On recognizable trace languages

Pascal Weil
(joint work with M. Kufleitner, Stuttgart)

LaBRI, Université de Bordeaux and CNRS

Computer Science and Engineering Department, IIT Delhi

Chennai, January 2009

Outline

- Trace languages, a model for concurrent behaviors. No justification needed for this distinguished audience

Outline

- Trace languages, a model for concurrent behaviors. No justification needed for this distinguished audience
- The algebraic approach to recognizability is an essential tool for word languages; is it also the case for trace languages?

Outline

- Trace languages, a model for concurrent behaviors. No justification needed for this distinguished audience
- The algebraic approach to recognizability is an essential tool for word languages; is it also the case for trace languages?
- the basic definitions: do trace languages have a syntactic monoid? how are varieties defined?

Outline

- Trace languages, a model for concurrent behaviors. No justification needed for this distinguished audience
- The algebraic approach to recognizability is an essential tool for word languages; is it also the case for trace languages?
- the basic definitions: do trace languages have a syntactic monoid? how are varieties defined?
- why have these definitions not given rise to a full-fledge theory as for word languages?

Outline

- Trace languages, a model for concurrent behaviors. No justification needed for this distinguished audience
- The algebraic approach to recognizability is an essential tool for word languages; is it also the case for trace languages?
- the basic definitions: do trace languages have a syntactic monoid? how are varieties defined?
- why have these definitions not given rise to a full-fledge theory as for word languages?
- we propose a new tool to discuss trace languages (built on the old framework of the syntactic monoid), we see how it allows us to give a robust framework of the classification of recognizable trace languages, and we give a first set of applications

Outline

- Trace languages, a model for concurrent behaviors. No justification needed for this distinguished audience
- The algebraic approach to recognizability is an essential tool for word languages; is it also the case for trace languages?
- the basic definitions: do trace languages have a syntactic monoid? how are varieties defined?
- why have these definitions not given rise to a full-fledge theory as for word languages?
- we propose a new tool to discuss trace languages (built on the old framework of the syntactic monoid), we see how it allows us to give a robust framework of the classification of recognizable trace languages, and we give a first set of applications
- this is on-going work (still rough around the edges...)

Traces: a classical model

- Independence alphabet (A, I): A is a finite alphabet, I an irreflexive and symmetric relation on A

Traces: a classical model

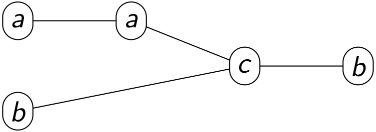
- Independence alphabet (A, I): A is a finite alphabet, I an irreflexive and symmetric relation on A
- the trace monoid $\mathbb{M}(A, I)$ is the quotient of A^* by the congruence generated by $ab = ba$ whenever $(a, b) \in I$

Traces: a classical model

- Independence alphabet (A, I): A is a finite alphabet, I an irreflexive and symmetric relation on A
- the trace monoid $\mathbb{M}(A, I)$ is the quotient of A^* by the congruence generated by $ab = ba$ whenever $(a, b) \in I$
- traces are one of the most important models used to represent concurrent behavior

Traces: a classical model

- Independence alphabet (A, I): A is a finite alphabet, I an irreflexive and symmetric relation on A
- the trace monoid $\mathbb{M}(A, I)$ is the quotient of A^* by the congruence generated by $ab = ba$ whenever $(a, b) \in I$
- traces are one of the most important models used to represent concurrent behavior
- Each trace is naturally represented as a poset. If $A = \{a, b, c\}$ and $I = \{(a, b), (b, a)\}$, then $abacb$ is represented by



Recognizable trace languages: a classical notion

Let $\mu\colon A^* \longrightarrow \mathbb{M}(A, I)$

- A trace language $L \subseteq \mathbb{M}(A, I)$ is recognizable if its set of linearizations is recognizable, that is, if $\mu^{-1}(L)$ is recognizable.

Recognizable trace languages: a classical notion

Let $\mu\colon A^* \longrightarrow \mathbb{M}(A, I)$

- A trace language $L \subseteq \mathbb{M}(A, I)$ is recognizable if its set of linearizations is recognizable, that is, if $\mu^{-1}(L)$ is recognizable.
- Equivalent to the monoid-theoretic notion: L is recognizable if there exists $\varphi\colon \mathbb{M}(A, I) \to M$ into a finite monoid such that $L = \varphi^{-1}\varphi(L)$. Trace languages have a syntactic monoid.

Recognizable trace languages: a classical notion

Let $\mu\colon A^* \longrightarrow \mathbb{M}(A, I)$

- A trace language $L \subseteq \mathbb{M}(A, I)$ is recognizable if its set of linearizations is recognizable, that is, if $\mu^{-1}(L)$ is recognizable.
- Equivalent to the monoid-theoretic notion: L is recognizable if there exists $\varphi\colon \mathbb{M}(A, I) \to M$ into a finite monoid such that $L = \varphi^{-1}\varphi(L)$. Trace languages have a syntactic monoid.
- Recognizability is equivalent with MSO-definability (Thomas, 1989)

