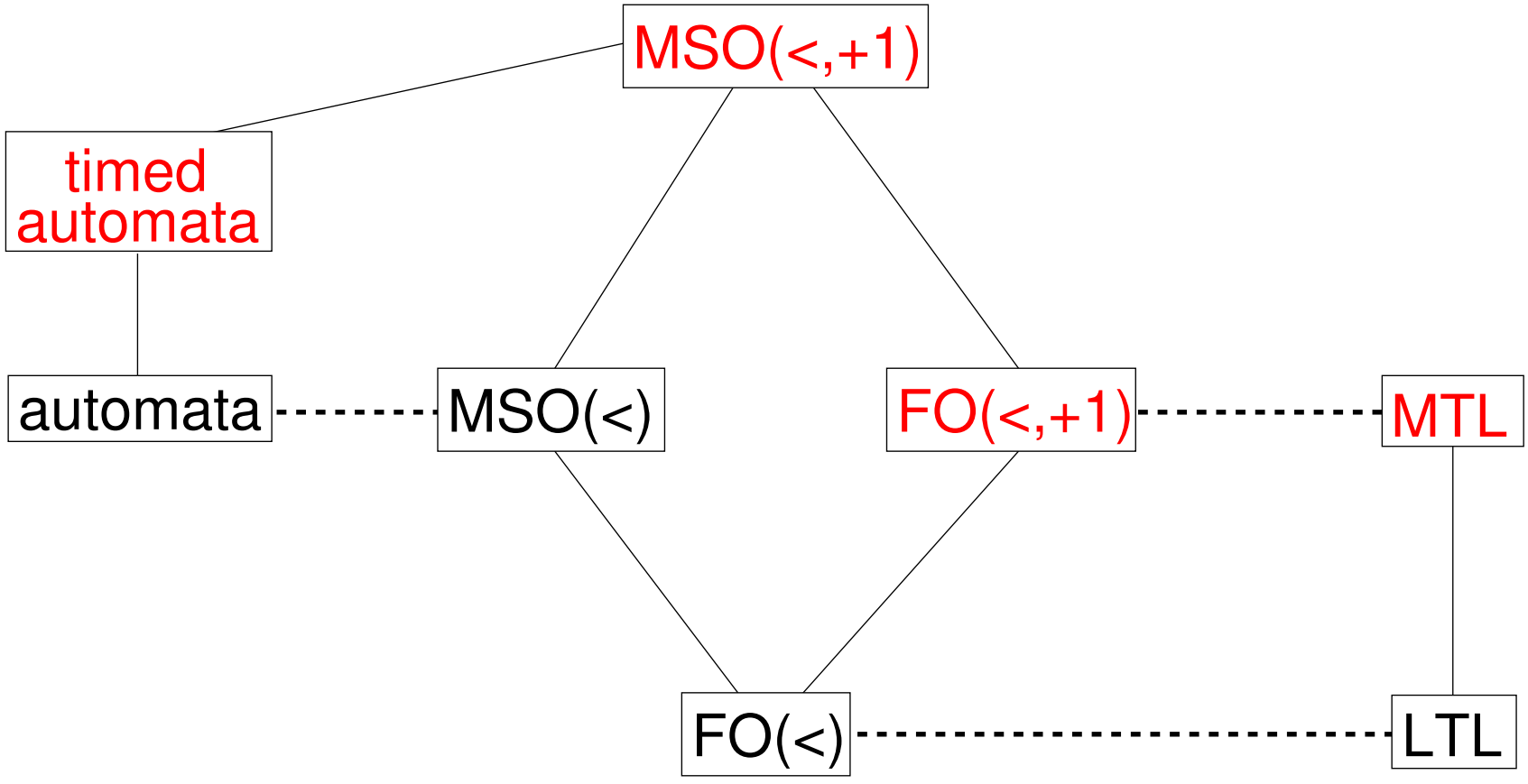
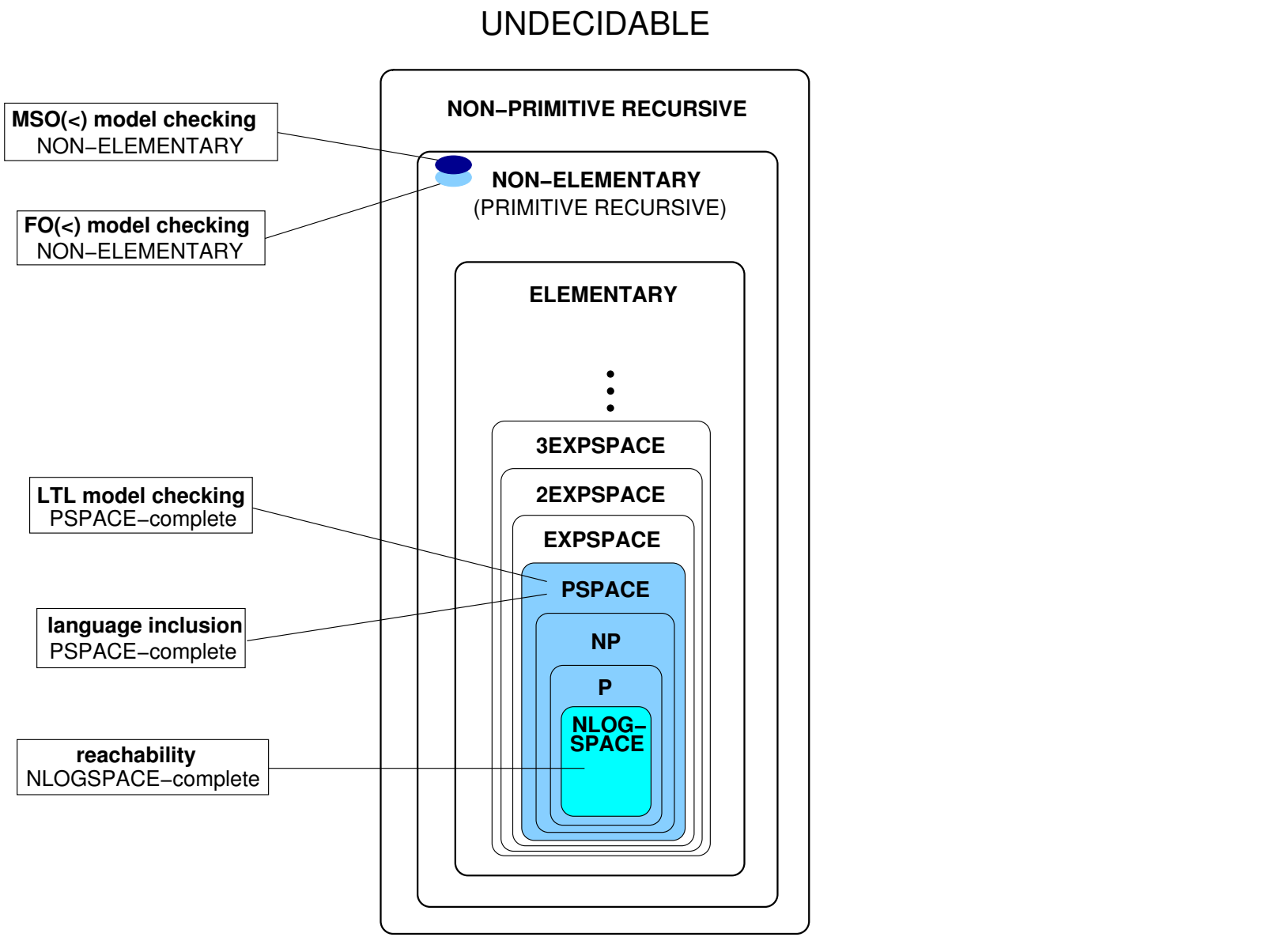


The Time-Bounded Theory: Complexity

