Counting CTL

Antoine Meyer (1)

Joint work with: François Laroussinie (2), Eudes Petonnet (2)

(1) LIGM, Univ. Paris Est – CNRS
(2) LIAFA / Univ. Paris 7 – CNRS

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Counting in formulas

Consider an ATM, and the property:
"Three mistakes forbid cash retrieval"

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- In CTL:

$$\neg \mathsf{EF}\Big(\mathtt{error} \wedge \mathsf{EXEF}\big(\mathtt{error} \wedge \mathsf{EXEF}(\mathtt{error} \wedge \mathsf{EF}\mathtt{money})\big)\Big)$$

or: $\neg \mathsf{EF}\Big(\mathtt{error} \wedge \mathsf{EF}_s\big(\mathtt{error} \wedge \mathsf{EF}_s(\mathtt{error} \wedge \mathsf{EF}\mathtt{money})\big)\Big)$

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- With counting:

$$\neg\mathsf{EF}_{[\sharp\mathtt{error}\geq 3]}\,\mathtt{money}$$

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$$\neg \mathsf{E} \neg \mathtt{error} \mathsf{U} \mathtt{lock} \wedge \neg \mathsf{E} \neg \mathtt{error} \mathsf{U} (\mathtt{error} \wedge \mathsf{EXE} \neg \mathtt{error} \mathsf{U} \mathtt{lock}) \wedge \neg \mathsf{E} \neg \mathtt{error} \mathsf{U} \big(\mathtt{error} \wedge \mathsf{EXE} \neg \mathtt{error} \mathsf{U} (\mathtt{error} \wedge \mathsf{EXE} \neg \mathtt{error} \mathsf{U} \mathtt{lock}) \big)$$

Counting in formulas

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- With counting:

$$\neg \mathsf{EF}_{[\sharp \mathtt{error} \leq 2]}\, \mathtt{lock}$$

Counting in formulas

Other examples:

$\mathsf{EF}_{[\sharp \mathsf{EX}\texttt{Pb} < 2 \,\wedge\, \sharp\texttt{ok} > 10]}\texttt{P}, \quad \mathsf{EF}_{[\sharp\texttt{ok} - \sharp\texttt{bad} > 10]}\texttt{P}, \quad \mathsf{AG}_{[10 \cdot \sharp\texttt{ok} < 300 \cdot \sharp\texttt{bad}]}\,\bot, \ldots$

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CCTL = CTL + counting constraints of the form

$$\sum_{i=1}^{\ell} \alpha_i \cdot \sharp\varphi_i - \sum_{i=1}^{m} \beta_i \cdot \sharp\psi_i \sim k$$

(and all sensible restrictions: $\ell = 1, m = 0, m = 1, \alpha_i = \beta_i = 1, \ldots$)

Counting temporal logics

- LTL with regular expressions containing quantitative constraints [Emerson,Trefler 97] $\rightsquigarrow$ exponential algorithms in $|\Phi|$ and the *value* of constants.
- CTL with constraints (with parameters) [Emerson, Trefler 99]
  Constraints as positive boolean combinations of $\sum_i P_i \leq c$
  - Model-checking $E\_U\_$ is NP-complete
  - Polynomial algorithm given for a restricted logic
- Branching-time temporal logic with general counting constraints (using freeze variables): undecidable [Yang, Mok, Wang 97].
- LTL and CTL with Presburger constraints [Bouajjani, Echahed, Habermehl 95] for infinite state processes
- (timed extensions. . .)