Recognizable trace languages: a classical notion

Let $\mu\colon A^* \longrightarrow \mathbb{M}(A, I)$

- A trace language $L \subseteq \mathbb{M}(A, I)$ is recognizable if its set of linearizations is recognizable, that is, if $\mu^{-1}(L)$ is recognizable.
- Equivalent to the monoid-theoretic notion: L is recognizable if there exists $\varphi\colon \mathbb{M}(A, I) \rightarrow M$ into a finite monoid such that $L = \varphi^{-1}\varphi(L)$. Trace languages have a syntactic monoid.
- Recognizability is equivalent with MSO-definability (Thomas, 1989)
- Automata: see the notion of diamond property in automata; equivalence with a beautiful model of automata which captures the notion of independence: Zielonka's automata

Recognizable trace languages: a classical notion

Let $\mu\colon A^* \longrightarrow \mathbb{M}(A, I)$

- A trace language $L \subseteq \mathbb{M}(A, I)$ is recognizable if its set of linearizations is recognizable, that is, if $\mu^{-1}(L)$ is recognizable.
- Equivalent to the monoid-theoretic notion: L is recognizable if there exists $\varphi\colon \mathbb{M}(A, I) \to M$ into a finite monoid such that $L = \varphi^{-1}\varphi(L)$. Trace languages have a syntactic monoid.
- Recognizability is equivalent with MSO-definability (Thomas, 1989)
- Automata: see the notion of diamond property in automata; equivalence with a beautiful model of automata which captures the notion of independence: Zielonka's automata
- Rational expressions: there is a problem. If a, b are independent letters, then $(ab)^*$ is not recognizable, see Ochmański's concurrent rational expressions.

A classification of recognizable trace languages?

- So we seem to be in a similar situation to word languages: the algebraic properties of the syntactic monoid should be useful to characterize and to decide significant classes of recognizable trace languages, in a framework similar to Eilenberg's variety theory.

A classification of recognizable trace languages?

- So we seem to be in a similar situation to word languages: the algebraic properties of the syntactic monoid should be useful to characterize and to decide significant classes of recognizable trace languages, in a framework similar to Eilenberg's variety theory.
- and indeed Guaiana, Restivo, Salemi showed that star-free trace languages are characterized by the aperiodicity of their syntactic monoid. Ebinger, Muscholl showed that this class coincides with FO-definable trace languages.

A classification of recognizable trace languages?

- So we seem to be in a similar situation to word languages: the algebraic properties of the syntactic monoid should be useful to characterize and to decide significant classes of recognizable trace languages, in a framework similar to Eilenberg's variety theory.
- and indeed Guaiana, Restivo, Salemi showed that star-free trace languages are characterized by the aperiodicity of their syntactic monoid. Ebinger, Muscholl showed that this class coincides with FO-definable trace languages.
- But that is essentially the only example of such a correspondence (until Kufleitner's 2006 result). There has been no satisfactory Eilenberg-like statement,... Why?

So what goes on in words?

- Schützenberger's result on star-free vs. aperiodic, is an instance of a general correspondence

So what goes on in words?

- Schützenberger's result on star-free vs. aperiodic, is an instance of a general correspondence
- Variety of languages: a class of recognizable languages closed under Boolean operations, left and right residuals and inverse morphisms

So what goes on in words?

- Schützenberger's result on star-free vs. aperiodic, is an instance of a general correspondence
- Variety of languages: a class of recognizable languages closed under Boolean operations, left and right residuals and inverse morphisms
- Pseudovariety of monoids: a class of finite monoids closed under submonoids, homomorphic images and finite direct products.

So what goes on in words?

- Schützenberger's result on star-free vs. aperiodic, is an instance of a general correspondence
- Variety of languages: a class of recognizable languages closed under Boolean operations, left and right residuals and inverse morphisms
- Pseudovariety of monoids: a class of finite monoids closed under submonoids, homomorphic images and finite direct products.
- Eilenberg's theorem: (a) the languages whose syntactic monoid lies in a given pseudovariety of monoids V form a variety of languages $\mathcal{V}$; (b) the correspondence $\mathsf{V} \mapsto \mathcal{V}$ is one-to-one and onto between pseudovarieties and varieties

So what goes on in words?

- Schützenberger's result on star-free vs. aperiodic, is an instance of a general correspondence
- Variety of languages: a class of recognizable languages closed under Boolean operations, left and right residuals and inverse morphisms
- Pseudovariety of monoids: a class of finite monoids closed under submonoids, homomorphic images and finite direct products.
- Eilenberg's theorem: (a) the languages whose syntactic monoid lies in a given pseudovariety of monoids V form a variety of languages $\mathcal{V}$; (b) the correspondence $\mathsf{V} \mapsto \mathcal{V}$ is one-to-one and onto between pseudovarieties and varieties
- a conceptual framework for many famous results: Simon on piecewise testable languages; Simon and McNaughton on locally testable languages; many others. . .

