Quantitative timed games

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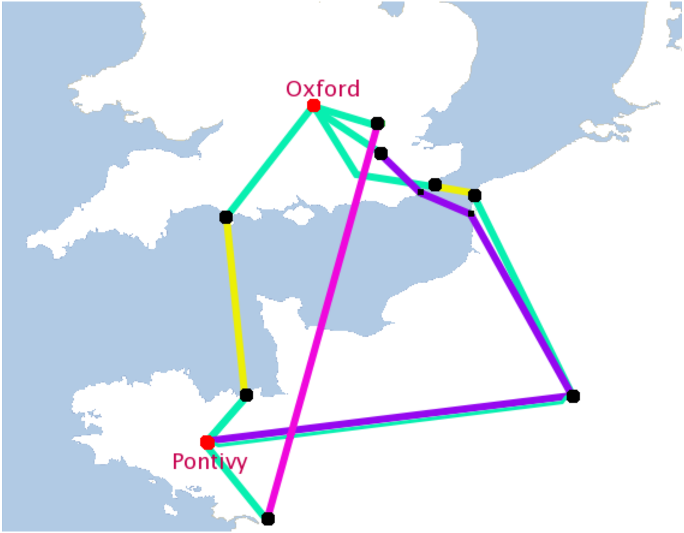
Based on joint works with Thomas Brihaye, Véronique Bruyère,
Uli Fahrenberg, Kim G. Larsen, Nicolas Markey,
Jean-François Raskin, Jirí Srba, and Jacob Illum Rasmussen

Outline

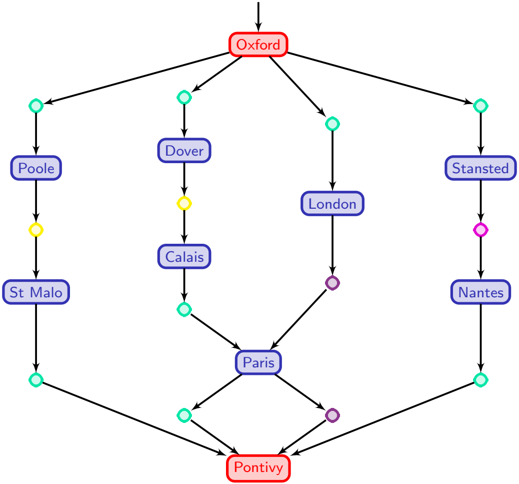
A starting example



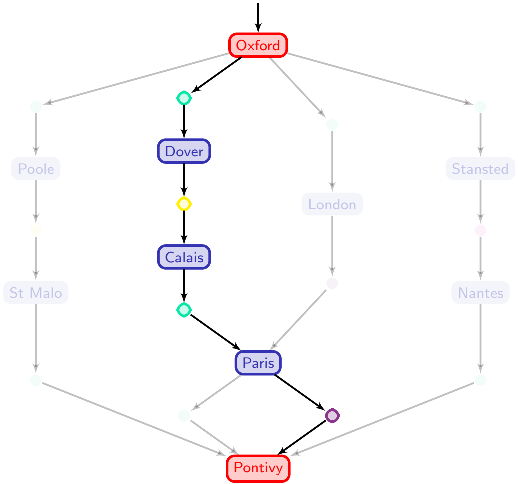
Natural questions

- **Can I reach Pontivy from Oxford?**
- **What is the minimal time to reach Pontivy from Oxford?**
- **What is the minimal fuel consumption to reach Pontivy from Oxford?**
- **What if there is an unexpected event?**
- **Can I use my computer all the way?**

A first model of the system

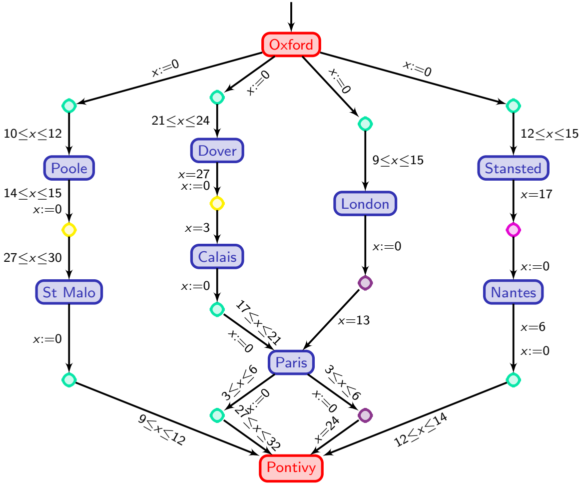


Can I reach Pontivy from Oxford?

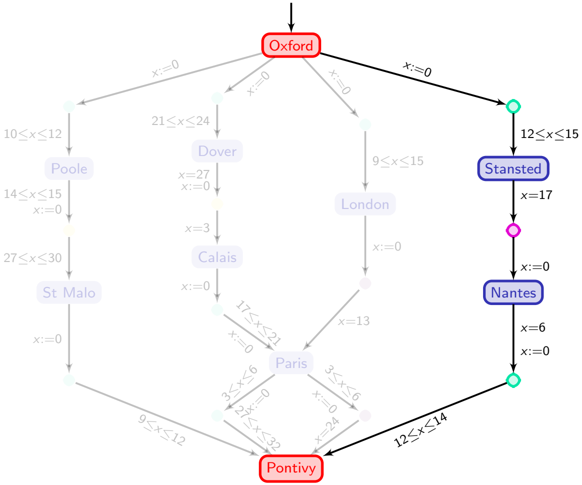


This is a reachability question in a finite graph: Yes, I can!

A second model of the system

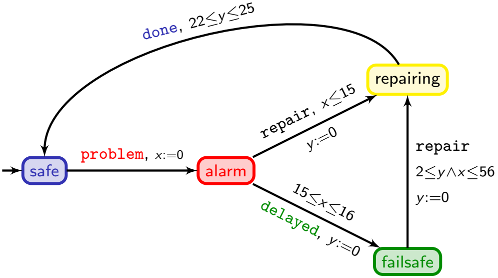


How long will that take?



It is a reachability (and optimization) question
in a timed automaton: at least 350mn = 5h50mn!

An example of a timed automaton



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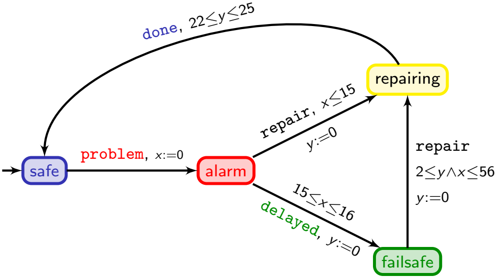

	safe
x	0
y	0

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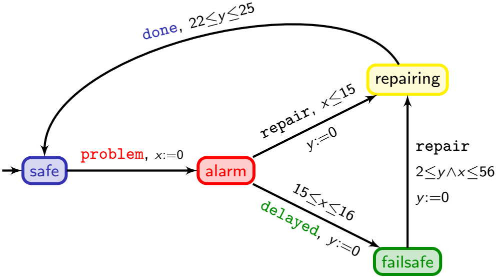
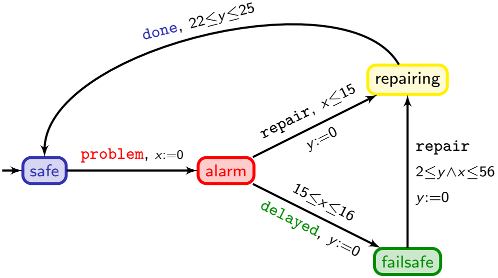


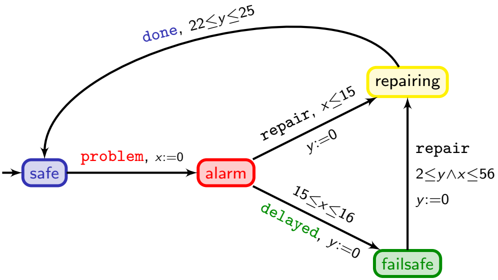
	safe	$\xrightarrow{23}$	safe
x	0		23
y	0		23

An example of a timed automaton



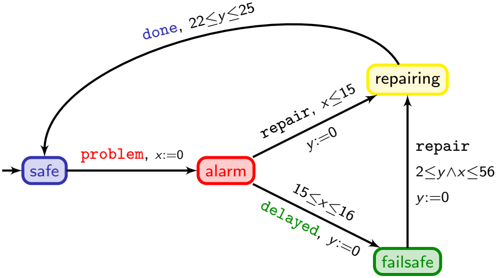
	safe	$\xrightarrow{23}$	safe	$\xrightarrow{\texttt{problem}}$	alarm
x	0		23		0
y	0		23		23

An example of a timed automaton



	safe	$\xrightarrow{23}$	safe	$\xrightarrow{\texttt{problem}}$	alarm	$\xrightarrow{15.6}$	alarm
x	0		23		0		15.6
y	0		23		23		38.6