Outline

1. CCTL
2. Expressiveness
3. Model checking
4. Freeze variables

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Counting CTL

Given $\ell, k \in \mathbb{N}$, $k' \in \mathbb{Z}$ and $\sim \in \{<, \leq, =, \geq, >\}$, we define:

$$\mathcal{C}_0 \ni C ::= \sharp\varphi \sim k$$

$$\mathcal{C}_1 \ni C ::= (\textstyle\sum_{i=1}^{\ell} \sharp\varphi_i) \sim k$$

$$\alpha\mathcal{C}_1 \ni C ::= (\textstyle\sum_{i=1}^{\ell} \alpha_i \cdot \sharp\varphi_i) \sim k$$

$$\alpha_i \in \mathbb{N}$$

$$\mathcal{C}_2 \ni C ::= (\sharp\varphi - \sharp\psi) \sim k'$$

$$\mathcal{C}_3 \ni C ::= (\textstyle\sum_{i=1}^{\ell} \pm\sharp\varphi_i) \sim k'$$

$$\alpha\mathcal{C}_3 \ni C ::= (\textstyle\sum_{i=1}^{\ell} \beta_i \cdot \sharp\varphi_i) \sim k$$

$$\beta_i \in \mathbb{Z}$$

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$$\alpha_i \in \mathbb{N} \qquad \beta_i \in \mathbb{Z}$$

For each $\mathcal{C}$, $\mathcal{B}(\mathcal{C})$ = boolean combinations of constraints in $\mathcal{C}$

Definition

Let $\mathcal{C}$ be a set of constraints as above, the syntax of $\mathrm{CCTL}_{\mathcal{C}}$ is:

$$\varphi, \psi ::= P \mid \varphi \wedge \psi \mid \neg\varphi \mid \mathsf{E}\varphi\mathsf{U}_{[C]}\psi \mid \mathsf{A}\varphi\mathsf{U}_{[C]}\psi$$

where $P \in \mathrm{AP}$ (atomic propositions), $C \in \mathcal{C}$

Counting CTL

CCTL formulas are interpreted over states of Kripke structures

$$\mathcal{S} = \langle Q, R, \ell \rangle$$

- Q is a finite set of states
- $R \subseteq Q \times Q$ is a complete edge relation
- $\ell : Q \to 2^{\mathsf{AP}}$ is a labeling of states with atomic propositions

No costs, no weights, (no probabilities), no time . . . !

Semantics of constraints

Let π be a finite run, $\pi \models C$ depends on the interpretation of $\sharp\varphi$ over π:

$$|\pi|_\varphi \stackrel{\text{def}}{=} \left|\{j \mid 0 \le j \le |\pi| \ \wedge\ \pi(j) \models \varphi\}\right|$$

CCTL semantics

$$\begin{array}{lll} q \models \mathsf{E}\varphi\mathsf{U}_{[C]}\psi & \text{iff} & \exists \rho \in \mathsf{Runs}(q),\ \exists k \ge 0, \rho(k) \models \psi, \\ & & \rho_{|k-1} \models C,\ \text{and } \forall 0 \le i < k,\ \rho(i) \models \varphi \\ q \models \mathsf{A}\varphi\mathsf{U}_{[C]}\psi & \text{iff} & \forall \rho \in \mathsf{Runs}(q),\ \exists k \ge 0,\ \rho(k) \models \psi, \\ & & \rho_{|k-1} \models C,\ \text{and } \forall 0 \le i < k,\ \rho(i) \models \varphi \end{array}$$

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- $\mathsf{E}\varphi\mathsf{U}_{<5}\psi \stackrel{\text{def}}{=} \mathsf{E}\varphi\mathsf{U}_{[\sharp\mathtt{tick}<5]}\psi$ – (TCTL over KSs with tick).

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- For an ATM: "it is not possible to get money when three mistakes are made in the same session":

$$\mathsf{AG}\left(\neg\mathsf{EF}_{[\sharp\mathtt{error}\geq 3\wedge\sharp\mathtt{reset}=0]}\mathtt{money}\right)$$

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- $\mathsf{AG}\,(\,\mathsf{EF}_{[\sharp(\mathsf{EX}\mathtt{alarm})\leq 5]}\mathtt{init}\,)$ "It is always possible to reach `init` along a path where less than 5 states have an `alarm` state as successor."

