Recent progress in birational geometry

Caucher Birkar

Tsinghua University

Chennai Mathematics Institute, online seminar, 21 Jan 2022

Dimension one: curves

Dimension one: curves

We work over $\mathbb{C}$ for simplicity.

Dimension one: curves

We work over $\mathbb{C}$ for simplicity.

For a smooth or normal variety X, K_X denotes the canonical divisor.

Dimension one: curves

We work over $\mathbb{C}$ for simplicity.

For a smooth or normal variety X, K_X denotes the canonical divisor.

Birational geometry is the classification theory of algebraic varieties using birational techniques.

Dimension one: curves

We work over $\mathbb{C}$ for simplicity.

For a smooth or normal variety X, K_X denotes the canonical divisor.

Birational geometry is the classification theory of algebraic varieties using birational techniques.

Statements are often of a biregular nature but proved by working on birational models of the spaces invovled.

Dimension one: curves

We work over $\mathbb{C}$ for simplicity.

For a smooth or normal variety X, K_X denotes the canonical divisor.

Birational geometry is the classification theory of algebraic varieties using birational techniques.

Statements are often of a biregular nature but proved by working on birational models of the spaces invovled.

Dimension one: curves

Dimension one: curves

Assume X is a smooth projective variety of dimension one, i.e. a curve.

Dimension one: curves

Assume X is a smooth projective variety of dimension one, i.e. a curve.

X is unique in its birational class.

Dimension one: curves

Assume X is a smooth projective variety of dimension one, i.e. a curve.

X is unique in its birational class.

The **genus** of X is

$$g = h^0(K_X) = h^1(\mathcal{O}_X) = \text{number of holes in } X$$

where X is considered as a compact Riemann surface.

Dimension one: curves

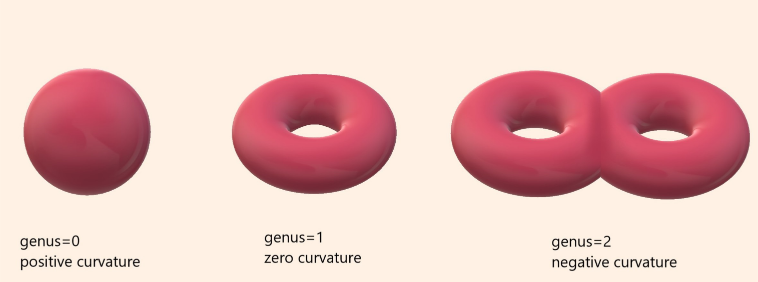
Assume X is a smooth projective variety of dimension one, i.e. a curve.

X is unique in its birational class.

The **genus** of X is

$$g = h^0(K_X) = h^1(\mathcal{O}_X) = \text{number of holes in } X$$

where X is considered as a compact Riemann surface.



Dimension one: curves

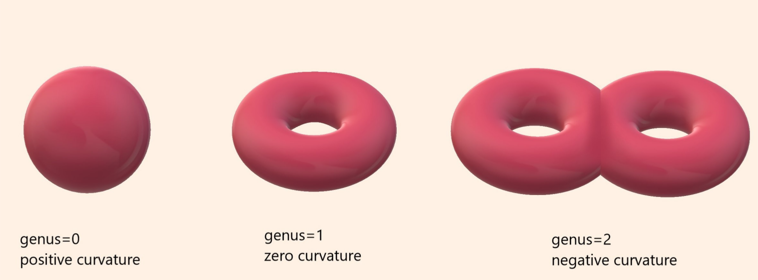
Assume X is a smooth projective variety of dimension one, i.e. a curve.

X is unique in its birational class.

The **genus** of X is

$$g = h^0(K_X) = h^1(\mathcal{O}_X) = \text{number of holes in } X$$

where X is considered as a compact Riemann surface.



Dimension one: curves

Dimension one: curves

We have

$$\begin{cases} g = 0 & \Leftrightarrow \deg K_X < 0 & \Leftrightarrow X \text{ Fano} & \Leftrightarrow X \simeq \mathbb{P}^1 \\ g = 1 & \Leftrightarrow \deg K_X = 0 & \Leftrightarrow X \text{ Calabi-Yau} & \Leftrightarrow X \text{ elliptic} \\ g \geq 2 & \Leftrightarrow \deg K_X > 0 & \Leftrightarrow X \text{ canonically polarised} & \Leftrightarrow X \text{ general type} \end{cases}$$

Dimension one: curves

We have

$$\begin{cases} g = 0 & \Leftrightarrow \deg K_X < 0 & \Leftrightarrow X \text{ Fano} & \Leftrightarrow X \simeq \mathbb{P}^1 \\ g = 1 & \Leftrightarrow \deg K_X = 0 & \Leftrightarrow X \text{ Calabi-Yau} & \Leftrightarrow X \text{ elliptic} \\ g \geq 2 & \Leftrightarrow \deg K_X > 0 & \Leftrightarrow X \text{ canonically polarised} & \Leftrightarrow X \text{ general type} \end{cases}$$

We can study such X individually and also collectively in families.

Dimension one: curves

We have

$$\begin{cases} g = 0 & \Leftrightarrow \deg K_X < 0 & \Leftrightarrow X \text{ Fano} & \Leftrightarrow X \simeq \mathbb{P}^1 \\ g = 1 & \Leftrightarrow \deg K_X = 0 & \Leftrightarrow X \text{ Calabi-Yau} & \Leftrightarrow X \text{ elliptic} \\ g \geq 2 & \Leftrightarrow \deg K_X > 0 & \Leftrightarrow X \text{ canonically polarised} & \Leftrightarrow X \text{ general type} \end{cases}$$

We can study such X individually and also collectively in families.

Moduli (Riemann): curves of genus $g \geq 2$ are parametrised by points of a **moduli space** M_g of dimension $3g - 3$.

Dimension one: curves

We have

$$\begin{cases} g = 0 & \Leftrightarrow \deg K_X < 0 & \Leftrightarrow X \text{ Fano} & \Leftrightarrow X \simeq \mathbb{P}^1 \\ g = 1 & \Leftrightarrow \deg K_X = 0 & \Leftrightarrow X \text{ Calabi-Yau} & \Leftrightarrow X \text{ elliptic} \\ g \geq 2 & \Leftrightarrow \deg K_X > 0 & \Leftrightarrow X \text{ canonically polarised} & \Leftrightarrow X \text{ general type} \end{cases}$$

We can study such X individually and also collectively in families.