An example of a timed automaton



	safe	$\xrightarrow{23}$	safe	$\xrightarrow{\texttt{problem}}$	alarm	$\xrightarrow{15.6}$	alarm	$\xrightarrow{\texttt{delayed}}$	failsafe	
x	0		23		0		15.6		15.6	···
y	0		23		23		38.6		0	

	failsafe
···	15.6
	0

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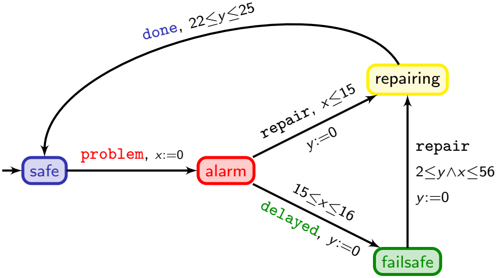
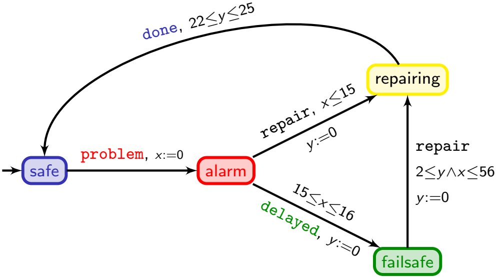


	safe	$\xrightarrow{23}$	safe	$\xrightarrow{\texttt{problem}}$	alarm	$\xrightarrow{15.6}$	alarm	$\xrightarrow{\texttt{delayed}}$	failsafe	
x	0		23		0		15.6		15.6	⋯
y	0		23		23		38.6		0	

	failsafe	$\xrightarrow{2.3}$	failsafe
⋯	15.6		17.9
	0		2.3

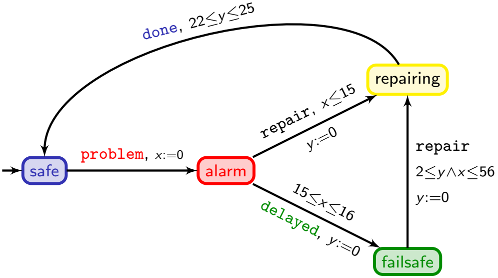
An example of a timed automaton



	safe	$\xrightarrow{23}$	safe	$\xrightarrow{\text{problem}}$	alarm	$\xrightarrow{15.6}$	alarm	$\xrightarrow{\text{delayed}}$	failsafe		
x	0		23		0		15.6		15.6	···
y	0		23		23		38.6		0	

	failsafe	$\xrightarrow{2.3}$	failsafe	$\xrightarrow{\text{repair}}$	repairing
···	15.6		17.9		17.9
	0		2.3		0

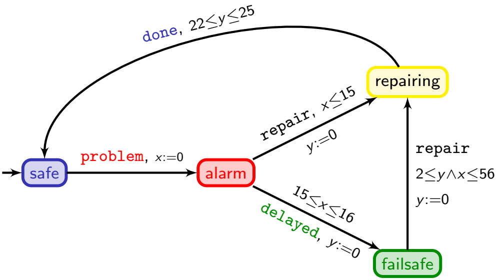
An example of a timed automaton



	safe	$\xrightarrow{23}$	safe	$\xrightarrow{\texttt{problem}}$	alarm	$\xrightarrow{15.6}$	alarm	$\xrightarrow{\texttt{delayed}}$	failsafe		
x	0		23		0		15.6		15.6		···
y	0		23		23		38.6		0		

	failsafe	$\xrightarrow{2.3}$	failsafe	$\xrightarrow{\texttt{repair}}$	repairing	$\xrightarrow{22.1}$	repairing
···	15.6		17.9		17.9		40
	0		2.3		0		22.1

An example of a timed automaton



	safe	$\xrightarrow{23}$	safe	$\xrightarrow{\texttt{problem}}$	alarm	$\xrightarrow{15.6}$	alarm	$\xrightarrow{\texttt{delayed}}$	failsafe		
x	0		23		0		15.6		15.6		⋯
y	0		23		23		38.6		0		

	failsafe	$\xrightarrow{2.3}$	failsafe	$\xrightarrow{\texttt{repair}}$	repairing	$\xrightarrow{22.1}$	repairing	$\xrightarrow{\texttt{done}}$	safe
⋯	15.6		17.9		17.9		40		40
	0		2.3		0		22.1		22.1

Timed automata

Theorem [AD90,CY92]

The (time-optimal) reachability problem is decidable (and PSPACE-complete) for timed automata.

[AD90] Alur, Dill. Automata for modeling real-time systems *(ICALP'90)*.
[CY92] Courcoubetis, Yannakakis. Minimum and maximum delay problems in real-time systems *(Formal Methods in System Design)*.

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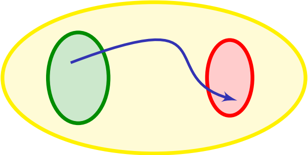
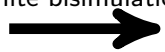
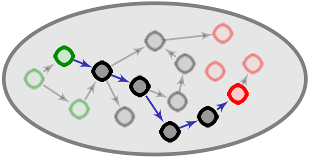


timed automaton





large (but finite) automaton
(region automaton)

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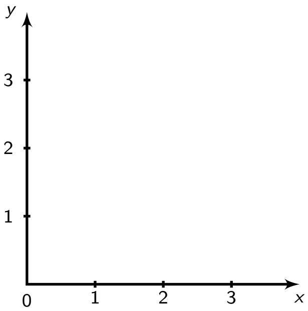
[CY92] Courcoubetis, Yannakakis. Minimum and maximum delay problems in real-time systems *(Formal Methods in System Design)*.

The region abstraction



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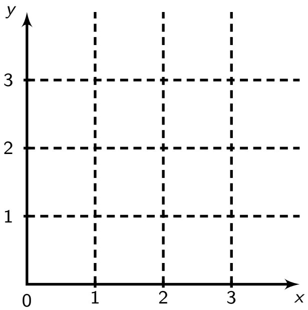
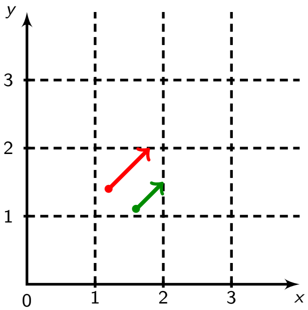


- "compatibility" between regions and constraints

The region abstraction



- "compatibility" between regions and constraints
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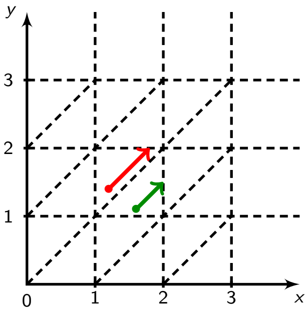
The region abstraction



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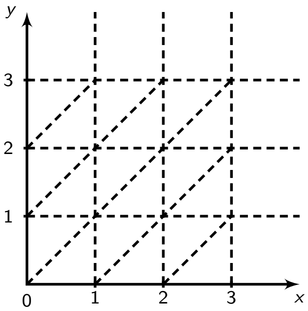
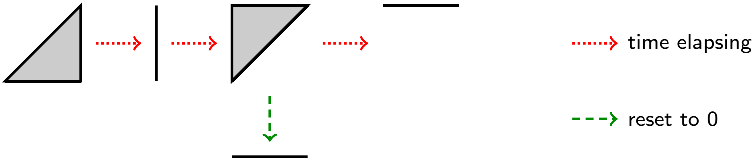


- "compatibility" between regions and constraints
- "compatibility" between regions and time elapsing

$\rightsquigarrow$ an equivalence of finite index
a time-abstract bisimulation

The region abstraction



Outline

Modelling resources in timed systems

- System resources might be relevant and even crucial information

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  - energy consumption,
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  - ...

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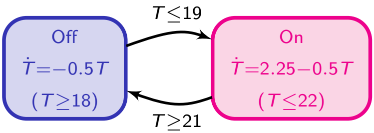
The thermostat example



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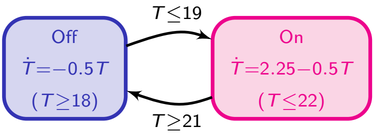

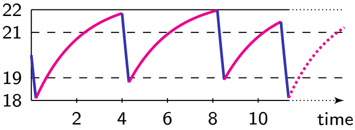

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Theorem [HKPV95]

The reachability problem is undecidable in hybrid automata.

[HKPV95] Henzinger, Kopke, Puri, Varaiya. What's decidable wbout hybrid automata? *(SToC'95).*

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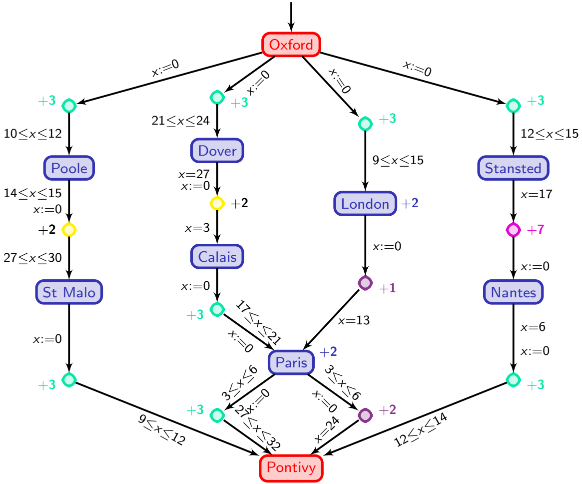
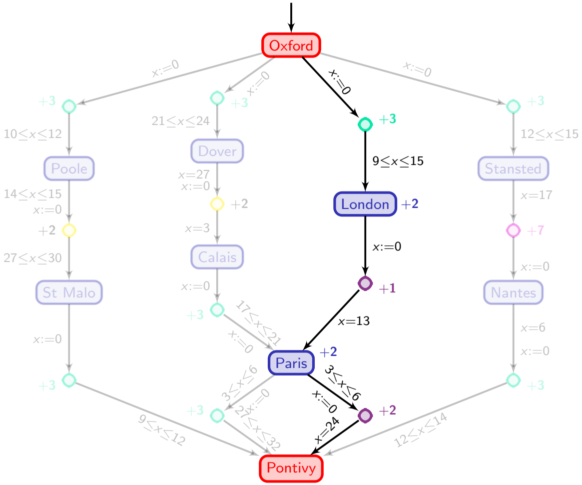
- An alternative: weighted/priced timed automata [ALP01,BFH+01]
  $\rightsquigarrow$ hybrid variables do not constrain the system

[ALP01] Alur, La Torre, Pappas. Optimal paths in weighted timed automata *(HSCC'01)*.
[BFH+01] Behrmann, Fehnker, Hune, Larsen, Pettersson, Romijn, Vaandrager. Minimum-cost reachability in priced timed automata *(HSCC'01)*.