Examples of formulas

- The bounded waiting property with bound 10 for a mutual exclusion algorithm with n processes:

$$\mathsf{AG} \bigwedge_i \left(\mathtt{request}_i \Rightarrow \neg \mathsf{EF}_{[\sum_{j \neq i} \sharp \mathtt{CS}_j > 10 \wedge \sharp \mathtt{CS}_i = 0]} \top\right)$$

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$$\mathsf{AG}_{[\sharp send - \sharp receive < 0]} \bot$$

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- Quantitative fairness: "The φ_i's occur infinitely often along every run and there is no sub-run where φ_1 holds for more than 10 states and φ_2 holds for less than 4 states":

$$\mathsf{AG}\,\mathsf{AF}_{[\bigwedge_i 5 \leq \sharp \varphi_i \leq 10]} \top$$

Outline

1. CCTL
2. Expressiveness
3. Model checking
4. Freeze variables

$\mathcal{B}(\alpha\mathcal{C}_1) : \bigwedge \bigvee \sum_{i=1}^{\ell} \alpha_i \cdot \sharp\varphi_i \sim k$ $\alpha\mathcal{C}_2 : \sharp\varphi - \sharp\psi \sim k$

Proposition

Any $CCTL_{\mathcal{B}(\alpha\mathcal{C}_1)}$ formula can be translated into CTL.

Idea: manually count occurrences of events using nested U modalities and consider all possible shuffles of such occurrences.

Expressiveness

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Proposition

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Proposition

The $CCTL_{\mathcal{C}_2}$ formula $\varphi = AG_{[\sharp A - \sharp B < 0]}\bot$ cannot be translated into CTL.

Idea: the set of models of any CTL formula can be recognized by an alternating tree automaton. This is not the case for φ.

Succinctness

$$\mathcal{B}(\alpha\mathcal{C}_1) : \bigwedge \bigvee \sum_{i=1}^{\ell} \alpha_i \cdot \sharp\varphi_i \sim k$$

$\mathsf{CCTL}_{\mathcal{B}(\alpha\mathcal{C}_1)}$ formulas can be translated into CTL, but in these constraints, there are three potential sources of concision:

- Binary encoding of constants
- Boolean combinations in constraints
- Sums of counting expressions

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- Binary encoding of constants
- Boolean combinations in constraints
- Sums of counting expressions

Only the first two yield an exponential improvement in succinctness

Succinctness – Binary encoding

$\mathcal{C}_0 : \sharp\varphi \sim k$

The previous translation of $\mathsf{EF}_{[\sharp A = k]} B$ into CTL yields an exponential formula (it uses k nested modalities)

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Proposition

$CCTL_{\mathcal{C}_0}$ can be exponentially more succinct than CTL

Idea: TCTL formulas $\mathsf{EF}_{<k} A$ and $\mathsf{EF}_{>k} A$ do not admit any equivalent CTL formula of temporal height less than k [Laroussinie, Schnoebelen, Turuani 01]

Succinctness – Boolean combinations

$\mathcal{B}(\mathcal{C}_0) : \bigwedge \bigvee \sharp\varphi \sim k$

Proposition

$CCTL_{\mathcal{B}(\mathcal{C}_0)}$ with unary encoding of integers can be exponentially more succinct than CTL.

Succinctness – Boolean combinations

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Proposition

$CCTL_{\mathcal{B}(\mathcal{C}_0)}$ with unary encoding of integers can be exponentially more succinct than CTL.

Idea: any CTL formula equivalent to ψ:

$$\psi \;=\; \mathsf{E}(\mathsf{F}\, P_0 \wedge \ldots \wedge \mathsf{F}\, P_n)$$

must be of length exponential in n [Wilke 99, Adler, Immerman 03]

$$\psi \;\equiv\; \mathsf{EF}_{[\bigwedge_i \sharp P_i \geq 1]} \top$$

(binary encoding of constants not needed)

Succinctness – Sums

$\mathcal{C}_1 : \sum_i \sharp\varphi_i \sim k$

Proposition

For every formula $\Phi \in CCTL_{\mathcal{C}_1}$ with unary encoding, there exists an equivalent CTL formula of DAG-size polynomial in $|\Phi|$.