Classical Theory

Time–Bounded Theory



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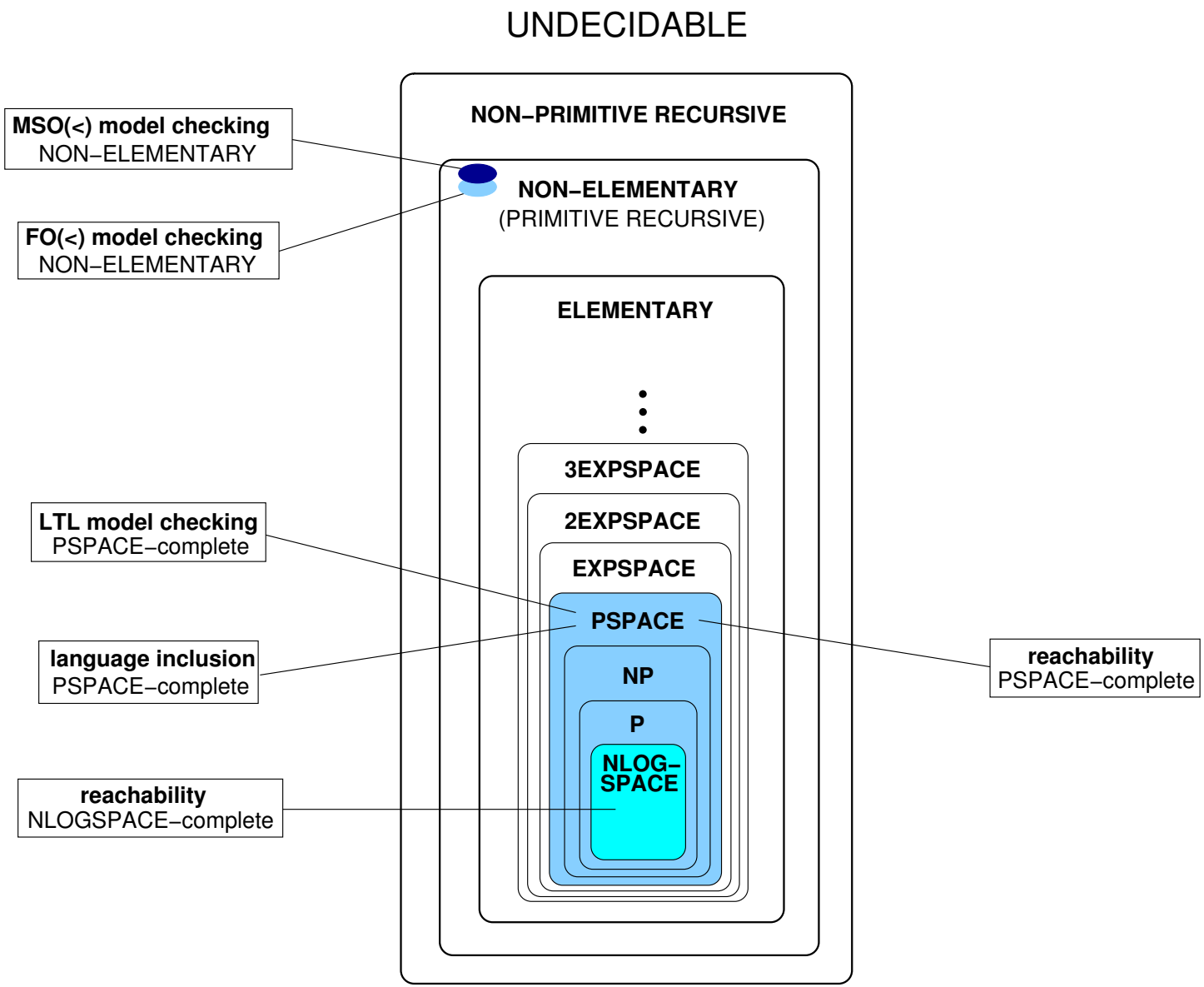