A third model of the system

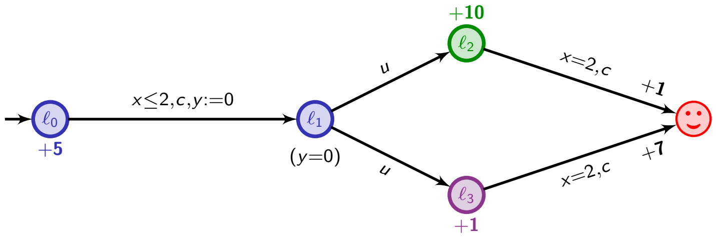


How much fuel will I use?



It is a quantitative (optimization) problem
in a priced/weighted timed automaton: at least 68 anti-planet units!

Weighted/priced timed automata [ALP01,BFH+01]



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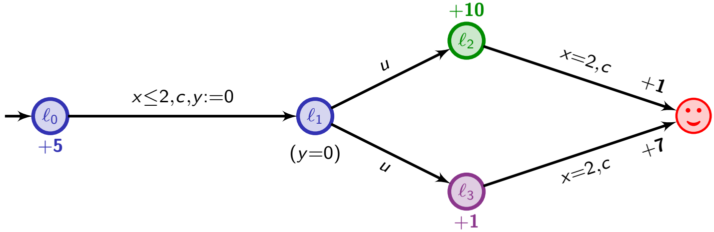

	ℓ_0	$\xrightarrow{1.3}$	ℓ_0	$\xrightarrow{c}$	ℓ_1	$\xrightarrow{u}$	ℓ_3	$\xrightarrow{0.7}$	ℓ_3	$\xrightarrow{c}$	☺
x	0		1.3		1.3		1.3		2		
y	0		1.3		0		0		0.7		

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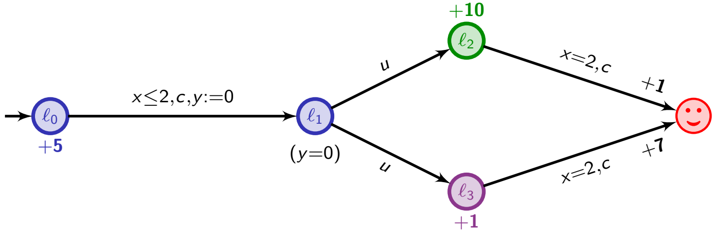

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x	0		1.3		1.3		1.3		2		
y	0		1.3		0		0		0.7		

cost :

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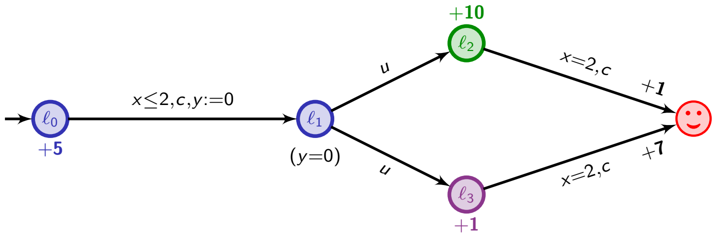

	ℓ_0	$\xrightarrow{1.3}$	ℓ_0	$\xrightarrow{c}$	ℓ_1	$\xrightarrow{u}$	ℓ_3	$\xrightarrow{0.7}$	ℓ_3	$\xrightarrow{c}$	☺
x	0		1.3		1.3		1.3		2		
y	0		1.3		0		0		0.7		

cost : 6.5

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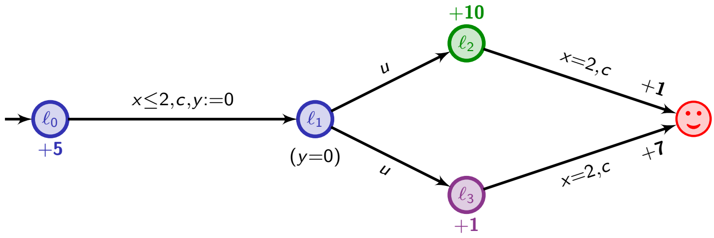

	ℓ_0	$\xrightarrow{1.3}$	ℓ_0	$\xrightarrow{c}$	ℓ_1	$\xrightarrow{u}$	ℓ_3	$\xrightarrow{0.7}$	ℓ_3	$\xrightarrow{c}$	☺
x	0		1.3		1.3		1.3		2		
y	0		1.3		0		0		0.7		

cost : 6.5 + 0

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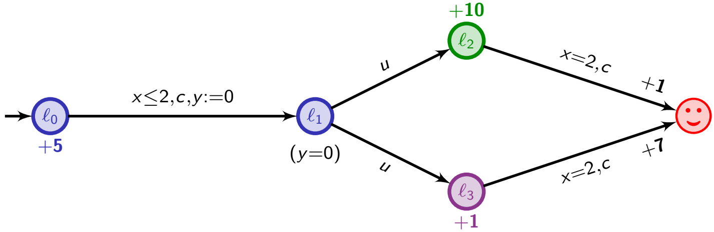

	ℓ_0	$\xrightarrow{1.3}$	ℓ_0	$\xrightarrow{c}$	ℓ_1	$\xrightarrow{u}$	ℓ_3	$\xrightarrow{0.7}$	ℓ_3	$\xrightarrow{c}$	☺
x	0		1.3		1.3		1.3		2		
y	0		1.3		0		0		0.7		
cost :		6.5	+	0	+	0					

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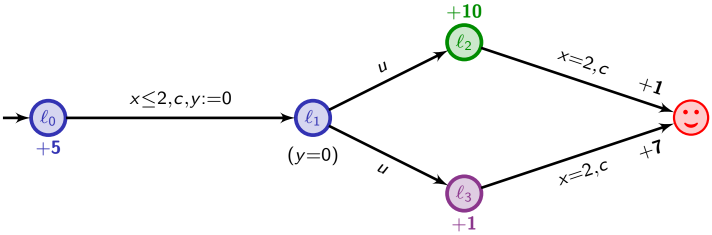

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x	0		1.3		1.3		1.3		2		
y	0		1.3		0		0		0.7		
cost :		6.5	+	0	+	0	+	0.7			

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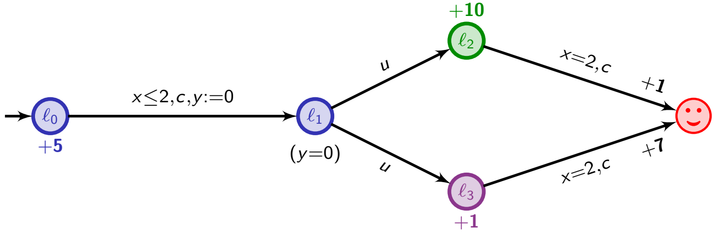

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x	0		1.3		1.3		1.3		2		
y	0		1.3		0		0		0.7		

cost : 6.5 + 0 + 0 + 0.7 + 7

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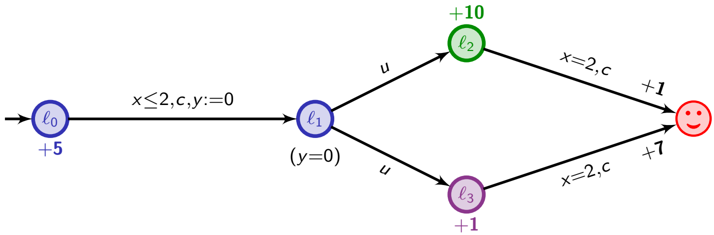

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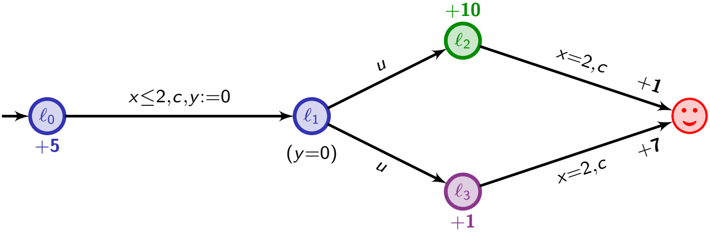
x	0		1.3		1.3		1.3		2				
y	0		1.3		0		0		0.7				
cost :		6.5	+	0	+	0	+	0.7	+	7		=	14.2

[ALP01] Alur, La Torre, Pappas. Optimal paths in weighted timed automata *(HSCC'01)*.
[BFH+01] Behrmann, Fehnker, Hune, Larsen, Pettersson, Romijn, Vaandrager. Minimum-cost reachability in priced timed automata *(HSCC'01)*.