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Example: $\Phi = \mathsf{EF}_{\sum_i \sharp P_i = K} A$ is equivalent to Ψ_K with:

$$\Psi_k \stackrel{\text{def}}{=} \mathsf{E}\big(\textstyle\bigwedge_i \bar{P}_i\big)\mathsf{U}\big(\textstyle\bigvee_i P_i \wedge \beta_{k,1,\bot}\big) \qquad (k > 0)$$

$$\Psi_0 \stackrel{\text{def}}{=} \mathsf{E}\big(\textstyle\bigwedge_i \bar{P}_i\big)\mathsf{U}\, A \qquad \Psi_{-1} \stackrel{\text{def}}{=} \bot$$

$$\beta_{k,i,\epsilon} \stackrel{\text{def}}{=} (P_i \wedge \beta_{k-1,i+1,\top}) \vee (\bar{P}_i \wedge \beta_{k,i+1,\epsilon}) \qquad (i < n)$$

$$\beta_{k,n,\top} \stackrel{\text{def}}{=} (P_n \wedge \mathsf{EX}\, \Psi_{k-1}) \vee (\bar{P}_n \wedge \mathsf{EX}\, \Psi_k)$$

$$\beta_{k,n,\bot} \stackrel{\text{def}}{=} P_n \wedge \mathsf{EX}\, \Psi_{k-1}$$

Comparison with Past

Counting constraints deal with past events !

We could use past-time modalities:

$$\mathsf{AG}\big(\texttt{money} \;\Rightarrow\; \neg\, \mathsf{F}_s^{-1}(\texttt{error} \wedge \mathsf{F}_s^{-1}(\texttt{error} \wedge \mathsf{F}_s^{-1}\texttt{error}))\big)$$

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+ Past-time modalities allow us to express properties over the ordering of the events.
+ They (often) increase the expressive power (compared to CTL).
+ Boolean combinations are directly handled. . .

− Counting constraints are still more succinct.
− Complexity (model-checking $\mathsf{CTL} + \mathsf{F}^{-1}$ is PSPACE-complete)

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Model checking $\mathrm{CCTL}_{\mathcal{C}_0}$ and $\mathrm{CCTL}_{\mathcal{C}_1}$

$\mathrm{CCTL}_{\mathcal{C}_0}$: $\sharp\varphi \sim k$

$\mathrm{CCTL}_{\mathcal{C}_1}$: $(\sum_{i=1}^{\ell} \sharp\varphi_i) \sim k$

Theorem

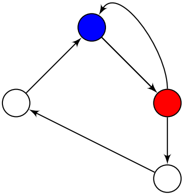
Model-checking $CCTL_{\mathcal{C}_1}$ and $CCTL_{\mathcal{C}_0}$ is P*-complete*

Idea: Reduction to a model-checking problem for TCTL formulas over Kripke structures with 0/1 durations

Model checking $\mathrm{CCTL}_{\mathcal{C}_0}$ and $\mathrm{CCTL}_{\mathcal{C}_1}$

Example: $\mathcal{S} = (Q, R, \ell)$, and $\Phi = \mathsf{E}\varphi\mathsf{U}_{[\sharp P_1 + \sharp P_2 \sim k]}\psi$

$\models P_1$ $\models P_1 \wedge P_2$



Model checking $\mathsf{CCTL}_{\mathcal{C}_0}$ and $\mathsf{CCTL}_{\mathcal{C}_1}$

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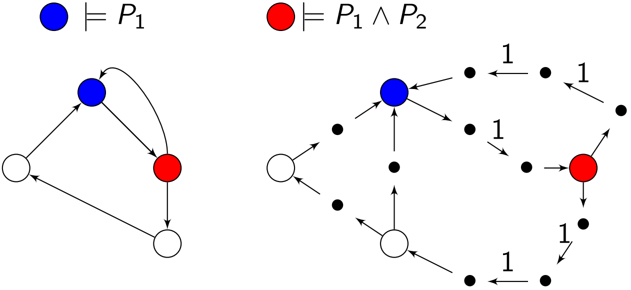

Model-checking $\mathrm{CCTL}_{\mathcal{C}_1}$

Proof: $\mathcal{S} = (Q, R, \ell)$, and $\Phi = \mathsf{E}\psi\mathsf{U}_{[C]}\psi'$ with $C \stackrel{\text{def}}{=} \sum_{i=1}^{\ell} \sharp\varphi_i \sim k$