Extensions and applications (for word languages)

- Eilenberg's theory extends to operations on varieties of languages

Extensions and applications (for word languages)

- Eilenberg's theory extends to operations on varieties of languages
- *Variants*: semigroups instead of monoids; ordered monoids and positive varieties; $\mathcal{C}$-varieties

Extensions and applications (for word languages)

- Eilenberg's theory extends to operations on varieties of languages
- *Variants*: semigroups instead of monoids; ordered monoids and positive varieties; $\mathcal{C}$-varieties
- Let $\mathsf{V} \mapsto \mathcal{V}$, let Pol$\mathcal{V}$ be the class of unions of products of the form $L_0 a_1 L_1 \cdots a_k L_k$, where $k \geq 0$, the a_i are letters and the L_i are in $\mathcal{V}$. And let UPolV be be the class of unions of unambiguous products of the same form

Extensions and applications (for word languages)

- Eilenberg's theory extends to operations on varieties of languages
- *Variants*: semigroups instead of monoids; ordered monoids and positive varieties; $\mathcal{C}$-varieties
- Let $\mathsf{V} \mapsto \mathcal{V}$, let Pol$\mathcal{V}$ be the class of unions of products of the form $L_0 a_1 L_1 \cdots a_k L_k$, where $k \geq 0$, the a_i are letters and the L_i are in $\mathcal{V}$. And let UPolV be be the class of unions of unambiguous products of the same form
- Then UPolV is a variety of languages, and the corresponding pseudovariety of monoids is LI ⓜ V (computable, decidable if V is, etc). And Pol$\mathcal{V}$ is a positive variety and the corresponding pseudovariety of ordered monoids is $[\![x^\omega y x^\omega \leq x^\omega]\!]$ ⓜ V.

And the results on traces?

- Except for the result on star-free trace languages (1992), no result until Kufleitner's dissertation (2006).

And the results on traces?

- Except for the result on star-free trace languages (1992), no result until Kufleitner's dissertation (2006).
- Kufleitner shows an analogue of the results on $\mathcal{V} \mapsto \mathsf{V}$ vs $\mathrm{Pol}\mathcal{V} \mapsto [\![x^\omega y x^\omega \leq x^\omega]\!] \circledm \mathsf{V}$ but only when V consists only of commutative monoids

And the results on traces?

- Except for the result on star-free trace languages (1992), no result until Kufleitner's dissertation (2006).
- Kufleitner shows an analogue of the results on $\mathcal{V} \mapsto \mathsf{V}$ vs $\mathsf{Pol}\mathcal{V} \mapsto [\![x^\omega y x^\omega \leq x^\omega]\!] \circledm \mathsf{V}$ but only when V consists only of commutative monoids
- and an analogue of the results on $\mathcal{V} \mapsto \mathsf{V}$ vs $\mathsf{UPol}\mathsf{V} \mapsto \mathsf{LI} \circledm \mathsf{V}$ but only when $\mathsf{V} = \mathsf{J}_1$ (idempotent and commutative monoids)

And the results on traces?

- Except for the result on star-free trace languages (1992), no result until Kufleitner's dissertation (2006).
- Kufleitner shows an analogue of the results on $\mathcal{V} \mapsto \mathsf{V}$ vs $\mathsf{Pol}\mathcal{V} \mapsto [\![x^\omega y x^\omega \leq x^\omega]\!] \circledcirc \mathsf{V}$ but only when V consists only of commutative monoids
- and an analogue of the results on $\mathcal{V} \mapsto \mathsf{V}$ vs $\mathsf{UPolV} \mapsto \mathsf{LI} \circledcirc \mathsf{V}$ but only when $\mathsf{V} = \mathsf{J}_1$ (idempotent and commutative monoids)
- In fact, there is no notion of variety of trace languages with an Eilenberg-like theorem, to provide a clean framework

Why doesn't the theory extend (smoothly) to trace languages?

- Proofs on recognizable trace languages that mimick the proofs on word languages, usually stumble on elementary technical lemmas, that are obvious for morphisms defined on A^* and fail on $\mathbb{M}(A, I)$.

Why doesn't the theory extend (smoothly) to trace languages?

- Proofs on recognizable trace languages that mimick the proofs on word languages, usually stumble on elementary technical lemmas, that are obvious for morphisms defined on A^* and fail on $\mathbb{M}(A, I)$.
- Our idea is that the monoid-theoretic framework is not sufficient to deal with trace languages,

Why doesn't the theory extend (smoothly) to trace languages?

- Proofs on recognizable trace languages that mimick the proofs on word languages, usually stumble on elementary technical lemmas, that are obvious for morphisms defined on A^* and fail on $\mathbb{M}(A, I)$.
- Our idea is that the monoid-theoretic framework is not sufficient to deal with trace languages,
- that the trace monoids have more than a monoid structure: they also have an independence structure.

Independent traces: not a new idea

- Given (A, I) an independence alphabet (I irreflexive and symmetric), extend I to $\mathbb{M}(A, I)$ by saying that traces u and v are independent if $\text{alph}(u) \times \text{alph}(v) \subseteq I$. Then

Independent traces: not a new idea

- Given (A, I) an independence alphabet (I irreflexive and symmetric), extend I to $\mathbb{M}(A, I)$ by saying that traces u and v are independent if $\mathrm{alph}(u) \times \mathrm{alph}(v) \subseteq I$. Then
- $(u, v) \in I \implies uv = vu$

Independent traces: not a new idea

- Given (A, I) an independence alphabet (I irreflexive and symmetric), extend I to $\mathbb{M}(A, I)$ by saying that traces u and v are independent if $\mathrm{alph}(u) \times \mathrm{alph}(v) \subseteq I$. Then
- $(u, v) \in I \Longrightarrow uv = vu$
- $(u, vw) \in I \Longleftrightarrow (u, v) \in I$ and $(u, w) \in I$

Independent traces: not a new idea

- Given (A, I) an independence alphabet (I irreflexive and symmetric), extend I to $\mathbb{M}(A, I)$ by saying that traces u and v are independent if $\mathrm{alph}(u) \times \mathrm{alph}(v) \subseteq I$. Then
- $(u, v) \in I \Longrightarrow uv = vu$
- $(u, vw) \in I \Longleftrightarrow (u, v) \in I$ and $(u, w) \in I$
- $(u, u) \in I$ iff $u = 1$. And $(u, 1) \in I$ for each u.