Weighted/priced timed automata [ALP01,BFH+01]



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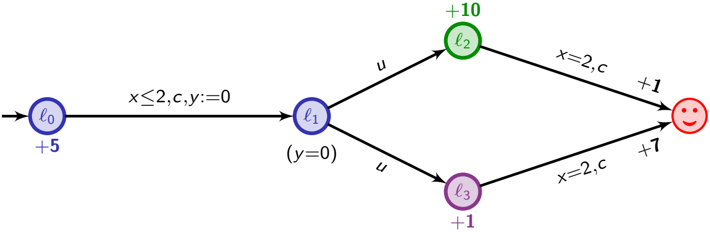

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$$5t + 10(2 - t) + 1$$

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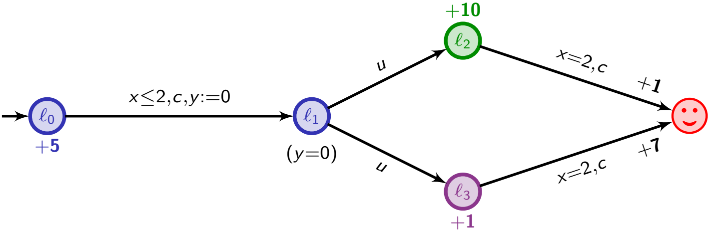

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$$5t + 10(2-t) + 1 \ , \ 5t + (2-t) + 7$$

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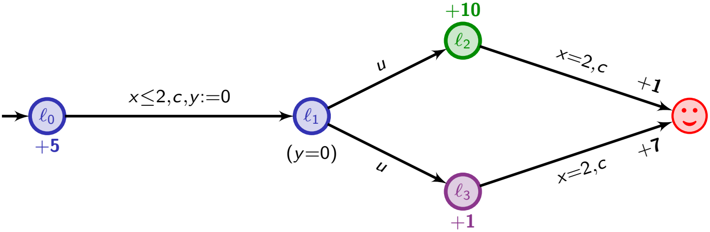

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$$\min \left(5t + 10(2-t) + 1 \;,\; 5t + (2-t) + 7 \right)$$

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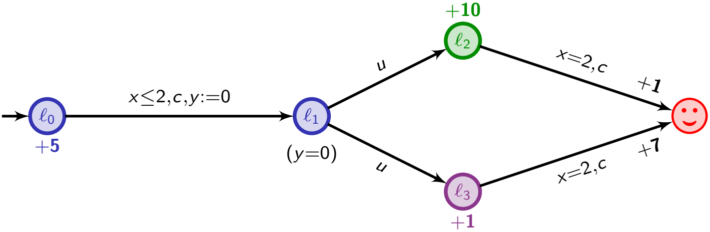

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$$\inf_{0 \leq t \leq 2} \min\left(5t + 10(2-t) + 1 \ ,\ 5t + (2-t) + 7 \right) = 9$$

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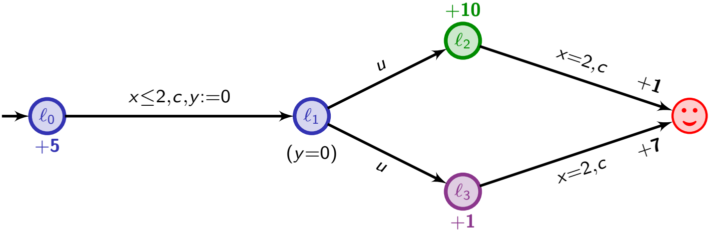

Question: what is the optimal cost for reaching ☺ ?

$$\inf_{0\leq t\leq 2} \min\left(5t + 10(2-t) + 1\ ,\ 5t + (2-t) + 7\right) = 9$$

⇝ *strategy:* leave immediately ℓ_0, go to ℓ_3, and wait there 2 t.u.

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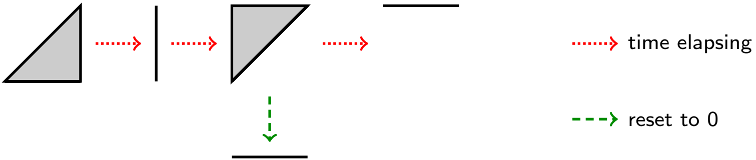
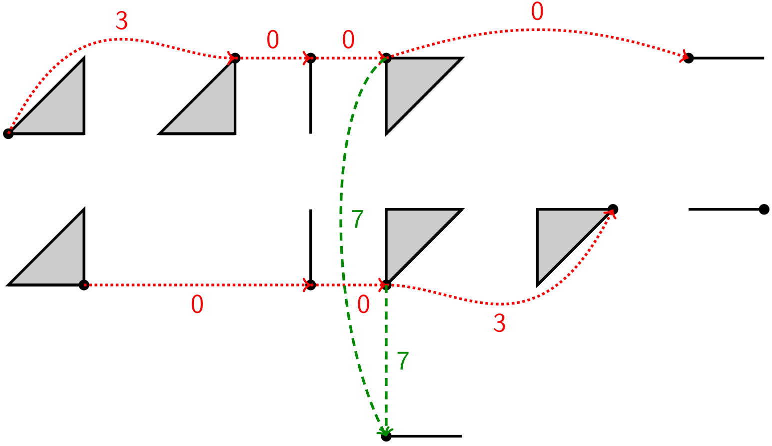
The region abstraction is not fine enough



The corner-point abstraction



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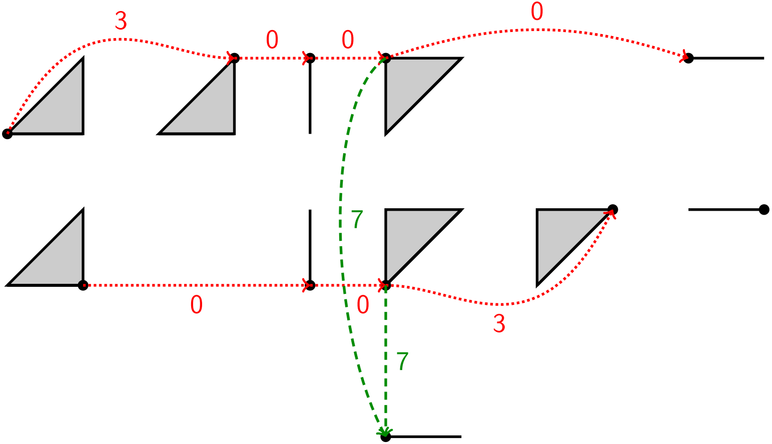


We can somehow discretize the behaviours...

From timed to discrete behaviours

Optimal reachability as a linear programming problem

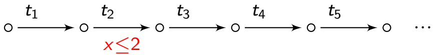
From timed to discrete behaviours

Optimal reachability as a linear programming problem

$$\circ \xrightarrow{t_1} \circ \xrightarrow{t_2} \circ \xrightarrow{t_3} \circ \xrightarrow{t_4} \circ \xrightarrow{t_5} \circ \quad \cdots$$

From timed to discrete behaviours

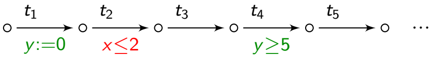
Optimal reachability as a linear programming problem



$$\left\{ \; t_1 + t_2 \leq 2 \right.$$

From timed to discrete behaviours

Optimal reachability as a linear programming problem



$$\begin{cases} t_1 + t_2 \leq 2 \\ t_2 + t_3 + t_4 \geq 5 \end{cases}$$

From timed to discrete behaviours

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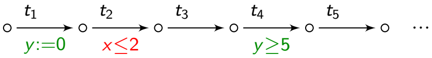

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Lemma

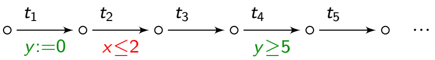
Let Z be a bounded zone and f be a function

$$f : (t_1, \ldots, t_n) \mapsto \sum_{i=1}^{n} c_i t_i + c$$

well-defined on $\overline{Z}$. Then $inf_Z f$ is obtained on the border of $\overline{Z}$ with integer coordinates.

From timed to discrete behaviours

Optimal reachability as a linear programming problem



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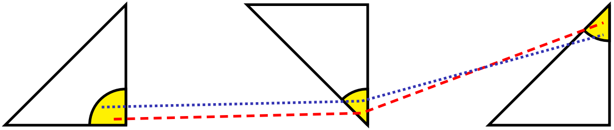
➜ for every finite path π in $\mathcal{A}$, there exists a path Π in $\mathcal{A}_{\text{cp}}$ such that

$$\text{cost}(\Pi)\leq\text{cost}(\pi)$$

[Π is a "corner-point projection" of π]

From discrete to timed behaviours

Approximation of abstract paths:



For any path Π of $\mathcal{A}_{\mathsf{cp}}$,

From discrete to timed behaviours

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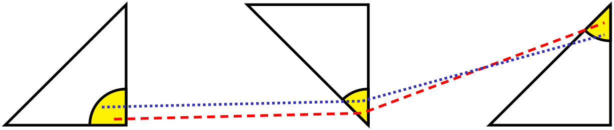
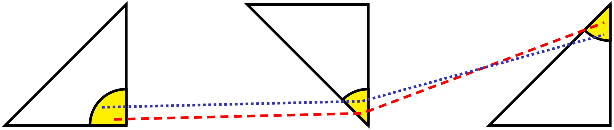


For any path Π of $\mathcal{A}_{\mathsf{cp}}$, for any $\varepsilon > 0$,

From discrete to timed behaviours

Approximation of abstract paths:



For any path Π of $\mathcal{A}_{\mathsf{cp}}$, for any $\varepsilon > 0$, there exists a path π_ε of $\mathcal{A}$ s.t.

$$\|\Pi - \pi_\varepsilon\|_\infty < \varepsilon$$

From discrete to timed behaviours

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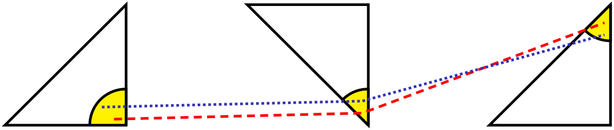

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For every $\eta > 0$, there exists $\varepsilon > 0$ s.t.

$$\|\Pi - \pi_\varepsilon\|_\infty < \varepsilon \Rightarrow |\mathsf{cost}(\Pi) - \mathsf{cost}(\pi_\varepsilon)| < \eta$$

Optimal-cost reachability

Theorem [ALP01,BFH+01,BBBR07]

The optimal-cost reachability problem is decidable (and PSPACE-complete) in (weighted) timed automata.