$\forall q \in Q$: $|q|_C \stackrel{\text{def}}{=} \big|\{i \mid q \models \varphi_i\}\big|$

We build the $\mathrm{DKS}^{0/1}$ $\mathcal{S}' = (Q', R', \ell')$ as follows:

- $Q' \stackrel{\text{def}}{=} Q \cup \bigcup_{q \in Q} \{q_i \mid 0 \leq i \leq |q|_C\}$,
- $R' \stackrel{\text{def}}{=} \{q \xrightarrow{0} q_0\} \cup \{q_i \xrightarrow{1} q_{i+1} \mid i < |q|_C\}$
  $\cup \{q_n \xrightarrow{0} q' \mid (q, q') \in R, n = |q|_C\}$,
- $\ell'(q_i) = \varnothing$ and $\ell'(q) = \ell(q) \cup \{\mathtt{ok}\}$

$\rho \models_{\mathcal{S}} \psi\mathsf{U}_{[C]}\psi'$ if and only if $\tilde{\rho} \models_{\mathcal{S}'} (\mathtt{ok} \Rightarrow \psi)\mathsf{U}_{[\sim k]}(\mathtt{ok} \wedge \psi')$

Model-checking $\mathsf{CCTL}_{\mathcal{C}_2}$

$$\mathsf{CCTL}_{\mathcal{C}_2} : (\sharp\varphi - \sharp\psi) \sim k$$

Theorem

The model-checking problem for $\mathsf{CCTL}_{\mathcal{C}_2}$ is P*-complete*

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Theorem

The model-checking problem for $CCTL_{\mathcal{C}_2}$ is P*-complete*

Let $\mathcal{S} \stackrel{\text{def}}{=} (Q, R, \ell)$

Case 1: $\Phi \stackrel{\text{def}}{=} \mathsf{E}\varphi' \mathsf{U}_{[C]} \psi'$ with $C \stackrel{\text{def}}{=} (\sharp\varphi - \sharp\psi) \sim k$

$\forall q \in Q$, we define $|q|_C \in \{-1, 0, 1\}$

Let $G_{\mathcal{S}} = (S', R', w)$ be the weighted graph such that:

- S' contains only S states satisfying $\mathsf{E}\varphi' \mathsf{U} \psi'$;
- R' is R restricted to $S' \times S'$;
- $w(q, q') \stackrel{\text{def}}{=} |q|_C$ if $q \models \varphi'$, and 0 otherwise

Model-checking $\mathrm{CCTL}_{\mathcal{C}_2}$

$C \stackrel{\text{def}}{=} (\sharp\varphi - \sharp\psi) \leq k$: shortest paths in $G_{\mathcal{S}}$ + reachability of negative cycles

$C \stackrel{\text{def}}{=} (\sharp\varphi - \sharp\psi) = k$: with $k \geq 0$

Compute $R_k \stackrel{\text{def}}{=} \{(q, q') \in S'^2 \mid \exists\sigma,\ |q\sigma q'|_C = k\}$ as follows:

- $R_k = R_{\lfloor k/2 \rfloor} \cdot R_{\lfloor k/2 \rfloor} \cdot R_{(k \mod 2)}$
- $R_1 \stackrel{\text{def}}{=} R_0 \cdot \xrightarrow{1} \cdot R_0$
- R_0 is the least solution of:

$$X = (\xrightarrow{0})^* \cup X \cdot (\xrightarrow{1} \cdot X \cdot \xrightarrow{-1} \cup \xrightarrow{-1} \cdot X \cdot \xrightarrow{1}) \cdot X$$

$\Rightarrow q \models \Phi$ iff $(q, q') \in R_k$ for some q' satisfying ψ'

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Case 2: $\Phi \stackrel{\text{def}}{=} \mathrm{EG}_{[C \wedge \sharp\varphi' = 0]}\psi'$: ...