Moduli (Riemann): curves of genus $g \geq 2$ are parametrised by points of a **moduli space** M_g of dimension $3g - 3$.

Moduli (Deligne-Mumford): M_g can be compactified by adding some curves with nodal singularities.

Dimension one: curves

We have

$$\begin{cases} g = 0 & \Leftrightarrow \deg K_X < 0 & \Leftrightarrow X \text{ Fano} & \Leftrightarrow X \simeq \mathbb{P}^1 \\ g = 1 & \Leftrightarrow \deg K_X = 0 & \Leftrightarrow X \text{ Calabi-Yau} & \Leftrightarrow X \text{ elliptic} \\ g \geq 2 & \Leftrightarrow \deg K_X > 0 & \Leftrightarrow X \text{ canonically polarised} & \Leftrightarrow X \text{ general type} \end{cases}$$

We can study such X individually and also collectively in families.

Moduli (Riemann): curves of genus $g \geq 2$ are parametrised by points of a **moduli space** M_g of dimension $3g - 3$.

Moduli (Deligne-Mumford): M_g can be compactified by adding some curves with nodal singularities.

Curves have been extensively studied from the 19th century to this date.

Dimension one: curves

We have

$$\begin{cases} g = 0 & \Leftrightarrow \deg K_X < 0 & \Leftrightarrow X \text{ Fano} & \Leftrightarrow X \simeq \mathbb{P}^1 \\ g = 1 & \Leftrightarrow \deg K_X = 0 & \Leftrightarrow X \text{ Calabi-Yau} & \Leftrightarrow X \text{ elliptic} \\ g \geq 2 & \Leftrightarrow \deg K_X > 0 & \Leftrightarrow X \text{ canonically polarised} & \Leftrightarrow X \text{ general type} \end{cases}$$

We can study such X individually and also collectively in families.

Moduli (Riemann): curves of genus $g \geq 2$ are parametrised by points of a **moduli space** M_g of dimension $3g - 3$.

Moduli (Deligne-Mumford): M_g can be compactified by adding some curves with nodal singularities.

Curves have been extensively studied from the 19th century to this date.

Dimension two: surfaces

Dimension two: surfaces

Let X be a smooth projective variety of dimension two, i.e. a surface.

Dimension two: surfaces

Let X be a smooth projective variety of dimension two, i.e. a surface.

Surfaces have a much more complicated geometry.

Dimension two: surfaces

Let X be a smooth projective variety of dimension two, i.e. a surface.

Surfaces have a much more complicated geometry.

X is birational to infinitely many other surfaces.

Dimension two: surfaces

Let X be a smooth projective variety of dimension two, i.e. a surface.

Surfaces have a much more complicated geometry.

X is birational to infinitely many other surfaces. K_X may not be uniformly negative, trivial, or positive.

Dimension two: surfaces

Let X be a smooth projective variety of dimension two, i.e. a surface.

Surfaces have a much more complicated geometry.

X is birational to infinitely many other surfaces. K_X may not be uniformly negative, trivial, or positive.

The idea is to transform X so that it has strong geometric properties.

Dimension two: surfaces

Let X be a smooth projective variety of dimension two, i.e. a surface.

Surfaces have a much more complicated geometry.

X is birational to infinitely many other surfaces. K_X may not be uniformly negative, trivial, or positive.

The idea is to transform X so that it has strong geometric properties.

There is a sequence of birational transformations

$$X = X_1 \to X_2 \to \cdots \to X_t =: Y$$

such that either

Dimension two: surfaces

Let X be a smooth projective variety of dimension two, i.e. a surface.

Surfaces have a much more complicated geometry.

X is birational to infinitely many other surfaces. K_X may not be uniformly negative, trivial, or positive.

The idea is to transform X so that it has strong geometric properties.

There is a sequence of birational transformations

$$X = X_1 \to X_2 \to \cdots \to X_t =: Y$$

such that either

- K_Y is negative along fibres of some fibration $Y \to Z$, or

Dimension two: surfaces

Let X be a smooth projective variety of dimension two, i.e. a surface.

Surfaces have a much more complicated geometry.

X is birational to infinitely many other surfaces. K_X may not be uniformly negative, trivial, or positive.

The idea is to transform X so that it has strong geometric properties.

There is a sequence of birational transformations

$$X = X_1 \to X_2 \to \cdots \to X_t =: Y$$

such that either

- K_Y is negative along fibres of some fibration $Y \to Z$, or
- K_Y is trivial along fibres of some fibration $Y \to Z$, or

Dimension two: surfaces

Let X be a smooth projective variety of dimension two, i.e. a surface.

Surfaces have a much more complicated geometry.

X is birational to infinitely many other surfaces. K_X may not be uniformly negative, trivial, or positive.

The idea is to transform X so that it has strong geometric properties.

There is a sequence of birational transformations

$$X = X_1 \to X_2 \to \cdots \to X_t =: Y$$

such that either

- K_Y is negative along fibres of some fibration $Y \to Z$, or
- K_Y is trivial along fibres of some fibration $Y \to Z$, or
- K_Y is positive.

Dimension two: surfaces

Let X be a smooth projective variety of dimension two, i.e. a surface.

Surfaces have a much more complicated geometry.

X is birational to infinitely many other surfaces. K_X may not be uniformly negative, trivial, or positive.

The idea is to transform X so that it has strong geometric properties.

There is a sequence of birational transformations

$$X = X_1 \to X_2 \to \cdots \to X_t =: Y$$

such that either

- K_Y is negative along fibres of some fibration $Y \to Z$, or
- K_Y is trivial along fibres of some fibration $Y \to Z$, or
- K_Y is positive.

Dimension two: surfaces

Dimension two: surfaces

The above process is the classical **minimal model program** (MMP) for surfaces.

Dimension two: surfaces

The above process is the classical **minimal model program** (MMP) for surfaces.

The next step is to study the outcomes Y in detail, e.g. form their moduli spaces.

Dimension two: surfaces

The above process is the classical **minimal model program** (MMP) for surfaces.

The next step is to study the outcomes Y in detail, e.g. form their moduli spaces.

The above classification scheme was developed in the 19th century and early 20th century by the Italian algebraic geometry school led by Castelnuovo and Enriques.

Dimension two: surfaces

The above process is the classical **minimal model program** (MMP) for surfaces.

The next step is to study the outcomes Y in detail, e.g. form their moduli spaces.

The above classification scheme was developed in the 19th century and early 20th century by the Italian algebraic geometry school led by Castelnuovo and Enriques.