[ALP01] Alur, La Torre, Pappas. Optimal paths in weighted timed automata *(HSCC'01)*.
[BFH+01] Behrmann, Fehnker, Hune, Larsen, Pettersson, Romijn, Vaandrager. Minimum-cost reachability in priced timed automata *(HSCC'01)*.
[BBBR07] Bouyer, Brihaye, Bruyère, Raskin. On the optimal reachability problem *(Formal Methods in System Design)*.

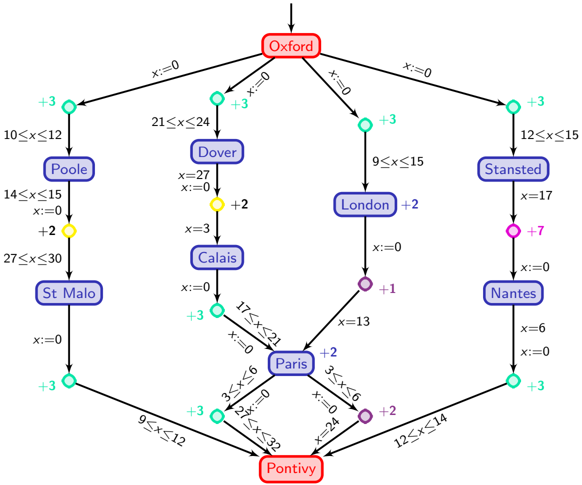
Going further

The corner-point abstraction can be used for the following problems:

- mean-cost optimization problem [BBL04,BBL08]
- discounted cost optimization problem [FL08]
- concavely-priced cost optimization problem [JT08]

[BBL04] Bouyer, Brinksma, Larsen. Staying alive as cheaply as possible *(HSCC'04)*.
[BBL08] Bouyer, Brinksma, Larsen. Optimal infinite scheduling for multi-priced timed automata *(Formal Methods in System Designs)*.
[FL08] Fahrenberg, Larsen. Discount-optimal infinite runs in priced timed automata *(INFINITY'08)*.
[JT08] Judziński, Trivedi. Concavely-priced timed automata *(FORMATS'08)*.

A fourth model of the system
What if there is an unexpected event?



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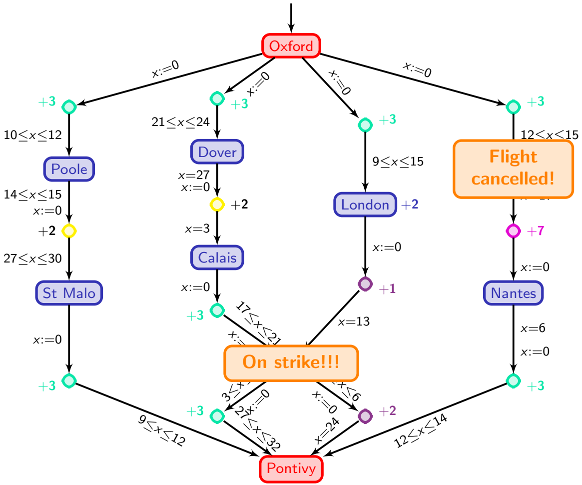

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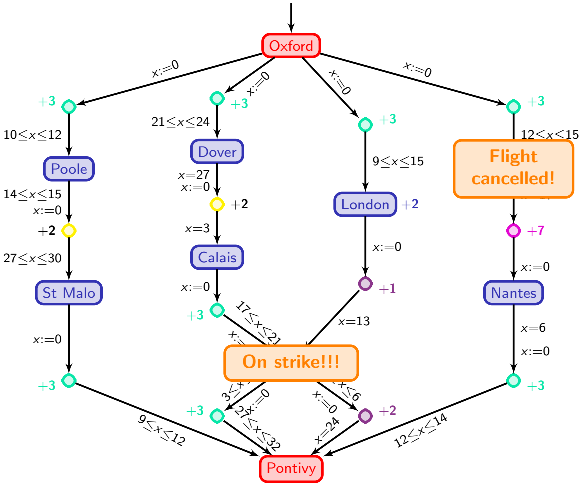
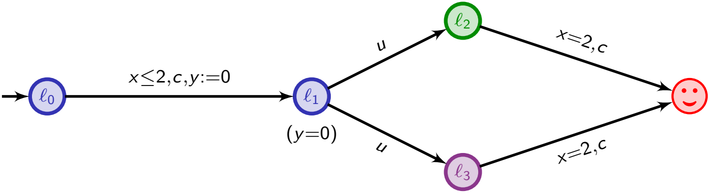


⇝ modelled as timed games

A simple example of timed game



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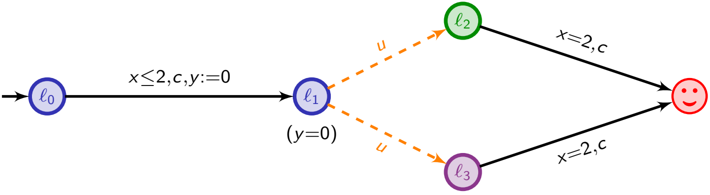
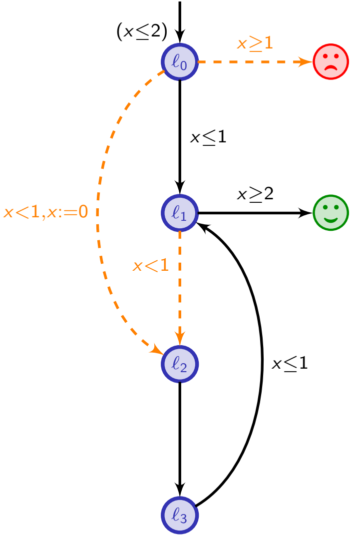


Another example



Decidability of timed games

Theorem [AMPS98,HK99]

Safety and reachability control in timed automata are decidable and EXPTIME-complete.

[AMPS98] Asarin, Maler, Pnueli, Sifakis. Controller synthesis for timed automata *(SSC'98)*.
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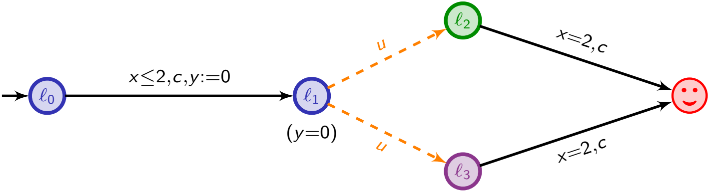
Theorem [AM99,BHPR07,JT07]

Optimal-time reachability timed games are decidable and EXPTIME-complete.

[AM99] Asarin, Maler. As soon as possible: time optimal control for timed automata *(HSCC'99)*.
[BHPR07] Brihaye, Henzinger, Prabhu, Raskin. Minimum-time reachability in timed games *(ICALP'07)*.
[JT07] Jurdziński, Trivedi. Reachability-time games on timed automata *(ICALP'07)*.

Back to the simple example



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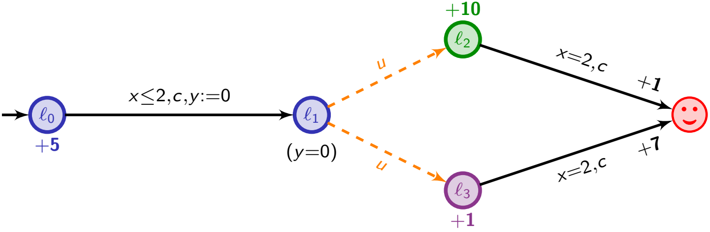

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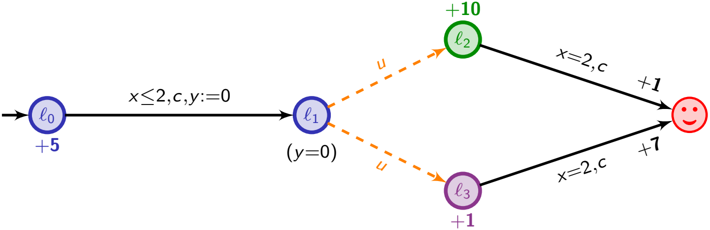

Question: what is the optimal cost we can ensure while reaching 🙂?

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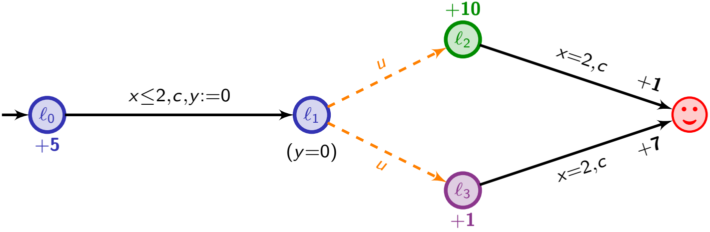

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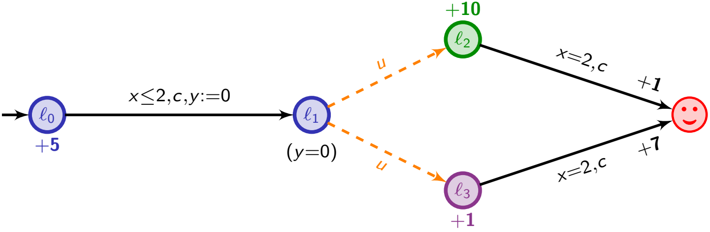

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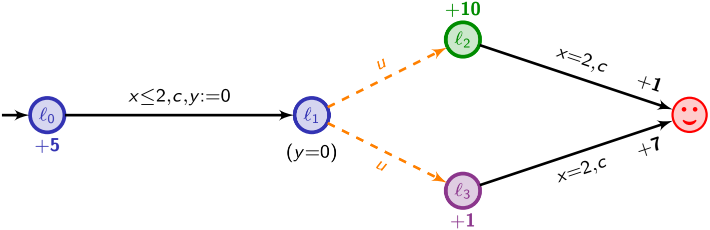