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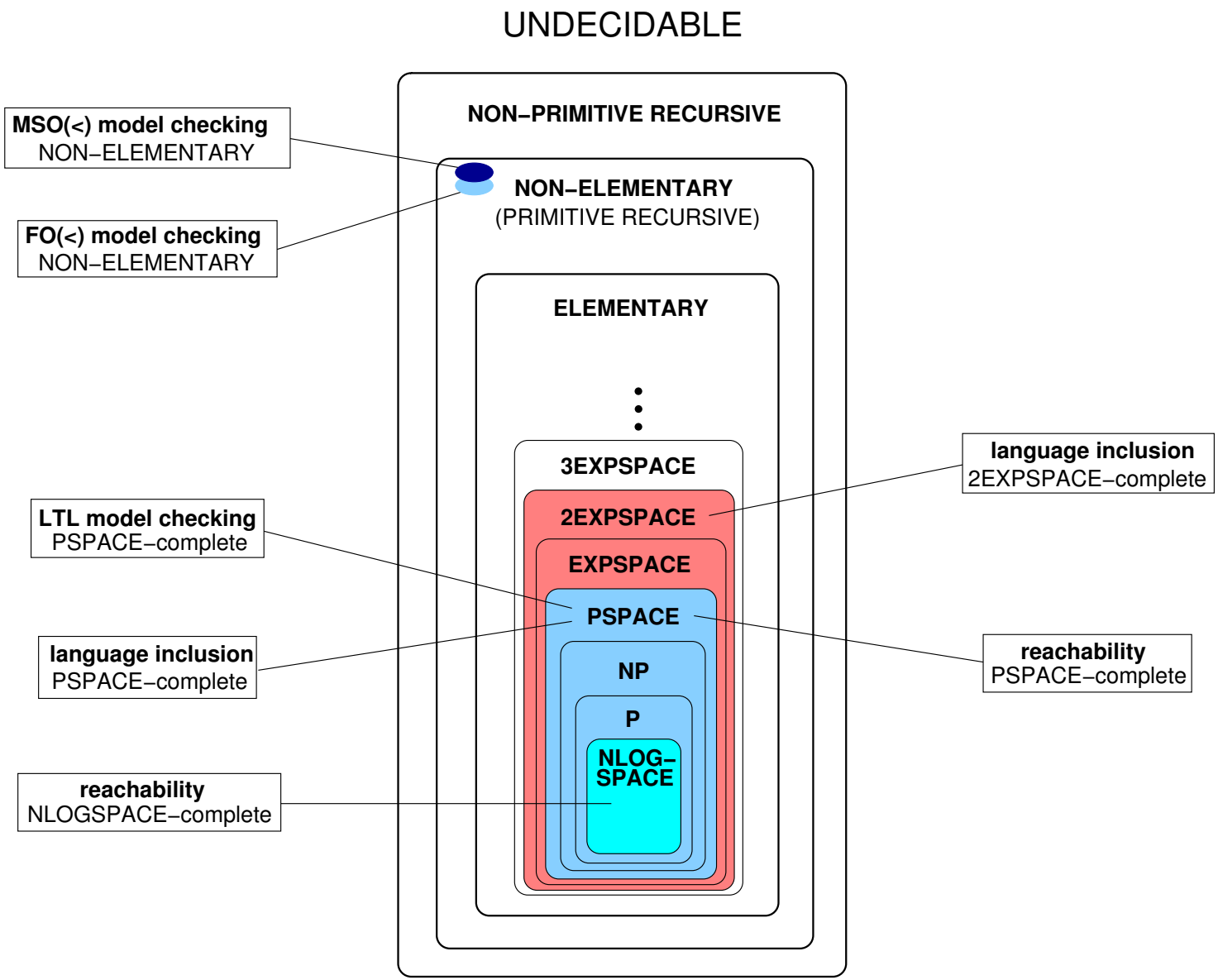