We propose to abstract out this notion

- On $\mathbb{M}(A, I)$, all the information on the independence relation is contained in the alphabetic information.

We propose to abstract out this notion

- On $\mathbb{M}(A, I)$, all the information on the independence relation is contained in the alphabetic information.
- It has been considered several times in the literature (Diekert, Gastin, Muscholl, Petit, ...), to transfer this alphabetic information onto the finite monoids recognizing trace languages: if $\varphi\colon \mathbb{M}(A, I) \to M$ recognizes L, then so does $\varphi'\colon \mathbb{M}(A, I) \to M \times 2^A$, where $\varphi'(u) = (\varphi(u), \mathrm{alph}(u))$

We propose to abstract out this notion

- On $\mathbb{M}(A, I)$, all the information on the independence relation is contained in the alphabetic information.
- It has been considered several times in the literature (Diekert, Gastin, Muscholl, Petit, ...), to transfer this alphabetic information onto the finite monoids recognizing trace languages: if $\varphi\colon \mathbb{M}(A, I) \to M$ recognizes L, then so does $\varphi'\colon \mathbb{M}(A, I) \to M \times 2^A$, where $\varphi'(u) = (\varphi(u), \mathsf{alph}(u))$
- in certain technical situation, this idea helps, but it increases the size of the recognizing monoid, and the algebraic structure that it yields is intrinsically alphabet-dependent.

We propose to abstract out this notion

- On $\mathbb{M}(A, I)$, all the information on the independence relation is contained in the alphabetic information.
- It has been considered several times in the literature (Diekert, Gastin, Muscholl, Petit, ...), to transfer this alphabetic information onto the finite monoids recognizing trace languages: if $\varphi\colon \mathbb{M}(A, I) \to M$ recognizes L, then so does $\varphi'\colon \mathbb{M}(A, I) \to M \times 2^A$, where $\varphi'(u) = (\varphi(u), \mathrm{alph}(u))$
- in certain technical situation, this idea helps, but it increases the size of the recognizing monoid, and the algebraic structure that it yields is intrinsically alphabet-dependent.
- To have a proper algebraic framework, abstract that out!

Independence monoids: definition

- An *independence monoid*, or I-monoid is a pair (M, I) such that M is a monoid and I is a symmetric relation satisfying

Independence monoids: definition

- An *independence monoid*, or I-monoid is a pair (M, I) such that M is a monoid and I is a symmetric relation satisfying
- $(u, v) \in I \Longrightarrow uv = vu$

Independence monoids: definition

- An *independence monoid*, or I-monoid is a pair (M, I) such that M is a monoid and I is a symmetric relation satisfying
- $(u, v) \in I \Longrightarrow uv = vu$
- $(u, vw) \in I \Longleftrightarrow (u, v) \in I$ and $(u, w) \in I$

Independence monoids: definition

- An *independence monoid*, or I-monoid is a pair (M, I) such that M is a monoid and I is a symmetric relation satisfying
- $(u, v) \in I \Longrightarrow uv = vu$
- $(u, vw) \in I \Longleftrightarrow (u, v) \in I$ and $(u, w) \in I$
- $(u, u) \in I$ iff $u = 1$. And $(u, 1) \in I$ for each u.

Independence monoids: definition

- An *independence monoid*, or I-monoid is a pair (M, I) such that M is a monoid and I is a symmetric relation satisfying
- $(u, v) \in I \Longrightarrow uv = vu$
- $(u, vw) \in I \Longleftrightarrow (u, v) \in I$ and $(u, w) \in I$
- $(u, u) \in I$ iff $u = 1$. And $(u, 1) \in I$ for each u.
- Morphisms of I-monoids: $\varphi\colon (M, I) \to (N, J)$ is an I-morphism if $\varphi\colon M \to N$ is a monoid morphism, and if $(u, v) \in I \Longrightarrow (\varphi(u), \varphi(v)) \in J$
  Already considered for morphisms between trace monoids (Bruyère, de Felice, Guaiana)

Independence monoids: definition

- An *independence monoid*, or I-monoid is a pair (M, I) such that M is a monoid and I is a symmetric relation satisfying
- $(u, v) \in I \Longrightarrow uv = vu$
- $(u, vw) \in I \Longleftrightarrow (u, v) \in I$ and $(u, w) \in I$
- $(u, u) \in I$ iff $u = 1$. And $(u, 1) \in I$ for each u.
- Morphisms of I-monoids: $\varphi\colon (M, I) \to (N, J)$ is an I-morphism if $\varphi\colon M \to N$ is a monoid morphism, and if $(u, v) \in I \Longrightarrow (\varphi(u), \varphi(v)) \in J$
  Already considered for morphisms between trace monoids (Bruyère, de Felice, Guaiana)
- Also consider *strong I-morphisms*, where $(u, v) \in I \Longleftrightarrow (\varphi(u), \varphi(v)) \in J$.