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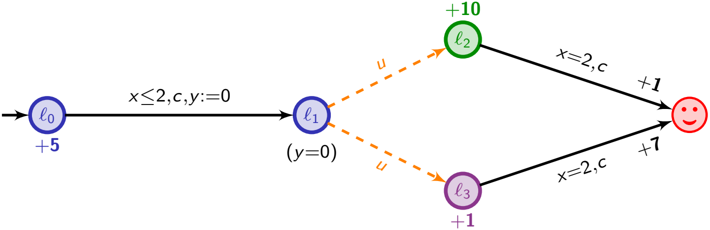
$$\max\left(\ 5t + 10(2-t) + 1\ ,\ 5t + (2-t) + 7\ \right)$$

Back to the simple example



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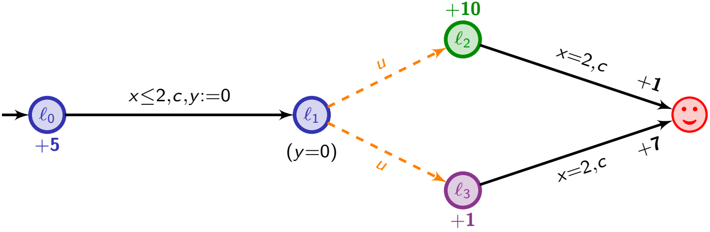

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$$\inf_{0\leq t\leq 2} \max\left(\, 5t + 10(2-t) + 1 \;,\; 5t + (2-t) + 7 \,\right) = 14 + \frac{1}{3}$$

$\rightsquigarrow$ *strategy:* wait in ℓ_0, and when $t = \frac{4}{3}$, go to ℓ_1

Optimal reachability in weighted timed games

This topic has been fairly hot these last couple of years...

e.g. [LMM02,ABM04,BCFL04]

[LMM02] La Torre, Mukhopadhyay, Murano. Optimal-reachability and control for acyclic weighted timed automata *(TCS@02)*.
[ABM04] Alur, Bernardsky, Madhusudan. Optimal reachability in weighted timed games *(ICALP'04)*.
[BCFL04] Bouyer, Cassez, Fleury, Larsen. Optimal strategies in priced timed game automata *(FSTTCS'04)*.

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Theorem [BBR05,BBM06]

Optimal timed games are undecidable, as soon as automata have three clocks or more.

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- Key: resetting the clock somehow resets the history...

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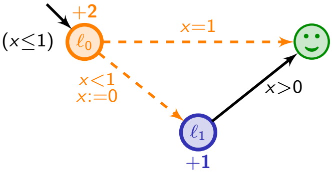

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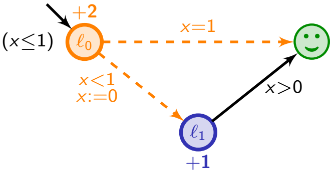

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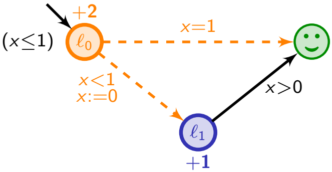

- However, by unfolding and removing one by one the locations,we can synthesize memoryless almost-optimal winning strategies.
- Rather involved proof of correctness for a simple algorithm.

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The negative side: why is that hard?

Given two clocks x and y, we can check whether $y = 2x$.

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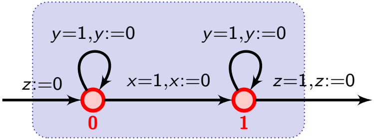
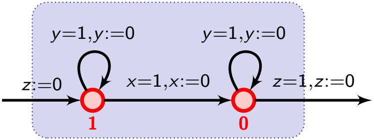
$\mathsf{Add}^+(x)$



The cost is increased by x_0

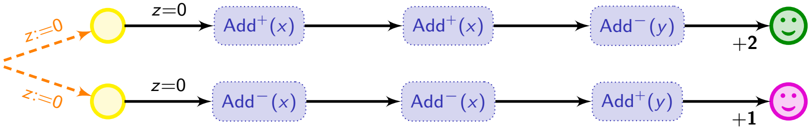
$\mathsf{Add}^-(x)$



The cost is increased by $1-x_0$

The negative side: why is that hard?

Given two clocks x and y, we can check whether $y = 2x$.



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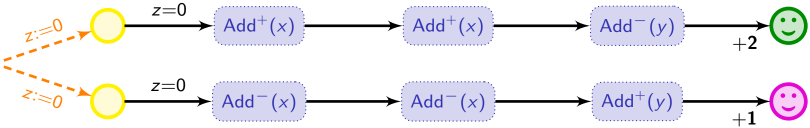

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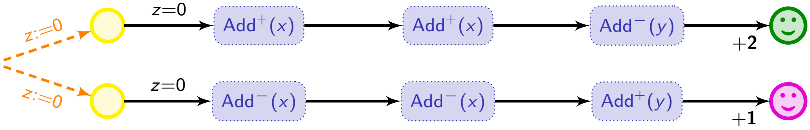

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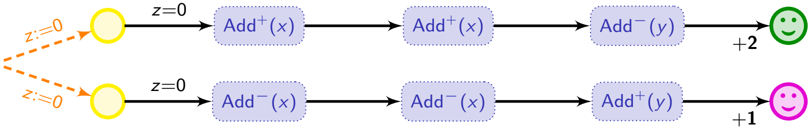

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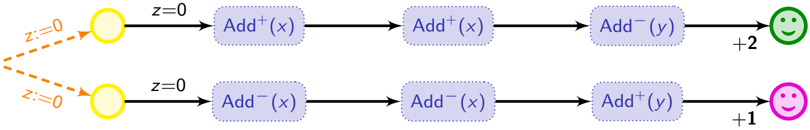

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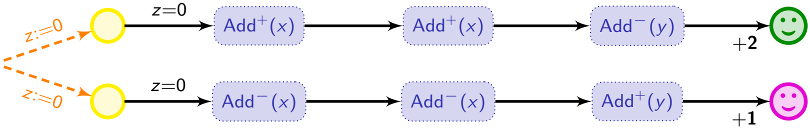

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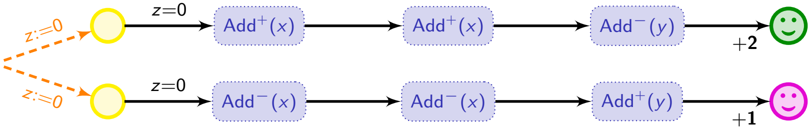

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- Player 1 has a winning strategy with cost ≤ 3 iff $y_0 = 2x_0$

The negative side: why is that hard?

Player 1 will simulate a two-counter machine:

- each instruction is encoded as a module;
- the values c_1 and c_2 of the counters are encoded by the values of two clocks:

$$x = \frac{1}{2^{c_1}} \qquad \text{and} \qquad y = \frac{1}{3^{c_2}}$$

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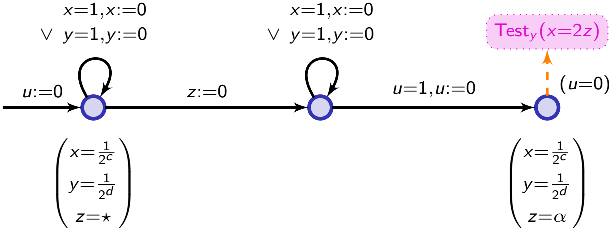
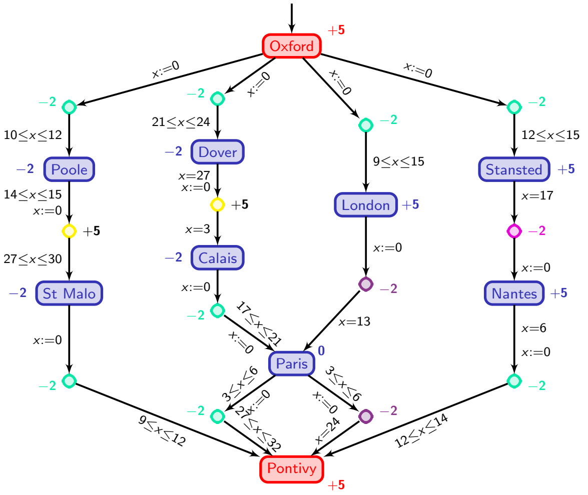
Globally, $(x \leq 1, y \leq 1, u \leq 1)$

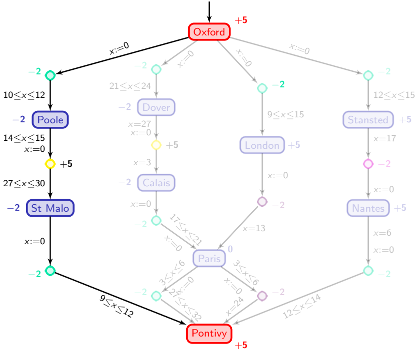


Outline

A fifth model of the system



Can I work with my computer all the way?