Model-checking $\mathsf{CCTL}_{\mathcal{C}_3}$

$$\mathsf{CCTL}_{\mathcal{C}_3} : \left(\sum_{i=1}^{\ell} \pm \cdot \sharp\varphi_i\right) \sim k$$

Theorem

The model-checking problem for and $CCTL_{\mathcal{C}_3}$ is P*-complete*

Each state contributes to a cost $d \in \{-M, \ldots, M\}$ with $M \leq |C|$: same technique as previously

Model-checking $\mathrm{CCTL}_{\mathcal{B}(\mathcal{C}_0)}$

$\mathcal{B}(\mathcal{C}_0) : \bigwedge \bigvee \sharp\varphi \sim k$

Theorem

The model-checking problem for $\mathrm{CCTL}_{\mathcal{B}(\mathcal{C}_0)}$ is Δ_2^{P}-hard

Reduction from SNSAT (derived from the reduction done for CTL^+ [Laroussinie, Markey, Schnoebelen 01]

SNSAT: collection of equations $z_i = \exists \bar{X}.\varphi_i(z_1, \ldots, z_{i-1}, \bar{X})$

Model-checking CCTL$_{\alpha\mathcal{C}_1}$

$\alpha\mathcal{C}_1 : (\sum_{i=1}^{\ell} \alpha_i \cdot \sharp\varphi_i) \sim k,\ \alpha_i \in \mathbb{N}$

Theorem

The model-checking problem for CCTL$_{\alpha\mathcal{C}_1}$ is Δ_2^{P}-hard

Model-checking CCTL$_{\alpha\mathcal{C}_1}$

$\alpha\mathcal{C}_1$: $(\sum_{i=1}^{\ell} \alpha_i \cdot \sharp\varphi_i) \sim k$, $\alpha_i \in \mathbb{N}$

Theorem

The model-checking problem for CCTL$_{\alpha\mathcal{C}_1}$ is Δ_2^{P}-hard

Reduction from the model-checking problem for TCTL over Kripke structures with integer durations (DKS)

Let $\mathcal{S} = (Q, R_{\mathcal{S}}, \ell)$ be a DKS
For every transition $q \xrightarrow{k} q'$ in $\mathcal{S}$, we add a new state between q and q' and labeled with only P_k
The TCTL formula $\mathsf{E}\varphi\mathsf{U}_{\sim m}\psi$ is replaced by:

$$\mathsf{E}(\mathtt{ok} \Rightarrow \widetilde{\varphi})\, \mathsf{U}_{[C]}\, (\mathtt{ok} \wedge \widetilde{\psi}) \qquad \text{with} \qquad C \stackrel{\text{def}}{=} \sum_{d \in W} d \cdot \sharp P_d \sim m$$

Model-checking $\mathrm{CCTL}_{\mathcal{B}(\alpha\mathcal{C}_1)}$

$\mathcal{B}(\alpha\mathcal{C}_1) : \bigvee\bigwedge(\sum_{i=1}^{\ell} \alpha_i \cdot \sharp\varphi_i) \sim k$, $\alpha_i \in \mathbb{N}$

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Based on the Parikh image of the runs satisfying $\mathrm{EF}_{[C]}\psi$

- we can assume that $|\rho|$ is in $O(|Q| \cdot 2^{|C|})$;
- check in polynomial time that a guessed Parikh image corresponds to some path;
- check that it verifies the formula

For $\mathrm{EG}_{[C]}\psi$ we are looking at infinite runs, but $(\sum \alpha_i \cdot \sharp\varphi_i) \sim k$ may change its truth value at most twice

Model-checking $\mathrm{CCTL}_{\mathcal{B}(\mathcal{C}_2)}$

$\mathcal{B}(\mathcal{C}_2) : \bigwedge\bigvee(\sharp\varphi - \sharp\psi) \sim k$

Theorem

The model-checking problem for $\mathrm{CCTL}_{\mathcal{B}(\mathcal{C}_2)}$ is undecidable