Free I-monoids and skeleton monoids

- An I-monoid (M, J) is generated (resp. strongly generated) by (A, I) if there exists a map $\psi\colon A \to M$ such that $\psi(A)$ generates M and $\psi(I) \subseteq J$ (resp. and in addition $\psi^{-1}(J) \subseteq I$)

Free I-monoids and skeleton monoids

- An I-monoid (M, J) is generated (resp. strongly generated) by (A, I) if there exists a map $\psi\colon A \to M$ such that $\psi(A)$ generates M and $\psi(I) \subseteq J$ (resp. and in addition $\psi^{-1}(J) \subseteq I$)
- There is a largest I-monoid generated by (A, I), and it is $\mathbb{M}(A, I)$ (an initial object in the category of (A, I)-generated I-monoids)

Free I-monoids and skeleton monoids

- An I-monoid (M, J) is generated (resp. strongly generated) by (A, I) if there exists a map $\psi\colon A \to M$ such that $\psi(A)$ generates M and $\psi(I) \subseteq J$ (resp. and in addition $\psi^{-1}(J) \subseteq I$)
- There is a largest I-monoid generated by (A, I), and it is $\mathbb{M}(A, I)$ (an initial object in the category of (A, I)-generated I-monoids)
- Every I-monoid (M, I) has a well-defined least strong homomorphic image $\mathbb{S}(M, I)$, called its skeleton

Free I-monoids and skeleton monoids

- An I-monoid (M, J) is generated (resp. strongly generated) by (A, I) if there exists a map $\psi\colon A \to M$ such that $\psi(A)$ generates M and $\psi(I) \subseteq J$ (resp. and in addition $\psi^{-1}(J) \subseteq I$)
- There is a largest I-monoid generated by (A, I), and it is $\mathbb{M}(A, I)$ (an initial object in the category of (A, I)-generated I-monoids)
- Every I-monoid (M, I) has a well-defined least strong homomorphic image $\mathbb{S}(M, I)$, called its skeleton
- Skeleton monoids are idempotent and commutative, and they encapsulate some fundamental information on the independence structure of the I-monoid

With morphisms comes recognizability

Let (M, I) be an I-monoid and $L \subseteq M$.

- L is recognizable if $L = \varphi^{-1}\varphi(L)$ for a monoid morphism φ into a finite monoid, I-recognizable if φ is an I-morphism, strongly I-recognizable if φ is a strong I-morphism.

With morphisms comes recognizability

Let (M, I) be an I-monoid and $L \subseteq M$.

- L is recognizable if $L = \varphi^{-1}\varphi(L)$ for a monoid morphism φ into a finite monoid, I-recognizable if φ is an I-morphism, strongly I-recognizable if φ is a strong I-morphism.
- For finitely generated I-monoids, and in particular for the trace monoid $\mathbb{M}(A, I)$, these 3 notions are equivalent (luckily!)

With morphisms comes recognizability

Let (M, I) be an I-monoid and $L \subseteq M$.

- L is recognizable if $L = \varphi^{-1}\varphi(L)$ for a monoid morphism φ into a finite monoid, I-recognizable if φ is an I-morphism, strongly I-recognizable if φ is a strong I-morphism.
- For finitely generated I-monoids, and in particular for the trace monoid $\mathbb{M}(A, I)$, these 3 notions are equivalent (luckily!)
- But a recognizable trace language $L \subseteq \mathbb{M}(A, I)$ has a syntactic I-monoid that may be different from its usual syntactic monoid.

With morphisms comes recognizability

Let (M, I) be an I-monoid and $L \subseteq M$.

- L is recognizable if $L = \varphi^{-1}\varphi(L)$ for a monoid morphism φ into a finite monoid, I-recognizable if φ is an I-morphism, strongly I-recognizable if φ is a strong I-morphism.
- For finitely generated I-monoids, and in particular for the trace monoid $\mathbb{M}(A, I)$, these 3 notions are equivalent (luckily!)
- But a recognizable trace language $L \subseteq \mathbb{M}(A, I)$ has a syntactic I-monoid that may be different from its usual syntactic monoid.
- Syntactic congruence of L. Let $u \sim_L v$ iff
  $xuy \in L \iff xvy \in L$ for all $x, y \in \mathbb{M}(A, I)$
  $(x, u) \in I \iff (x, v) \in I$ for all $x \in \mathbb{M}(A, I)$

With morphisms comes recognizability

Let (M, I) be an I-monoid and $L \subseteq M$.

- L is recognizable if $L = \varphi^{-1}\varphi(L)$ for a monoid morphism φ into a finite monoid, I-recognizable if φ is an I-morphism, strongly I-recognizable if φ is a strong I-morphism.
- For finitely generated I-monoids, and in particular for the trace monoid $\mathbb{M}(A, I)$, these 3 notions are equivalent (luckily!)
- But a recognizable trace language $L \subseteq \mathbb{M}(A, I)$ has a syntactic I-monoid that may be different from its usual syntactic monoid.
- Syntactic congruence of L. Let $u \sim_L v$ iff
  $xuy \in L \iff xvy \in L$ for all $x, y \in \mathbb{M}(A, I)$
  $(x, u) \in I \iff (x, v) \in I$ for all $x \in \mathbb{M}(A, I)$
- the syntactic I-morphism $\mathbb{M}(A, I) \to \mathbb{M}(A, I)/\sim_L$ is always a strong I-morphism