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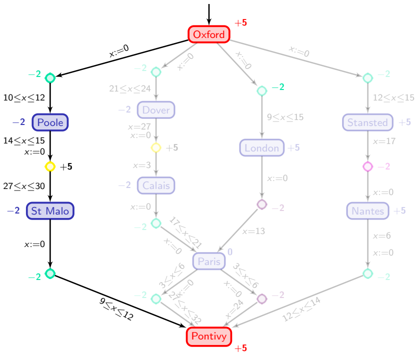
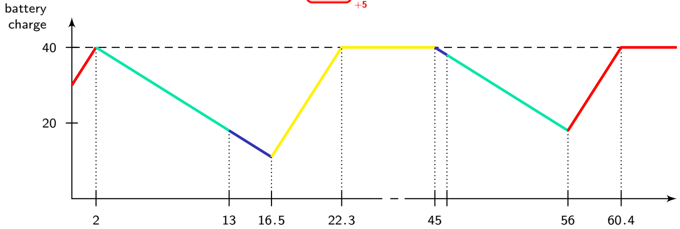
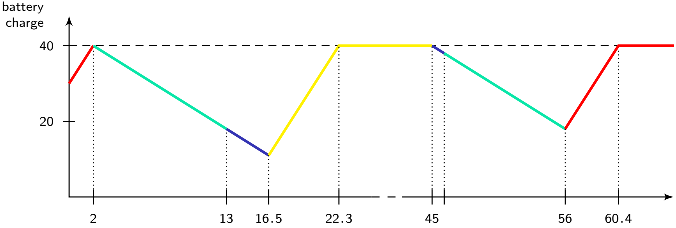




Can I work with my computer all the way?

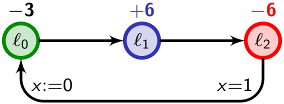
Energy is not only consumed, but can be regained.

$\rightsquigarrow$ the aim is to continuously satisfy some energy constraints.



An example of resource management

Globally $(x \leq 1)$



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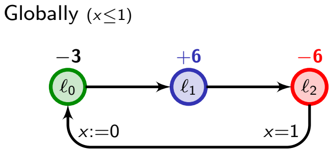

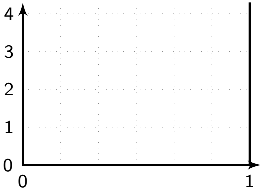

- Lower-bound problem: can we stay above 0?

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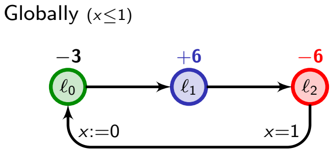

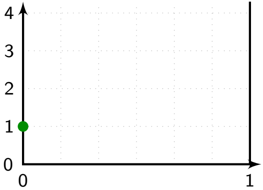

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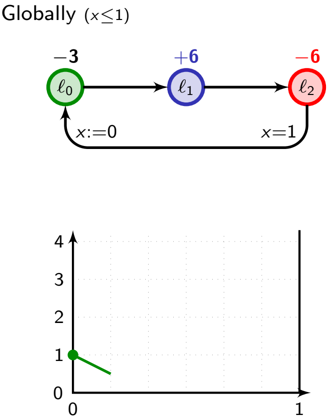

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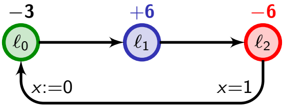

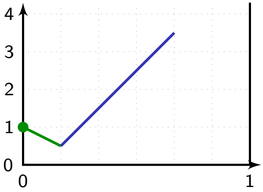

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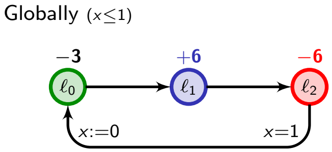

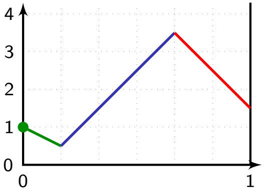

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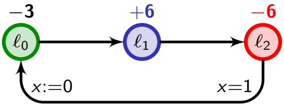

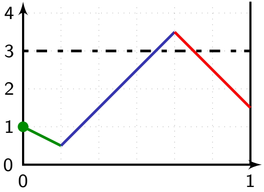

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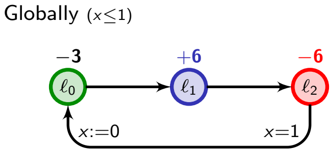

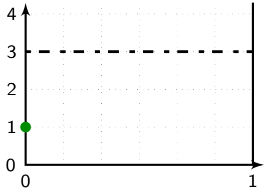

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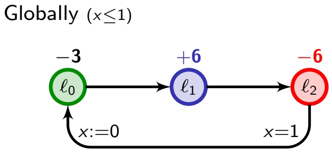

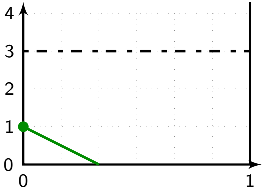

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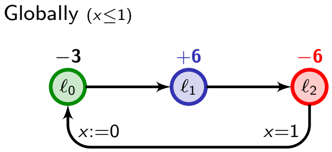

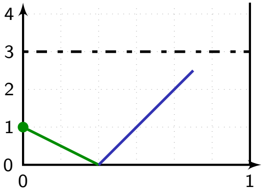

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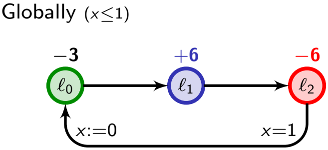

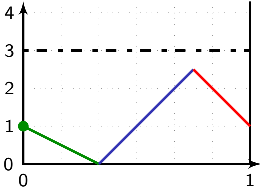

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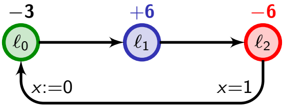

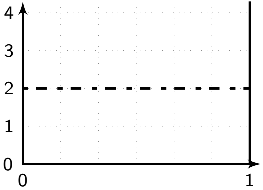

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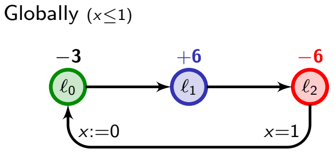

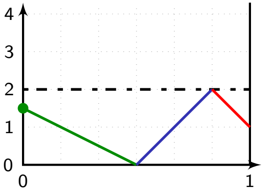

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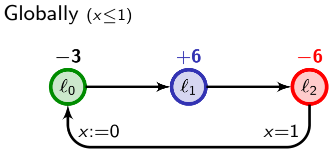

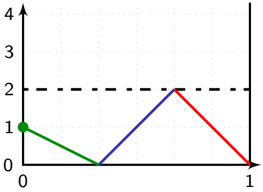

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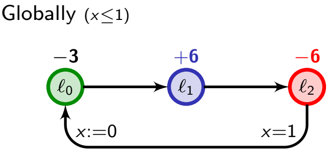

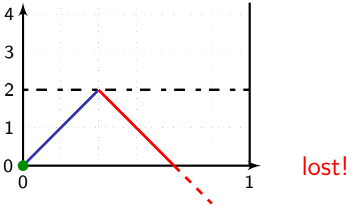

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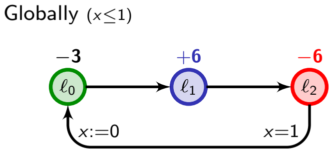

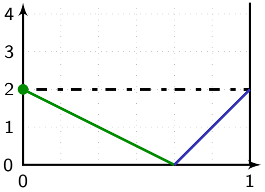

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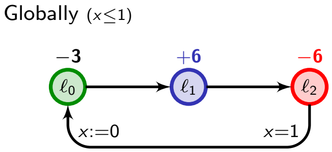

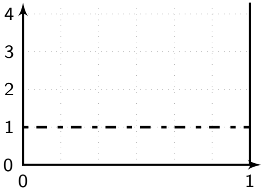

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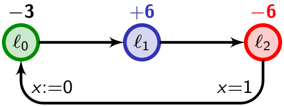
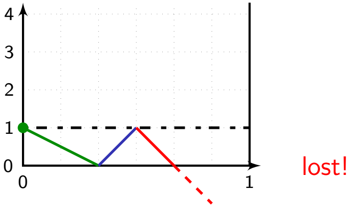
An example of resource management

Globally $(x \leq 1)$





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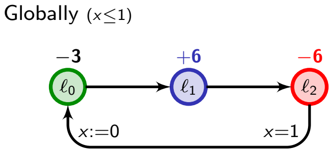
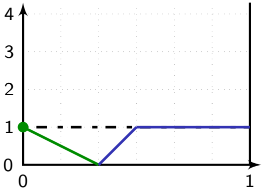
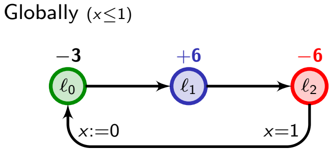
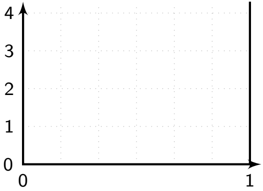




- Lower-bound problem
- Lower-upper-bound problem
- Lower-weak-upper-bound problem: can we "weakly" stay within bounds?

An example of resource management





- Lower–bound problem $\rightsquigarrow$ **L**
- Lower-upper-bound problem $\rightsquigarrow$ **L+U**
- Lower-weak-upper-bound problem $\rightsquigarrow$ **L+W**

Only partial results so far [BFLMS08]

0 clock!	**exist. problem**	**univ. problem**	**games**
L	$\in$ PTIME	$\in$ PTIME	$\in$ UP $\cap$ co-UP PTIME-hard
L+W	$\in$ PTIME	$\in$ PTIME	$\in$ NP $\cap$ co-NP PTIME-hard
L+U	$\in$ PSPACE NP-hard	$\in$ PTIME	EXPTIME-c.

[BFLMS08] Bouyer, Fahrenberg, Larsen, Markey, Srba. Infinite runs in weighted timed automata with energy constraints *(FORMATS'08)*.

Only partial results so far [BFLMS08]

1 clock	**exist. problem**	**univ. problem**	**games**
L	$\in$ PTIME	$\in$ PTIME	?
L+W	$\in$ PTIME	$\in$ PTIME	?
L+U	?	?	undecidable

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Only partial results so far [BFLMS08]

n clocks	**exist. problem**	**univ. problem**	**games**
L	?	?	?
L+W	?	?	?
L+U	?	?	undecidable

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- from **L**-games to mean-payoff games: transform the game as follows:

Single-clock **L**+**U**-games

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The single-clock **L**+**U**-games are undecidable.