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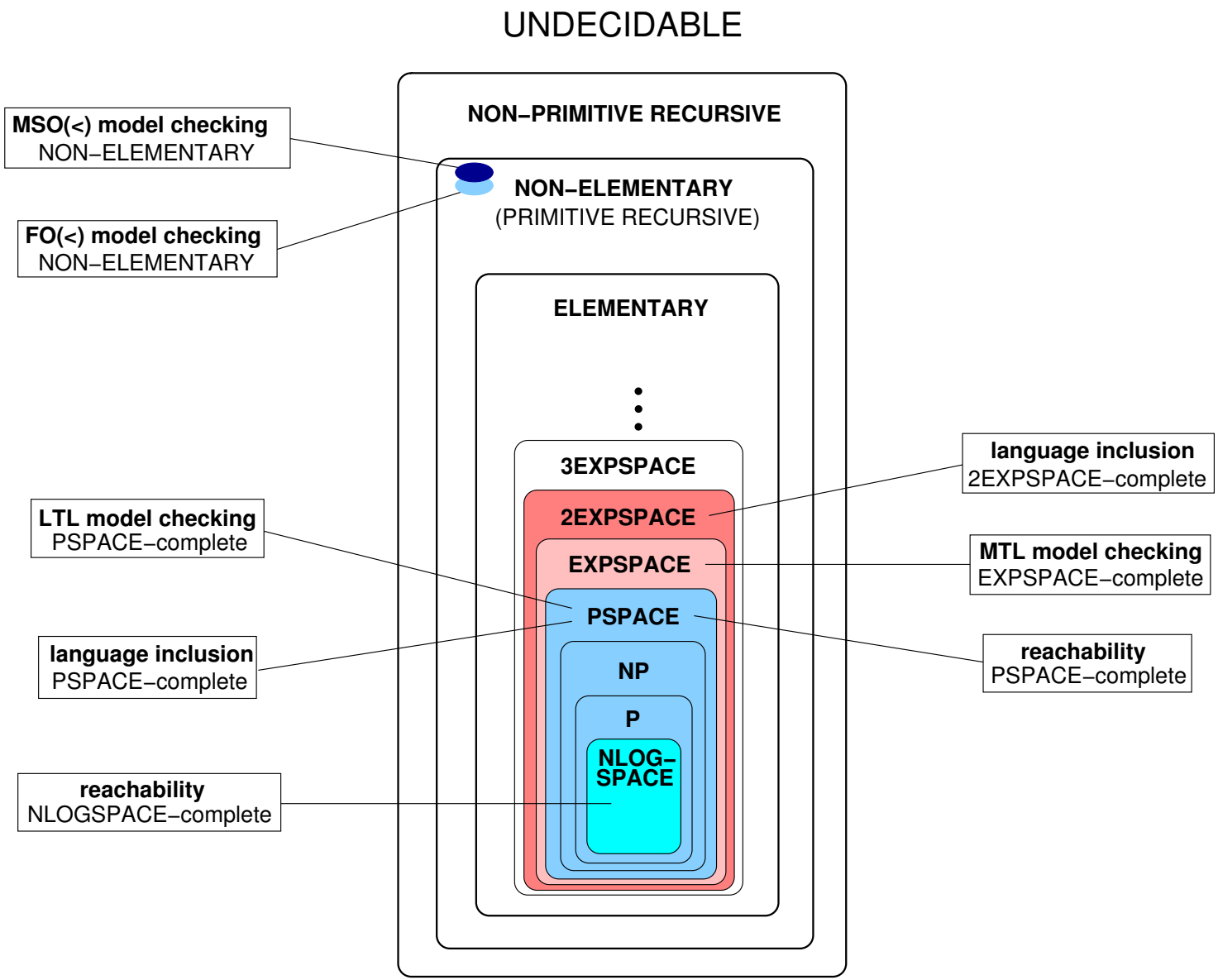

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