An Eilenberg theorem for trace languages

Let us (re)define varieties in a now natural fashion

- Let a variety of trace languages be a collection $\mathcal{V} = (\mathcal{V}(A, I))_{(A,I)}$ of recognizable trace languages closed under Boolean operations, left and right residuals and inverse I-morphisms

An Eilenberg theorem for trace languages

Let us (re)define varieties in a now natural fashion

- Let a variety of trace languages be a collection $\mathcal{V} = (\mathcal{V}(A, I))_{(A,I)}$ of recognizable trace languages closed under Boolean operations, left and right residuals and inverse I-morphisms
- Let a pseudovariety of I-monoids be a class V of finite I-monoids closed under taking sub-I-monoids, finite direct products and images under strong I-morphisms. We also require that V contains all skeleton monoids

An Eilenberg theorem for trace languages

Let us (re)define varieties in a now natural fashion

- Let a variety of trace languages be a collection $\mathcal{V} = (\mathcal{V}(A, I))_{(A,I)}$ of recognizable trace languages closed under Boolean operations, left and right residuals and inverse I-morphisms
- Let a pseudovariety of I-monoids be a class V of finite I-monoids closed under taking sub-I-monoids, finite direct products and images under strong I-morphisms. We also require that V contains all skeleton monoids
- If V is a pseudovariety, let $\mathcal{V}$ = all trace languages recognized by an I-monoid in V. Then $\mathcal{V}$ = all trace languages whose syntactic I-monoid is in V and

An Eilenberg theorem for trace languages

Let us (re)define varieties in a now natural fashion

- Let a variety of trace languages be a collection $\mathcal{V} = (\mathcal{V}(A, I))_{(A,I)}$ of recognizable trace languages closed under Boolean operations, left and right residuals and inverse I-morphisms
- Let a pseudovariety of I-monoids be a class V of finite I-monoids closed under taking sub-I-monoids, finite direct products and images under strong I-morphisms. We also require that V contains all skeleton monoids
- If V is a pseudovariety, let $\mathcal{V}$ = all trace languages recognized by an I-monoid in V. Then $\mathcal{V}$ = all trace languages whose syntactic I-monoid is in V and
- V $\mapsto \mathcal{V}$ is a one-to-one and onto correspondence between varieties of trace languages and pseudovarieties of I-monoids

Identities for pseudovarieties of I-monoids

- Pseudovarieties of I-monoids can be defined by sequences of identities (and probably pseudo-identities in the profinite completion)

Identities for pseudovarieties of I-monoids

- Pseudovarieties of I-monoids can be defined by sequences of identities (and probably pseudo-identities in the profinite completion)
- Example: $[\![yxyzy = yzyxy]\!]$, which is in fact equal to $[\![xz = zx]\!]$

Identities for pseudovarieties of I-monoids

- Pseudovarieties of I-monoids can be defined by sequences of identities (and probably pseudo-identities in the profinite completion)
- Example: $[\![yxyzy = yzyxy]\!]$, which is in fact equal to $[\![xz = zx]\!]$
- but also $[\![yxyzy = yzyxy]\!]_{(x,z)\in I}$, which is a different pseudoidentity

Varieties of word languages vs varieties of trace languages

- A word language $L \subseteq A^*$ is a particular case of a trace language (over (A, I) with $I = \emptyset$)

Varieties of word languages vs varieties of trace languages

- A word language $L \subseteq A^*$ is a particular case of a trace language (over (A, I) with $I = \emptyset$)
- Say that a variety $\mathcal{W}$ of trace languages is *independence-blind* if $L \in \mathcal{W}(A, I)$ iff $\mu^{-1}(L) \in \mathcal{W}(A, \emptyset)$, where $\mu \colon A^* \to \mathbb{M}(A, I)$

Varieties of word languages vs varieties of trace languages

- A word language $L \subseteq A^*$ is a particular case of a trace language (over (A, I) with $I = \emptyset$)
- Say that a variety $\mathcal{W}$ of trace languages is *independence-blind* if $L \in \mathcal{W}(A, I)$ iff $\mu^{-1}(L) \in \mathcal{W}(A, \emptyset)$, where $\mu\colon A^* \to \mathbb{M}(A, I)$
- Let V be a pseudovariety of monoids (containing J_1) and $\mathcal{V}$ the corresponding variety of word languages. Let V^{ind} be class of all I-monoids (M, I) such that $M \in \mathsf{V}$ and let $\mathcal{V}^{ind}$ be the corresponding variety of trace languages

Varieties of word languages vs varieties of trace languages

- A word language $L \subseteq A^*$ is a particular case of a trace language (over (A, I) with $I = \emptyset$)
- Say that a variety $\mathcal{W}$ of trace languages is *independence-blind* if $L \in \mathcal{W}(A, I)$ iff $\mu^{-1}(L) \in \mathcal{W}(A, \emptyset)$, where $\mu \colon A^* \to \mathbb{M}(A, I)$
- Let V be a pseudovariety of monoids (containing J_1) and $\mathcal{V}$ the corresponding variety of word languages. Let V^{ind} be class of all I-monoids (M, I) such that $M \in \mathsf{V}$ and let $\mathcal{V}^{ind}$ be the corresponding variety of trace languages
- Then $L \in \mathcal{V}^{ind}$ iff $\mu^{-1}(L) \in \mathcal{V}$