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There is an infinite execution in the two-counter machine iff there is a strategy in the single-clock timed game under which the energy level remains between 0 and 5.

Single-clock **L**+**U**-games

Theorem

The single-clock **L**+**U**-games are undecidable.

We encode the behaviour of a two-counter machine:

- each instruction is encoded as a module;
- the values c_1 and c_2 of the counters are encoded by the energy level

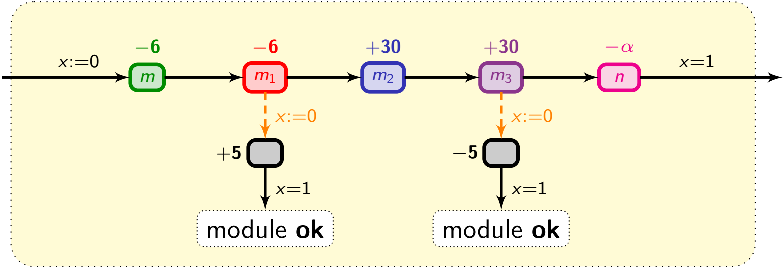
$$e = 5 - \frac{1}{2^{c_1} \cdot 3^{c_2}}$$

when entering the corresponding module.

There is an infinite execution in the two-counter machine iff there is a strategy in the single-clock timed game under which the energy level remains between 0 and 5.

$\rightsquigarrow$ We present a generic construction for incrementing/decrementing the counters.

Generic module for incrementing/decrementing



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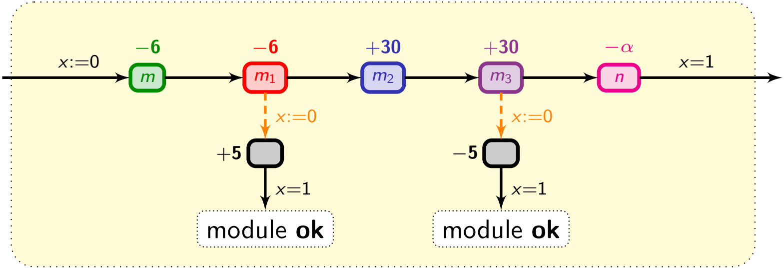

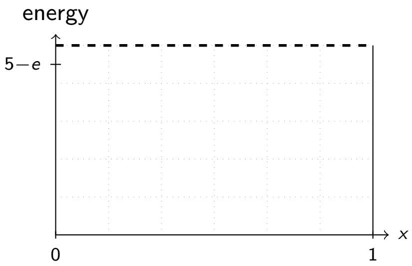

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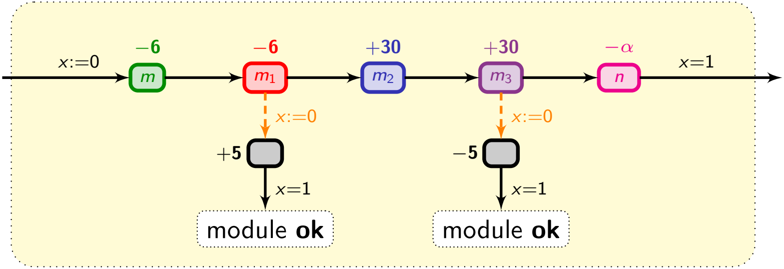

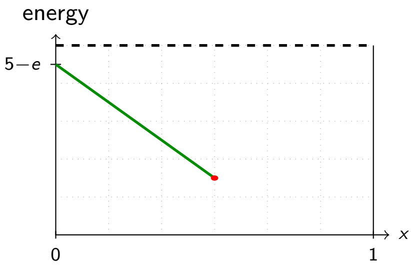

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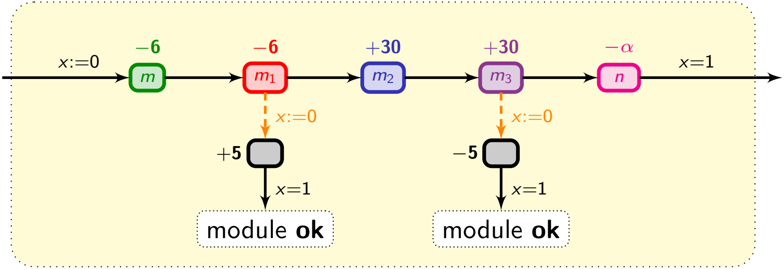

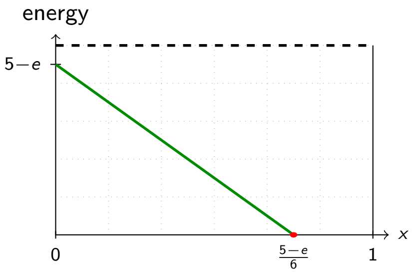

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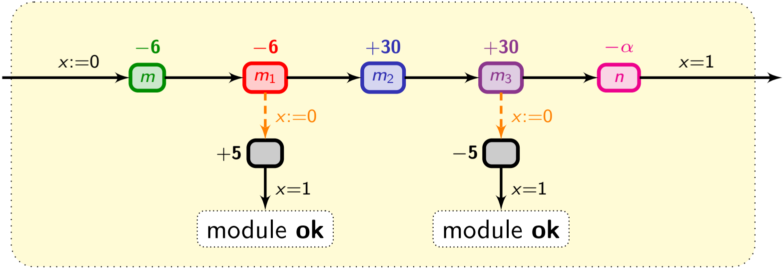

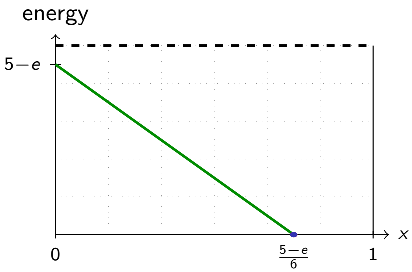

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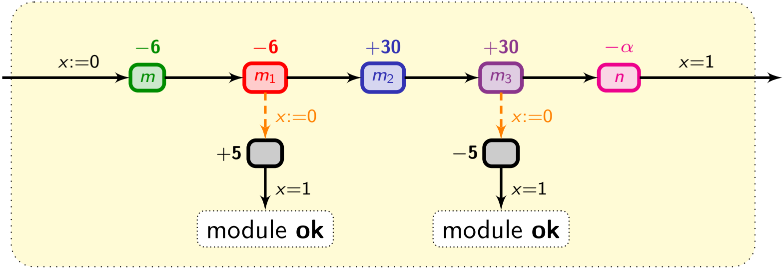

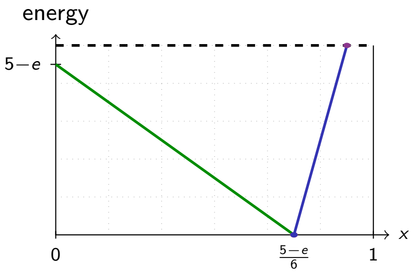

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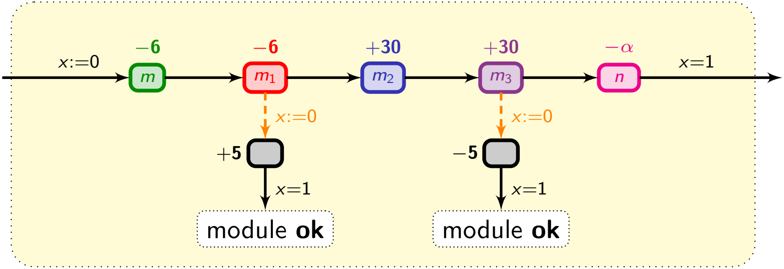
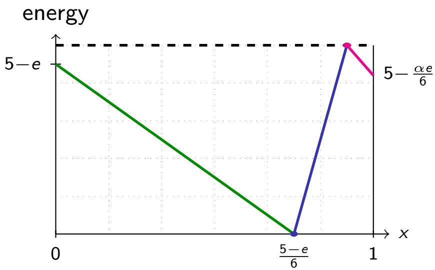
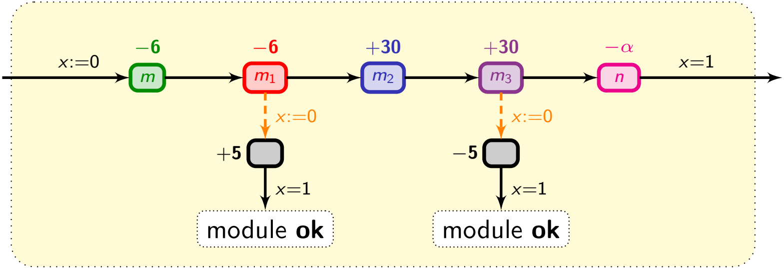
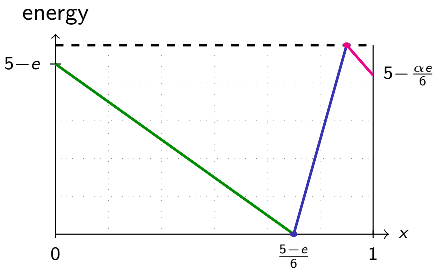




Generic module for incrementing/decrementing





- $\alpha=3$: increment c_1
- $\alpha=2$: increment c_2
- $\alpha=12$: decrement c_1
- $\alpha=18$: decrement c_2

Outline

Conclusion

- Priced/weighted timed automata, a model for representing quantitative constraints on timed systems:
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  ... and not all of them have been answered!

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    - weighted strong reset hybrid games [BBJLR07]
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- Current and further work:
  - computation of approximate optimal values
  - further investigation of safe games + several cost variables?
  - discounted-time optimal games
  - link between discounted-time games and mean-cost games?
  - computation of equilibria
  - ...