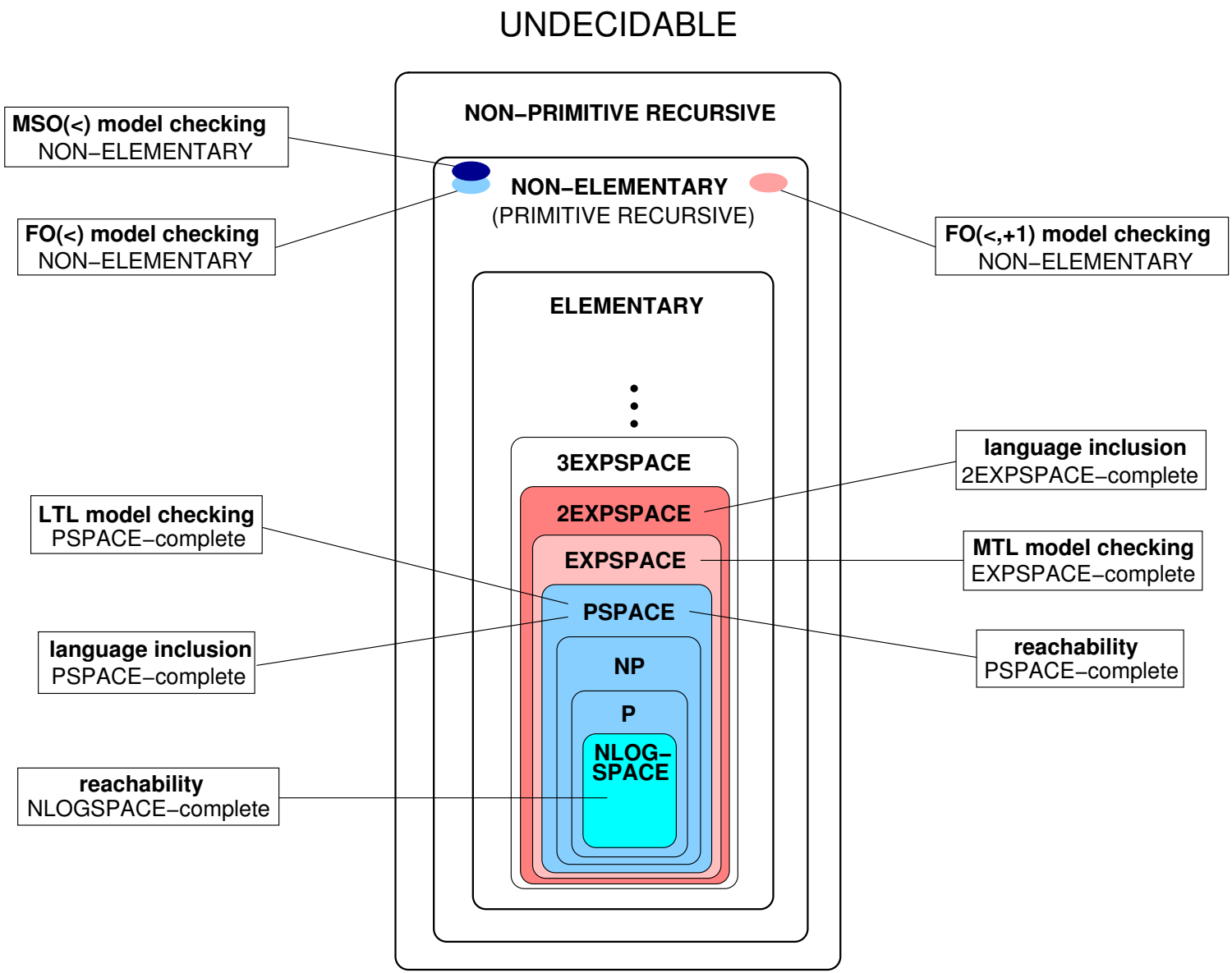

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