It is still ongoing research.

Dimension two: surfaces

The above process is the classical **minimal model program** (MMP) for surfaces.

The next step is to study the outcomes Y in detail, e.g. form their moduli spaces.

The above classification scheme was developed in the 19th century and early 20th century by the Italian algebraic geometry school led by Castelnuovo and Enriques.

It is still ongoing research.

Dimension three

Dimension three

Fano and others studied varieties of dimension 3 in early 20th century.

Dimension three

Fano and others studied varieties of dimension 3 in early 20th century.

It took many decades until a clear picture in dimension 3 emerged. In the 70's there was a wave of developments.

Dimension three

Fano and others studied varieties of dimension 3 in early 20th century.

It took many decades until a clear picture in dimension 3 emerged. In the 70's there was a wave of developments.

Iskovskikh-Manin developed techniques of Noether, Fano, Segre, giving counter-example to Lüroth problem by showing 3-fold quartics are non-rational.

Dimension three

Fano and others studied varieties of dimension 3 in early 20th century.

It took many decades until a clear picture in dimension 3 emerged. In the 70's there was a wave of developments.

Iskovskikh-Manin developed techniques of Noether, Fano, Segre, giving counter-example to Lüroth problem by showing 3-fold quartics are non-rational.

Iitaka attempted to classify varieties according to their Kodaira dimension. He also studied open varieties.

Dimension three

Fano and others studied varieties of dimension 3 in early 20th century.

It took many decades until a clear picture in dimension 3 emerged. In the 70's there was a wave of developments.

Iskovskikh-Manin developed techniques of Noether, Fano, Segre, giving counter-example to Lüroth problem by showing 3-fold quartics are non-rational.

Iitaka attempted to classify varieties according to their Kodaira dimension. He also studied open varieties.

Mori proved Hartshorne conjecture: $\mathbb{P}^n$ are the only smooth projective varieties with ample tangent bundle T_X.

Dimension three

Fano and others studied varieties of dimension 3 in early 20th century.

It took many decades until a clear picture in dimension 3 emerged. In the 70's there was a wave of developments.

Iskovskikh-Manin developed techniques of Noether, Fano, Segre, giving counter-example to Lüroth problem by showing 3-fold quartics are non-rational.

Iitaka attempted to classify varieties according to their Kodaira dimension. He also studied open varieties.

Mori proved Hartshorne conjecture: $\mathbb{P}^n$ are the only smooth projective varieties with ample tangent bundle T_X.

Mori introduced bend and break technique and extremal rays.

Dimension three

Fano and others studied varieties of dimension 3 in early 20th century.

It took many decades until a clear picture in dimension 3 emerged. In the 70's there was a wave of developments.

Iskovskikh-Manin developed techniques of Noether, Fano, Segre, giving counter-example to Lüroth problem by showing 3-fold quartics are non-rational.

Iitaka attempted to classify varieties according to their Kodaira dimension. He also studied open varieties.

Mori proved Hartshorne conjecture: $\mathbb{P}^n$ are the only smooth projective varieties with ample tangent bundle T_X.

Mori introduced bend and break technique and extremal rays.

By the early 90's the classification theory of 3-folds was well-developed.

Dimension three

Fano and others studied varieties of dimension 3 in early 20th century.

It took many decades until a clear picture in dimension 3 emerged. In the 70's there was a wave of developments.

Iskovskikh-Manin developed techniques of Noether, Fano, Segre, giving counter-example to Lüroth problem by showing 3-fold quartics are non-rational.

Iitaka attempted to classify varieties according to their Kodaira dimension. He also studied open varieties.

Mori proved Hartshorne conjecture: $\mathbb{P}^n$ are the only smooth projective varieties with ample tangent bundle T_X.

Mori introduced bend and break technique and extremal rays.

By the early 90's the classification theory of 3-folds was well-developed.

Dimension three

Fano and others studied varieties of dimension 3 in early 20th century.

It took many decades until a clear picture in dimension 3 emerged. In the 70's there was a wave of developments.

Iskovskikh-Manin developed techniques of Noether, Fano, Segre, giving counter-example to Lüroth problem by showing 3-fold quartics are non-rational.

Iitaka attempted to classify varieties according to their Kodaira dimension. He also studied open varieties.

Mori proved Hartshorne conjecture: $\mathbb{P}^n$ are the only smooth projective varieties with ample tangent bundle T_X.

Mori introduced bend and break technique and extremal rays.

By the early 90's the classification theory of 3-folds was well-developed.

Special varieties

Special varieties

Let X be a projective variety with "good" singularities.

Special varieties

Let X be a projective variety with "good" singularities.

$$\text{We say } X \text{ is } \begin{cases} \textbf{Fano} & \text{if } K_X \text{ is anti-ample, eg } \mathbb{P}^n \\ \textbf{Calabi-Yau} & \text{if } K_X \text{ is trivial, eg abelian varieties} \\ \textbf{canonically polarised} & \text{if } K_X \text{ is ample} \end{cases}$$

Special varieties

Let X be a projective variety with "good" singularities.

We say X is $\begin{cases} \textbf{Fano} & \text{if } K_X \text{ is anti-ample, eg } \mathbb{P}^n \\ \textbf{Calabi-Yau} & \text{if } K_X \text{ is trivial, eg abelian varieties} \\ \textbf{canonically polarised} & \text{if } K_X \text{ is ample} \end{cases}$

Example: $X \subset \mathbb{P}^n$ a hypersurface of degree r:

Special varieties

Let X be a projective variety with "good" singularities.

$$\text{We say } X \text{ is } \begin{cases} \textbf{Fano} & \text{if } K_X \text{ is anti-ample, eg } \mathbb{P}^n \\ \textbf{Calabi-Yau} & \text{if } K_X \text{ is trivial, eg abelian varieties} \\ \textbf{canonically polarised} & \text{if } K_X \text{ is ample} \end{cases}$$

Example: $X \subset \mathbb{P}^n$ a hypersurface of degree r:

$$\begin{cases} r < n+1 & \Rightarrow X \text{ is Fano} \\ r = n+1 & \Rightarrow X \text{ is Calabi-Yau} \\ r > n+1 & \Rightarrow X \text{ is canonically polarised} \end{cases}$$

Special varieties

Let X be a projective variety with "good" singularities.