Reduction from the halting problem of a two-counter machine $\mathcal{M}$: $\mathcal{M}$ does not halt *if and only if* $q_1 \models_{\mathcal{S}_\mathcal{M}} \mathsf{EG}_{[C]}\bot$ with:

$$
\begin{aligned}
C &\stackrel{\text{def}}{=} (\sharp\mathsf{halt} \geq 1) \ \vee C_{bad} \\
C_{bad} &\stackrel{\text{def}}{=} \bigvee_{X \in \{\mathtt{C},\mathtt{D}\}}\big(\quad (\sharp\varphi_X^+ - \sharp\varphi_X^- < 0) \\
&\qquad\qquad \vee (\sharp\varphi_X^+ - \sharp\varphi_X^- > 0 \wedge \sharp\mathsf{ko}_X - \sharp\mathsf{ok}_X > 0)\big)
\end{aligned}
$$

Outline

1. CCTL
2. Expressiveness
3. Model checking
4. Freeze variables

CCTL with freeze variables

Definition

Let V be a set of variables.

$$\mathsf{CCTL}^{\mathsf{v}} \ni \varphi, \psi ::= P \mid \varphi \wedge \psi \mid \neg\varphi \mid z[\psi].\varphi \mid C \mid \mathsf{E}\varphi\mathsf{U}\psi \mid \mathsf{A}\varphi\mathsf{U}\psi$$

where $P \in \mathsf{AP}$ and C is a constraint $\sum_{i=1}^{\ell} \alpha_i \cdot z_i \sim c$
with $z_i \in V$, $\alpha_i, c \in \mathbb{N}$, and $\sim \in \{<, \leq, =, \geq, >\}$.

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For example:

$$\mathsf{EF}_{[\sharp P \leq 5 \wedge \sharp P' > 2]} A \equiv z[P].z'[P'].\mathsf{EF}(z \leq 5 \wedge z' > 2 \wedge A)$$

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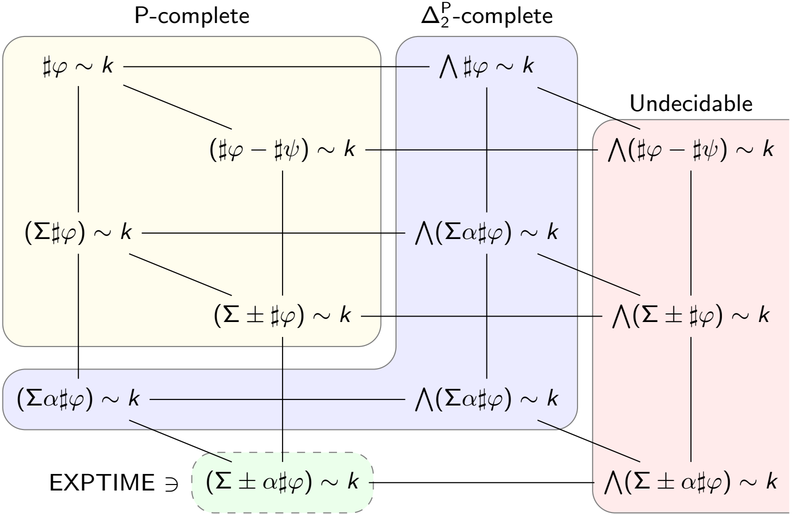
Theorem

Model checking closed $CCTL^{\mathsf{v}}$ formulas is PSPACE*-complete.*

Conclusion


P-complete
Δ_2^P-complete
Undecidable
$\sharp\varphi \sim k$
$\bigwedge \sharp\varphi \sim k$
$(\sharp\varphi - \sharp\psi) \sim k$
$\bigwedge(\sharp\varphi - \sharp\psi) \sim k$
$(\Sigma\sharp\varphi) \sim k$
$\bigwedge(\Sigma\alpha\sharp\varphi) \sim k$
$(\Sigma \pm \sharp\varphi) \sim k$
$\bigwedge(\Sigma \pm \sharp\varphi) \sim k$
$(\Sigma\alpha\sharp\varphi) \sim k$
$\bigwedge(\Sigma\alpha\sharp\varphi) \sim k$
EXPTIME $\ni$
$(\Sigma \pm \alpha\sharp\varphi) \sim k$
$\bigwedge(\Sigma \pm \alpha\sharp\varphi) \sim k$