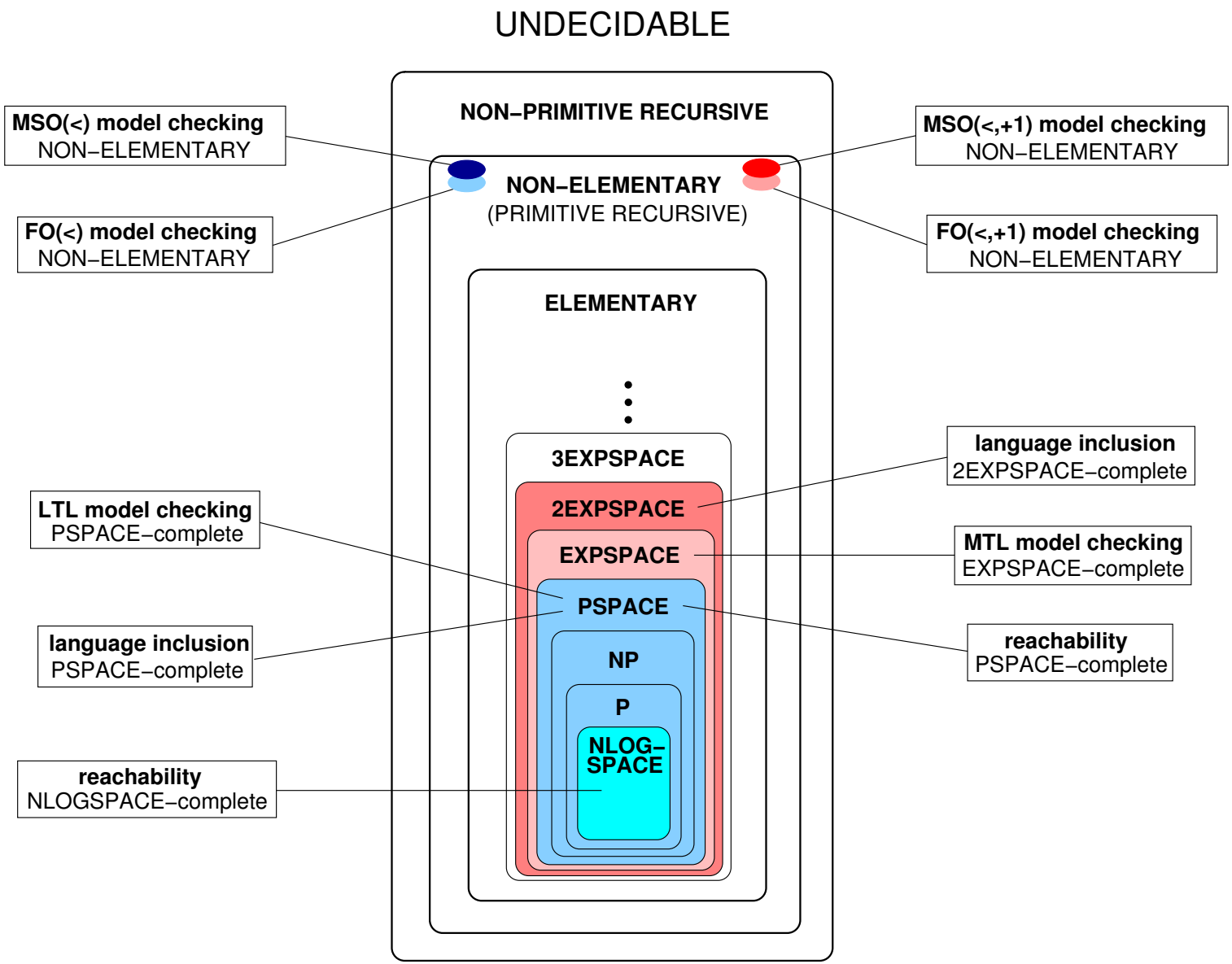

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Future work:

- Timed alternating automata
- Data complexity
- Time-bounded reachability for hybrid systems
- Expressiveness issues
- Implementation