We say X is
$$\begin{cases} \textbf{Fano} & \text{if } K_X \text{ is anti-ample, eg } \mathbb{P}^n \\ \textbf{Calabi-Yau} & \text{if } K_X \text{ is trivial, eg abelian varieties} \\ \textbf{canonically polarised} & \text{if } K_X \text{ is ample} \end{cases}$$

Example: $X \subset \mathbb{P}^n$ a hypersurface of degree r:

$$\begin{cases} r < n+1 & \Rightarrow X \text{ is Fano} \\ r = n+1 & \Rightarrow X \text{ is Calabi-Yau} \\ r > n+1 & \Rightarrow X \text{ is canonically polarised} \end{cases}$$

Example: X a smooth projective curve:

Special varieties

Let X be a projective variety with "good" singularities.

$$\text{We say } X \text{ is } \begin{cases} \textbf{Fano} & \text{if } K_X \text{ is anti-ample, eg } \mathbb{P}^n \\ \textbf{Calabi-Yau} & \text{if } K_X \text{ is trivial, eg abelian varieties} \\ \textbf{canonically polarised} & \text{if } K_X \text{ is ample} \end{cases}$$

Example: $X \subset \mathbb{P}^n$ a hypersurface of degree r:

$$\begin{cases} r < n+1 & \Rightarrow X \text{ is Fano} \\ r = n+1 & \Rightarrow X \text{ is Calabi-Yau} \\ r > n+1 & \Rightarrow X \text{ is canonically polarised} \end{cases}$$

Example: X a smooth projective curve:

$$\begin{cases} genus = 0 \iff X \text{ Fano} & \iff X \simeq \mathbb{P}^1 \\ genus = 1 \iff X \text{ Calabi-Yau} & \iff X \text{ elliptic curve} \\ genus \geq 2 \iff X \text{ canonically polarised} & \iff X \text{ general type.} \end{cases}$$

Higher dimension

Higher dimension

Let X be a smooth projective variety.

Higher dimension

Let X be a smooth projective variety.

We like to run MMP giving a sequence of birational transformations

$$X = X_1 \overset{\text{div contraction}}{\dashrightarrow} X_2 \overset{\text{flip}}{\dashrightarrow} X_3 \dashrightarrow \cdots \dashrightarrow X_t = Y$$

Higher dimension

Let X be a smooth projective variety.

We like to run MMP giving a sequence of birational transformations

$$X = X_1 \overset{\text{div contraction}}{\dashrightarrow} X_2 \overset{\text{flip}}{\dashrightarrow} X_3 \dashrightarrow \cdots \dashrightarrow X_t = Y$$

This is a generalisation of the classical MMP for surfaces.

Higher dimension

Let X be a smooth projective variety.

We like to run MMP giving a sequence of birational transformations

$$X = X_1 \overset{\text{div contraction}}{\dashrightarrow} X_2 \overset{\text{flip}}{\dashrightarrow} X_3 \dashrightarrow \cdots \dashrightarrow X_t = Y$$

This is a generalisation of the classical MMP for surfaces.

It is expected that either

Higher dimension

Let X be a smooth projective variety.

We like to run MMP giving a sequence of birational transformations

$$X = X_1 \overset{\text{div contraction}}{\dashrightarrow} X_2 \overset{\text{flip}}{\dashrightarrow} X_3 \dashrightarrow \cdots \dashrightarrow X_t = Y$$

This is a generalisation of the classical MMP for surfaces.

It is expected that either

- we have a Fano fibration $Y \to Z$, or

Higher dimension

Let X be a smooth projective variety.

We like to run MMP giving a sequence of birational transformations

$$X = X_1 \overset{\text{div contraction}}{\dashrightarrow} X_2 \overset{\text{flip}}{\dashrightarrow} X_3 \dashrightarrow \cdots \dashrightarrow X_t = Y$$

This is a generalisation of the classical MMP for surfaces.

It is expected that either

- we have a Fano fibration $Y \to Z$, or
- we have a Calabi-Yau fibration $Y \to Z$, or

Higher dimension

Let X be a smooth projective variety.

We like to run MMP giving a sequence of birational transformations

$$X = X_1 \overset{\text{div contraction}}{\dashrightarrow} X_2 \overset{\text{flip}}{\dashrightarrow} X_3 \dashrightarrow \cdots \dashrightarrow X_t = Y$$

This is a generalisation of the classical MMP for surfaces.

It is expected that either

- we have a Fano fibration $Y \to Z$, or
- we have a Calabi-Yau fibration $Y \to Z$, or
- Y is canonically polarised.

Higher dimension

Let X be a smooth projective variety.

We like to run MMP giving a sequence of birational transformations

$$X = X_1 \overset{\text{div contraction}}{\dashrightarrow} X_2 \overset{\text{flip}}{\dashrightarrow} X_3 \dashrightarrow \cdots \dashrightarrow X_t = Y$$

This is a generalisation of the classical MMP for surfaces.

It is expected that either

- we have a Fano fibration $Y \to Z$, or
- we have a Calabi-Yau fibration $Y \to Z$, or
- Y is canonically polarised.

Running the MMP requires many local and global ingredients.

Higher dimension

Let X be a smooth projective variety.

We like to run MMP giving a sequence of birational transformations

$$X = X_1 \overset{\text{div contraction}}{\dashrightarrow} X_2 \overset{\text{flip}}{\dashrightarrow} X_3 \dashrightarrow \cdots \dashrightarrow X_t = Y$$

This is a generalisation of the classical MMP for surfaces.

It is expected that either

- we have a Fano fibration $Y \to Z$, or
- we have a Calabi-Yau fibration $Y \to Z$, or
- Y is canonically polarised.

Running the MMP requires many local and global ingredients.

Higher dimension

Higher dimension

Contractions: their existence was established by Kawamata and Shokurov.

Higher dimension

Contractions: their existence was established by Kawamata and Shokurov.

Flips: Their existence was established by Mori in dimension 3, by Shokurov in dimensions 3,4, and by Hacon, McKernan, Birkar, Cascini in any dimension.

Higher dimension

Contractions: their existence was established by Kawamata and Shokurov.

Flips: Their existence was established by Mori in dimension 3, by Shokurov in dimensions 3,4, and by Hacon, M^{c}Kernan, Birkar, Cascini in any dimension.

Finite generation: existence of flips is equivalent to a local version of this fact:

$$\bigoplus_{m \geq 0} \mathcal{O}_X(mK_X)$$

is a finitely generated $\mathbb{C}$-algebra.

Higher dimension

Contractions: their existence was established by Kawamata and Shokurov.