Varieties of word languages vs varieties of trace languages

- A word language $L \subseteq A^*$ is a particular case of a trace language (over (A, I) with $I = \emptyset$)
- Say that a variety $\mathcal{W}$ of trace languages is *independence-blind* if $L \in \mathcal{W}(A, I)$ iff $\mu^{-1}(L) \in \mathcal{W}(A, \emptyset)$, where $\mu\colon A^* \to \mathbb{M}(A, I)$
- Let V be a pseudovariety of monoids (containing J_1) and $\mathcal{V}$ the corresponding variety of word languages. Let V^{ind} be class of all I-monoids (M, I) such that $M \in \mathsf{V}$ and let $\mathcal{V}^{ind}$ be the corresponding variety of trace languages
- Then $L \in \mathcal{V}^{ind}$ iff $\mu^{-1}(L) \in \mathcal{V}$
- $\mathsf{V} \mapsto \mathsf{V}^{ind}$ maps injectively the lattice of pseudovarieties of monoids into the lattice of pseudovarieties of I-monoids, and the corresponding map $\mathcal{V} \mapsto \mathcal{V}^{ind}$ is onto the independence-blind varieties of trace languages

What can we hope to do with this theory?

- Of course, this variety-theoretic framework is appropriate to account for the known correspondence Star-free = FO[$<$]-definable trace languages = Ap^{ind}

What can we hope to do with this theory?

- Of course, this variety-theoretic framework is appropriate to account for the known correspondence Star-free = FO[$<$]-definable trace languages = Ap^{ind}
- More examples can be derived, and we have concentrated on discussing operations on trace language varieties

What can we hope to do with this theory?

- Of course, this variety-theoretic framework is appropriate to account for the known correspondence Star-free = FO[$<$]-definable trace languages = Ap^{ind}
- More examples can be derived, and we have concentrated on discussing operations on trace language varieties
- The hope is that all the (algebraic) hard work has already been done in the theory of word languages, and we can re-use the same concepts, techniques and proofs for trace languages

What can we hope to do with this theory?

- Of course, this variety-theoretic framework is appropriate to account for the known correspondence Star-free = FO[$<$]-definable trace languages = Ap^{ind}
- More examples can be derived, and we have concentrated on discussing operations on trace language varieties
- The hope is that all the (algebraic) hard work has already been done in the theory of word languages, and we can re-use the same concepts, techniques and proofs for trace languages
- On-going work on wreath products

What can we hope to do with this theory?

- Of course, this variety-theoretic framework is appropriate to account for the known correspondence Star-free = FO[$<$]-definable trace languages = Ap^{ind}
- More examples can be derived, and we have concentrated on discussing operations on trace language varieties
- The hope is that all the (algebraic) hard work has already been done in the theory of word languages, and we can re-use the same concepts, techniques and proofs for trace languages
- On-going work on wreath products
- Some positive results on Malcev products

It works for Malcev products and polynomial closure! (1/2)

- Let $\mathcal{V}$ be a variety of trace languages, and let $\mathrm{Pol}\mathcal{V}(A, I)$ consist of all unions of products of the form $L_0 a_1 L_1 \cdots a_n L_n$ (as in the word case)

It works for Malcev products and polynomial closure! (1/2)

- Let $\mathcal{V}$ be a variety of trace languages, and let $\mathrm{Pol}\mathcal{V}(A, I)$ consist of all unions of products of the form $L_0 a_1 L_1 \cdots a_n L_n$ (as in the word case)
- in the word/monoid case, $\mathrm{Pol}\mathcal{V}$ is a positive variety, and the corresponding ordered pseudovariety is described by a Malcev product. Recall the following

It works for Malcev products and polynomial closure! (1/2)

- Let $\mathcal{V}$ be a variety of trace languages, and let $\mathrm{Pol}\mathcal{V}(A, I)$ consist of all unions of products of the form $L_0 a_1 L_1 \cdots a_n L_n$ (as in the word case)
- in the word/monoid case, $\mathrm{Pol}\mathcal{V}$ is a positive variety, and the corresponding ordered pseudovariety is described by a Malcev product. Recall the following
- a monoid M is in $\mathsf{W} \,ⓜ\, \mathsf{V}$ if there exists a relational morphism $\tau\colon M \to N$ with $N \in \mathsf{V}$ and $\tau^{-1}(e) \in \mathsf{W}$ for each $e = e^2 \in N\ldots$

It works for Malcev products and polynomial closure! (1/2)

- Let $\mathcal{V}$ be a variety of trace languages, and let $\mathrm{Pol}\mathcal{V}(A, I)$ consist of all unions of products of the form $L_0 a_1 L_1 \cdots a_n L_n$ (as in the word case)
- in the word/monoid case, $\mathrm{Pol}\mathcal{V}$ is a positive variety, and the corresponding ordered pseudovariety is described by a Malcev product. Recall the following
- a monoid M is in $\mathsf{W} \circledcirc \mathsf{V}$ if there exists a relational morphism $\tau\colon M \to N$ with $N \in \mathsf{V}$ and $\tau^{-1}(e) \in \mathsf{W}$ for each $e = e^2 \in N$...
- ...and a relational morphism $\tau\colon M \to N$ is a relation such that $\mathrm{graph}(\tau)$ is a submonoid of $M \times N$ whose first projection is onto M

It works for Malcev products and polynomial closure! (2/2)