Flips: Their existence was established by Mori in dimension 3, by Shokurov in dimensions 3,4, and by Hacon, M^{c}Kernan, Birkar, Cascini in any dimension.

Finite generation: existence of flips is equivalent to a local version of this fact:

$$\bigoplus_{m \geq 0} \mathcal{O}_X(mK_X)$$

is a finitely generated $\mathbb{C}$-algebra.

Historical summary:

Higher dimension

Contractions: their existence was established by Kawamata and Shokurov.

Flips: Their existence was established by Mori in dimension 3, by Shokurov in dimensions 3,4, and by Hacon, M^c^Kernan, Birkar, Cascini in any dimension.

Finite generation: existence of flips is equivalent to a local version of this fact:

$$\bigoplus_{m\geq 0} \mathcal{O}_X(mK_X)$$

is a finitely generated $\mathbb{C}$-algebra.

Historical summary:

- dimension 2: (Castelnuovo, Enriques)(Zariski, Kodaira, Shafarevich, etc) 1900,

Higher dimension

Contractions: their existence was established by Kawamata and Shokurov.

Flips: Their existence was established by Mori in dimension 3, by Shokurov in dimensions 3,4, and by Hacon, M^{c}Kernan, Birkar, Cascini in any dimension.

Finite generation: existence of flips is equivalent to a local version of this fact:

$$\bigoplus_{m \geq 0} \mathcal{O}_X(mK_X)$$

is a finitely generated $\mathbb{C}$-algebra.

Historical summary:

- dimension 2: (Castelnuovo, Enriques)(Zariski, Kodaira, Shafarevich, etc) 1900,
- dimension 3 (Kawamata, Kollár, Miyaoka, Mori, Reid, Shokurov)(Fano, Hironaka, Iitaka, Iskovskikh, Manin, etc) 1970's-1990's,

Higher dimension

Contractions: their existence was established by Kawamata and Shokurov.

Flips: Their existence was established by Mori in dimension 3, by Shokurov in dimensions 3,4, and by Hacon, M^{c}Kernan, Birkar, Cascini in any dimension.

Finite generation: existence of flips is equivalent to a local version of this fact:

$$\bigoplus_{m \geq 0} \mathcal{O}_X(mK_X)$$

is a finitely generated $\mathbb{C}$-algebra.

Historical summary:

- dimension 2: (Castelnuovo, Enriques)(Zariski, Kodaira, Shafarevich, etc) 1900,
- dimension 3 (Kawamata, Kollár, Miyaoka, Mori, Reid, Shokurov)(Fano, Hironaka, Iitaka, Iskovskikh, Manin, etc) 1970's-1990's,
- any dimension for X of general type (BCHM=B-Cascini-Hacon-M^{c}Kernan, after Shokurov, etc) 2006.

Birational geometry of pairs

Birational geometry of pairs

Calssification of varieties is best understood in the context of birational geometry.

Birational geometry of pairs

Calssification of varieties is best understood in the context of birational geometry.

A **pair** (X, B) consists of a normal variety X and a boundary divisor B with coefficients in $[0, 1]$.

Birational geometry of pairs

Calssification of varieties is best understood in the context of birational geometry.

A **pair** (X, B) consists of a normal variety X and a boundary divisor B with coefficients in $[0, 1]$.

Singularities of (X, B) are defined by taking a log resolution $\phi\colon W \to X$ and writing $K_W + B_W = \phi^*(K_X + B)$.

Birational geometry of pairs

Calssification of varieties is best understood in the context of birational geometry.

A **pair** (X, B) consists of a normal variety X and a boundary divisor B with coefficients in $[0, 1]$.

Singularities of (X, B) are defined by taking a log resolution $\phi\colon W \to X$ and writing $K_W + B_W = \phi^*(K_X + B)$.

We say (X, B) is **lc** if every coefficient of B_W is ≤ 1.

Birational geometry of pairs

Calssification of varieties is best understood in the context of birational geometry.

A **pair** (X, B) consists of a normal variety X and a boundary divisor B with coefficients in $[0, 1]$.

Singularities of (X, B) are defined by taking a log resolution $\phi\colon W \to X$ and writing $K_W + B_W = \phi^*(K_X + B)$.

We say (X, B) is **lc** if every coefficient of B_W is ≤ 1.

Given a projective lc pair (X, B), standard conjectures of birational geometry say that there is a birational transformation

$$(X, B) \dashrightarrow (X', B')$$

such that either

- (X', B') admits a **Mori-Fano** fibration, or
- (X', B') is a **good minimal model**.

Birational geometry of pairs

Calssification of varieties is best understood in the context of birational geometry.

A **pair** (X, B) consists of a normal variety X and a boundary divisor B with coefficients in $[0, 1]$.

Singularities of (X, B) are defined by taking a log resolution $\phi\colon W \to X$ and writing $K_W + B_W = \phi^*(K_X + B)$.

We say (X, B) is **lc** if every coefficient of B_W is ≤ 1.

Given a projective lc pair (X, B), standard conjectures of birational geometry say that there is a birational transformation

$$(X, B) \dashrightarrow (X', B')$$

such that either

- (X', B') admits a **Mori-Fano** fibration, or
- (X', B') is a **good minimal model**.

The latter means $m(K_{X'} + B')$ is generated by global sections for some $m \in \mathbb{N}$, defining a Calabi-Yau fibration $(X', B') \to Z$

Birational geometry of pairs

Calssification of varieties is best understood in the context of birational geometry.

A **pair** (X, B) consists of a normal variety X and a boundary divisor B with coefficients in $[0, 1]$.

Singularities of (X, B) are defined by taking a log resolution $\phi\colon W \to X$ and writing $K_W + B_W = \phi^*(K_X + B)$.

We say (X, B) is **lc** if every coefficient of B_W is ≤ 1.

Given a projective lc pair (X, B), standard conjectures of birational geometry say that there is a birational transformation

$$(X, B) \dashrightarrow (X', B')$$

such that either

- (X', B') admits a **Mori-Fano** fibration, or
- (X', B') is a **good minimal model**.

The latter means $m(K_{X'} + B')$ is generated by global sections for some $m \in \mathbb{N}$, defining a Calabi-Yau fibration $(X', B') \to Z$

Birational geometry of pairs

Birational geometry of pairs

To construst moduli spaces one usually restricts attention to good minimal models.

Birational geometry of pairs

To construst moduli spaces one usually restricts attention to good minimal models.

Given a good minimal model (X, B), its **Kodaira dimension** $\kappa(X, B)$ is the dimension of the base Z of the corresponding Calabi-Yau fibration.

Birational geometry of pairs

To construst moduli spaces one usually restricts attention to good minimal models.

Given a good minimal model (X, B), its **Kodaira dimension** $\kappa(X, B)$ is the dimension of the base Z of the corresponding Calabi-Yau fibration.

$\kappa(X, B) \in \{0, 1, \ldots, \dim X\}$.

Birational geometry of pairs

To construst moduli spaces one usually restricts attention to good minimal models.

Given a good minimal model (X, B), its **Kodaira dimension** $\kappa(X, B)$ is the dimension of the base Z of the corresponding Calabi-Yau fibration.

$\kappa(X, B) \in \{0, 1, \ldots, \dim X\}$.

In dimension one, (X, B) being a good minimal model means $\deg K_X + B \geq 0$.

Birational geometry of pairs

To construst moduli spaces one usually restricts attention to good minimal models.

Given a good minimal model (X, B), its **Kodaira dimension** $\kappa(X, B)$ is the dimension of the base Z of the corresponding Calabi-Yau fibration.

$$\kappa(X, B) \in \{0, 1, \dots, \dim X\}.$$

In dimension one, (X, B) being a good minimal model means $\deg K_X + B \geq 0$.

- $\kappa(X, B) = 0$ iff $\deg K_X + B = 0$,
- $\kappa(X, B) = 1$ iff $\deg K_X + B > 0$.

In dimension two: for a good minimal model $(X, B) \to Z$,

Birational geometry of pairs

To construst moduli spaces one usually restricts attention to good minimal models.

Given a good minimal model (X, B), its **Kodaira dimension** $\kappa(X, B)$ is the dimension of the base Z of the corresponding Calabi-Yau fibration.

$\kappa(X, B) \in \{0, 1, \ldots, \dim X\}$.

In dimension one, (X, B) being a good minimal model means $\deg K_X + B \geq 0$.

- $\kappa(X, B) = 0$ iff $\deg K_X + B = 0$,
- $\kappa(X, B) = 1$ iff $\deg K_X + B > 0$.

In dimension two: for a good minimal model $(X, B) \to Z$,

- $\kappa(X, B) = 0$ iff $K_X + B \equiv 0$ iff (X, B) is Calabi-Yau,
- $\kappa(X, B) = 1$ iff $(X + B) \to Z$ is an elliptic fibration,
- $\kappa(X, B) = 2$ iff $(X + B) \to Z$ is birational.

Birational geometry of pairs

To construst moduli spaces one usually restricts attention to good minimal models.

Given a good minimal model (X, B), its **Kodaira dimension** $\kappa(X, B)$ is the dimension of the base Z of the corresponding Calabi-Yau fibration.

$\kappa(X, B) \in \{0, 1, \ldots, \dim X\}$.

In dimension one, (X, B) being a good minimal model means $\deg K_X + B \geq 0$.

- $\kappa(X, B) = 0$ iff $\deg K_X + B = 0$,
- $\kappa(X, B) = 1$ iff $\deg K_X + B > 0$.

In dimension two: for a good minimal model $(X, B) \to Z$,

- $\kappa(X, B) = 0$ iff $K_X + B \equiv 0$ iff (X, B) is Calabi-Yau,
- $\kappa(X, B) = 1$ iff $(X + B) \to Z$ is an elliptic fibration,
- $\kappa(X, B) = 2$ iff $(X + B) \to Z$ is birational.

Birational geometry of pairs

To construst moduli spaces one usually restricts attention to good minimal models.

Given a good minimal model (X, B), its **Kodaira dimension** $\kappa(X, B)$ is the dimension of the base Z of the corresponding Calabi-Yau fibration.

$\kappa(X, B) \in \{0, 1, \ldots, \dim X\}$.

In dimension one, (X, B) being a good minimal model means $\deg K_X + B \geq 0$.

- $\kappa(X, B) = 0$ iff $\deg K_X + B = 0$,
- $\kappa(X, B) = 1$ iff $\deg K_X + B > 0$.

In dimension two: for a good minimal model $(X, B) \to Z$,

- $\kappa(X, B) = 0$ iff $K_X + B \equiv 0$ iff (X, B) is Calabi-Yau,
- $\kappa(X, B) = 1$ iff $(X + B) \to Z$ is an elliptic fibration,
- $\kappa(X, B) = 2$ iff $(X + B) \to Z$ is birational.

Boundedness

Boundedness

A class of varieties is **bounded** if can be parametrised by a finite dimensional space.

Boundedness

A class of varieties is **bounded** if can be parametrised by a finite dimensional space.

For example, the set of smooth projective curves of genus g is bounded.

Boundedness

A class of varieties is **bounded** if can be parametrised by a finite dimensional space.

For example, the set of smooth projective curves of genus g is bounded.

Boundedness often helps with proving statements, e.g. the groups $\mathrm{Bir}(\mathbb{P}^d)$ are Jordan.

Boundedness

A class of varieties is **bounded** if can be parametrised by a finite dimensional space.

For example, the set of smooth projective curves of genus g is bounded.

Boundedness often helps with proving statements, e.g. the groups $\mathrm{Bir}(\mathbb{P}^d)$ are Jordan.

It is also usually the first step of construction of moduli spaces.

Boundedness

A class of varieties is **bounded** if can be parametrised by a finite dimensional space.

For example, the set of smooth projective curves of genus g is bounded.

Boundedness often helps with proving statements, e.g. the groups $\mathrm{Bir}(\mathbb{P}^d)$ are Jordan.

It is also usually the first step of construction of moduli spaces.

A classical boundedness result is that Fano surfaces form a bounded family.

Boundedness

A class of varieties is **bounded** if can be parametrised by a finite dimensional space.

For example, the set of smooth projective curves of genus g is bounded.

Boundedness often helps with proving statements, e.g. the groups $\mathrm{Bir}(\mathbb{P}^d)$ are Jordan.

It is also usually the first step of construction of moduli spaces.

A classical boundedness result is that Fano surfaces form a bounded family.

But singular Fano surfaces do not form a bounded family, e.g. consider projective cones over $\mathbb{P}^1$.

Boundedness

A class of varieties is **bounded** if can be parametrised by a finite dimensional space.

For example, the set of smooth projective curves of genus g is bounded.

Boundedness often helps with proving statements, e.g. the groups $\mathrm{Bir}(\mathbb{P}^d)$ are Jordan.