- If (M, I) and (N, J) are I-monoids, define a relational I-morphism $\tau\colon (M, I) \to (N, J)$ to be a relational morphism such that $(u, v) \in I$ implies $\tau(u) \times \tau(v) \subseteq J$

It works for Malcev products and polynomial closure! (2/2)

- If (M, I) and (N, J) are I-monoids, define a relational I-morphism $\tau\colon (M, I) \to (N, J)$ to be a relational morphism such that $(u, v) \in I$ implies $\tau(u) \times \tau(v) \subseteq J$
- and if V, W are pseudovarieites of (ordered) I-monoids, let (M, I) be an element of $\mathsf{W} \circledm \mathsf{V}$ if there exists a relational I-morphism $\tau\colon (M, I) \to (N, J)$ with $(N, J) \in \mathsf{V}$ and $(\tau^{-1}(e), I) \in \mathsf{W}$ for each $e = e^2 \in N$

It works for Malcev products and polynomial closure! (2/2)

- If (M, I) and (N, J) are I-monoids, define a relational I-morphism $\tau\colon (M, I) \to (N, J)$ to be a relational morphism such that $(u, v) \in I$ implies $\tau(u) \times \tau(v) \subseteq J$
- and if V, W are pseudovarieites of (ordered) I-monoids, let (M, I) be an element of $\mathsf{W} \circledm \mathsf{V}$ if there exists a relational I-morphism $\tau\colon (M, I) \to (N, J)$ with $(N, J) \in \mathsf{V}$ and $(\tau^{-1}(e), I) \in \mathsf{W}$ for each $e = e^2 \in N$
- Then we get the exact same statement as in the word case, with essentially the same proof ideas

It works for Malcev products and polynomial closure! (2/2)

- If (M, I) and (N, J) are I-monoids, define a relational I-morphism $\tau\colon (M, I) \to (N, J)$ to be a relational morphism such that $(u, v) \in I$ implies $\tau(u) \times \tau(v) \subseteq J$
- and if V, W are pseudovarieites of (ordered) I-monoids, let (M, I) be an element of $\mathsf{W} \ⓜ \mathsf{V}$ if there exists a relational I-morphism $\tau\colon (M, I) \to (N, J)$ with $(N, J) \in \mathsf{V}$ and $(\tau^{-1}(e), I) \in \mathsf{W}$ for each $e = e^2 \in N$
- Then we get the exact same statement as in the word case, with essentially the same proof ideas
- If $\mathcal{V} \mapsto \mathsf{V}$, then $\mathrm{Pol}\mathcal{V} \mapsto [\![x^\omega y x^\omega \leq x^\omega]\!] \ⓜ \mathsf{V}$ generalizing Kufleitner's earlier results (only for V commutative)

As in the word case, logical consequences

- Let $\Sigma_n[E]$ be the class of first-order formulas in normal prenex form, with n blocks of quantifier, starting with a block of existential quantifiers — where E is the edge relation in the dependence graph of a trace. Let $\Sigma_n[E]$ also denote the class of trace languages definable by such formulas

As in the word case, logical consequences

- Let $\Sigma_n[E]$ be the class of first-order formulas in normal prenex form, with n blocks of quantifier, starting with a block of existential quantifiers — where E is the edge relation in the dependence graph of a trace. Let $\Sigma_n[E]$ also denote the class of trace languages definable by such formulas
- Then $\Sigma_n[E]$ is a positive variety of trace languages, $\mathrm{Bool}(\Sigma_n[E])$ is a variety of trace languages, and $\Sigma_{n+1} = \mathrm{PolBool}(\Sigma_n[E])$

As in the word case, logical consequences

- Let $\Sigma_n[E]$ be the class of first-order formulas in normal prenex form, with n blocks of quantifier, starting with a block of existential quantifiers — where E is the edge relation in the dependence graph of a trace. Let $\Sigma_n[E]$ also denote the class of trace languages definable by such formulas
- Then $\Sigma_n[E]$ is a positive variety of trace languages, $\mathsf{Bool}(\Sigma_n[E])$ is a variety of trace languages, and $\Sigma_{n+1} = \mathsf{PolBool}(\Sigma_n[E])$
- Moreover and if $\mathsf{V}_n, \mathsf{BV}_n$ are the corresponding pseudovarieties of (ordered) I-monoids, then $V_{n+1} = [\![x^\omega y x^\omega \leq x^\omega]\!] ⓜ \mathsf{BV}_n$

As in the word case, logical consequences

- Let $\Sigma_n[E]$ be the class of first-order formulas in normal prenex form, with n blocks of quantifier, starting with a block of existential quantifiers — where E is the edge relation in the dependence graph of a trace. Let $\Sigma_n[E]$ also denote the class of trace languages definable by such formulas
- Then $\Sigma_n[E]$ is a positive variety of trace languages, $\mathsf{Bool}(\Sigma_n[E])$ is a variety of trace languages, and $\Sigma_{n+1} = \mathsf{PolBool}(\Sigma_n[E])$
- Moreover and if $\mathsf{V}_n, \mathsf{BV}_n$ are the corresponding pseudovarieties of (ordered) I-monoids, then $V_{n+1} = [\![x^\omega y x^\omega \leq x^\omega]\!] \circledast \mathsf{BV}_n$
- Decidability results for the lower levels (up to V_2) should follow

Thank you for your attention!