It is also usually the first step of construction of moduli spaces.

A classical boundedness result is that Fano surfaces form a bounded family.

But singular Fano surfaces do not form a bounded family, e.g. consider projective cones over $\mathbb{P}^1$.

It is also well-known that K3 surfaces do not form a bounded family.

Boundedness

Boundedness

In the last decade there has been tremendous progress on boundedness of various classes.

Boundedness

In the last decade there has been tremendous progress on boundedness of various classes.

Hacon-M^{c}Kernan-Xu: canonically polarised varieties Y of fixed dimension d and fixed volume K_Y^d form a bounded family.

Boundedness

In the last decade there has been tremendous progress on boundedness of various classes.

Hacon-McKernan-Xu: canonically polarised varieties Y of fixed dimension d and fixed volume K_Y^d form a bounded family.

Birkar: Fano varieties of fixed dimension d with ϵ-lc singularities form a bounded family ($\epsilon > 0$).

Boundedness

In the last decade there has been tremendous progress on boundedness of various classes.

Hacon-McKernan-Xu: canonically polarised varieties Y of fixed dimension d and fixed volume K_Y^d form a bounded family.

Birkar: Fano varieties of fixed dimension d with ϵ-lc singularities form a bounded family ($\epsilon > 0$).

Birkar: Calabi-Yau varieties of fixed dimension d polarised by ample divisor A of fixed volume A^d form a bounded family.

Boundedness

In the last decade there has been tremendous progress on boundedness of various classes.

Hacon-McKernan-Xu: canonically polarised varieties Y of fixed dimension d and fixed volume K_Y^d form a bounded family.

Birkar: Fano varieties of fixed dimension d with ϵ-lc singularities form a bounded family ($\epsilon > 0$).

Birkar: Calabi-Yau varieties of fixed dimension d polarised by ample divisor A of fixed volume A^d form a bounded family.

Birkar-Di Cerbo-Svaldi: strict Calabi-Yau manifolds of fixed dimension d admitting an elliptic fibration form a bounded family up to isomorphism in codimension one.

Moduli

Moduli

After the MMP the idea is to classify the outcomes Y, e.g. form their moduli spaces.

Moduli

After the MMP the idea is to classify the outcomes Y, e.g. form their moduli spaces.

The above results on boundedness is the first step.

Moduli

After the MMP the idea is to classify the outcomes Y, e.g. form their moduli spaces.

The above results on boundedness is the first step.

Until recently, moduli theory was limited to dimension two and very special varieties in higher dimension, e.g. abelian varieties.

Moduli

After the MMP the idea is to classify the outcomes Y, e.g. form their moduli spaces.

The above results on boundedness is the first step.

Until recently, moduli theory was limited to dimension two and very special varieties in higher dimension, e.g. abelian varieties.

Kollár, Alexeev, Viehweg, etc have developed a general theory of moduli of varieties.

Moduli

After the MMP the idea is to classify the outcomes Y, e.g. form their moduli spaces.

The above results on boundedness is the first step.

Until recently, moduli theory was limited to dimension two and very special varieties in higher dimension, e.g. abelian varieties.

Kollár, Alexeev, Viehweg, etc have developed a general theory of moduli of varieties.

Combining MMP, boundedness and moduli theory gives: moduli spaces exist for canonically polarised varieties and for polarised Calabi-Yau varieties.

Moduli

After the MMP the idea is to classify the outcomes Y, e.g. form their moduli spaces.

The above results on boundedness is the first step.

Until recently, moduli theory was limited to dimension two and very special varieties in higher dimension, e.g. abelian varieties.

Kollár, Alexeev, Viehweg, etc have developed a general theory of moduli of varieties.

Combining MMP, boundedness and moduli theory gives: moduli spaces exist for canonically polarised varieties and for polarised Calabi-Yau varieties.

We try to make this more precise.

Moduli of stable pairs

Moduli of stable pairs

It is well-known that there is a moduli space M_g for smooth projective curves of genus g.

Moduli of stable pairs

It is well-known that there is a moduli space M_g for smooth projective curves of genus g.

To compactify M_g one considers stable curves: these are connected curves of genus g with nodal singularities and some positivity condition.

Moduli of stable pairs

It is well-known that there is a moduli space M_g for smooth projective curves of genus g.

To compactify M_g one considers stable curves: these are connected curves of genus g with nodal singularities and some positivity condition.

Stable pairs are higher dimensional generalisations of stable curves.

Moduli of stable pairs

It is well-known that there is a moduli space M_g for smooth projective curves of genus g.

To compactify M_g one considers stable curves: these are connected curves of genus g with nodal singularities and some positivity condition.

Stable pairs are higher dimensional generalisations of stable curves.

Fix $d \in \mathbb{N}$ and $c, v \in \mathbb{Q}^{>0}$.

Moduli of stable pairs

It is well-known that there is a moduli space M_g for smooth projective curves of genus g.

To compactify M_g one considers stable curves: these are connected curves of genus g with nodal singularities and some positivity condition.

Stable pairs are higher dimensional generalisations of stable curves.

Fix $d \in \mathbb{N}$ and $c, v \in \mathbb{Q}^{>0}$.

A (d, c, v)-**KSBA-stable pair** is a connected projective pure dim pair (X, B) with

Moduli of stable pairs

It is well-known that there is a moduli space M_g for smooth projective curves of genus g.

To compactify M_g one considers stable curves: these are connected curves of genus g with nodal singularities and some positivity condition.

Stable pairs are higher dimensional generalisations of stable curves.

Fix $d \in \mathbb{N}$ and $c, v \in \mathbb{Q}^{>0}$.

A (d, c, v)-**KSBA-stable pair** is a connected projective pure dim pair (X, B) with

- (X, B) slc of dimension d,

Moduli of stable pairs

It is well-known that there is a moduli space M_g for smooth projective curves of genus g.

To compactify M_g one considers stable curves: these are connected curves of genus g with nodal singularities and some positivity condition.

Stable pairs are higher dimensional generalisations of stable curves.

Fix $d \in \mathbb{N}$ and $c, v \in \mathbb{Q}^{>0}$.

A (d, c, v)-**KSBA-stable pair** is a connected projective pure dim pair (X, B) with

- (X, B) slc of dimension d,
- $B = cD$ for some integral divisor D,

Moduli of stable pairs

It is well-known that there is a moduli space M_g for smooth projective curves of genus g.

To compactify M_g one considers stable curves: these are connected curves of genus g with nodal singularities and some positivity condition.

Stable pairs are higher dimensional generalisations of stable curves.

Fix $d \in \mathbb{N}$ and $c, v \in \mathbb{Q}^{>0}$.

A (d, c, v)-**KSBA-stable pair** is a connected projective pure dim pair (X, B) with

- (X, B) slc of dimension d,
- $B = cD$ for some integral divisor D,
- $K_X + B$ is ample with volume $\mathrm{vol}(K_X + B) := (K_X + B)^d = v$.

Moduli of stable pairs

It is well-known that there is a moduli space M_g for smooth projective curves of genus g.

To compactify M_g one considers stable curves: these are connected curves of genus g with nodal singularities and some positivity condition.

Stable pairs are higher dimensional generalisations of stable curves.

Fix $d \in \mathbb{N}$ and $c, v \in \mathbb{Q}^{>0}$.

A (d, c, v)-**KSBA-stable pair** is a connected projective pure dim pair (X, B) with

- (X, B) slc of dimension d,
- $B = cD$ for some integral divisor D,
- $K_X + B$ is ample with volume $\mathrm{vol}(K_X + B) := (K_X + B)^d = v$.

Moduli of stable pairs

Moduli of stable pairs

It takes more work to define (d, c, v)-stable families $(X, B) \to S$. Roughly this is a flat projective family with (d, c, v)-stable log fibres.

Moduli of stable pairs

It takes more work to define (d, c, v)-stable families $(X, B) \to S$. Roughly this is a flat projective family with (d, c, v)-stable log fibres.

Example: $(X, B = \sum x_i)$ an n-marked stable curve of genus g.

Moduli of stable pairs

It takes more work to define (d, c, v)-stable families $(X, B) \to S$. Roughly this is a flat projective family with (d, c, v)-stable log fibres.

Example: $(X, B = \sum x_i)$ an n-marked stable curve of genus g. Then (X, B) is a $(1, 1, v)$-stable pair with $v = \deg K_X + B$.

Moduli of stable pairs

It takes more work to define (d, c, v)-stable families $(X, B) \to S$. Roughly this is a flat projective family with (d, c, v)-stable log fibres.

Example: $(X, B = \sum x_i)$ an n-marked stable curve of genus g. Then (X, B) is a $(1, 1, v)$-stable pair with $v = \deg K_X + B$.

Example: $X \subset \mathbb{P}^{d+1}$ hypersurface of degree r, $B \subset X$ general hyperplane section.

Moduli of stable pairs

It takes more work to define (d, c, v)-stable families $(X, B) \to S$. Roughly this is a flat projective family with (d, c, v)-stable log fibres.

Example: $(X, B = \sum x_i)$ an n-marked stable curve of genus g. Then (X, B) is a $(1, 1, v)$-stable pair with $v = \deg K_X + B$.

Example: $X \subset \mathbb{P}^{d+1}$ hypersurface of degree r, $B \subset X$ general hyperplane section. Then (X, B) is a $(d, 1, v)$-stable pair with $v = (r - d - 1)^d r$.

Moduli of stable pairs

It takes more work to define (d, c, v)-stable families $(X, B) \to S$. Roughly this is a flat projective family with (d, c, v)-stable log fibres.

Example: $(X, B = \sum x_i)$ an n-marked stable curve of genus g. Then (X, B) is a $(1, 1, v)$-stable pair with $v = \deg K_X + B$.

Example: $X \subset \mathbb{P}^{d+1}$ hypersurface of degree r, $B \subset X$ general hyperplane section. Then (X, B) is a $(d, 1, v)$-stable pair with $v = (r - d - 1)^d r$.

Kollár, Alexeev, et al: There is a projective coarse moduli space for (d, c, v)-KSBA-stable pairs.

Moduli of stable pairs

It takes more work to define (d, c, v)-stable families $(X, B) \to S$. Roughly this is a flat projective family with (d, c, v)-stable log fibres.

Example: $(X, B = \sum x_i)$ an n-marked stable curve of genus g. Then (X, B) is a $(1, 1, v)$-stable pair with $v = \deg K_X + B$.

Example: $X \subset \mathbb{P}^{d+1}$ hypersurface of degree r, $B \subset X$ general hyperplane section. Then (X, B) is a $(d, 1, v)$-stable pair with $v = (r - d - 1)^d r$.

Kollár, Alexeev, et al: There is a projective coarse moduli space for (d, c, v)-KSBA-stable pairs.

Moduli of stable pairs

It takes more work to define (d, c, v)-stable families $(X, B) \to S$. Roughly this is a flat projective family with (d, c, v)-stable log fibres.

Example: $(X, B = \sum x_i)$ an n-marked stable curve of genus g. Then (X, B) is a $(1, 1, v)$-stable pair with $v = \deg K_X + B$.

Example: $X \subset \mathbb{P}^{d+1}$ hypersurface of degree r, $B \subset X$ general hyperplane section. Then (X, B) is a $(d, 1, v)$-stable pair with $v = (r - d - 1)^d r$.

Kollár, Alexeev, et al: There is a projective coarse moduli space for (d, c, v)-KSBA-stable pairs.

Moduli of stable Calabi-Yau pairs

Moduli of stable Calabi-Yau pairs

Next we treat moduli of Calabi-Yau pairs.

Moduli of stable Calabi-Yau pairs

Next we treat moduli of Calabi-Yau pairs.

Calabi-Yau pairs do not carry a natural ample divisor, so we need to add one.

Moduli of stable Calabi-Yau pairs

Next we treat moduli of Calabi-Yau pairs.

Calabi-Yau pairs do not carry a natural ample divisor, so we need to add one.

Fix $d \in \mathbb{N}$ and $c, v \in \mathbb{Q}^{>0}$.

Moduli of stable Calabi-Yau pairs

Next we treat moduli of Calabi-Yau pairs.

Calabi-Yau pairs do not carry a natural ample divisor, so we need to add one.

Fix $d \in \mathbb{N}$ and $c, v \in \mathbb{Q}^{>0}$.

A (d, c, v)**-stable Calabi-Yau pair** $(X, B), A$ is defined by the data:

Moduli of stable Calabi-Yau pairs

Next we treat moduli of Calabi-Yau pairs.

Calabi-Yau pairs do not carry a natural ample divisor, so we need to add one.

Fix $d \in \mathbb{N}$ and $c, v \in \mathbb{Q}^{>0}$.

A (d, c, v)**-stable Calabi-Yau pair** $(X, B), A$ is defined by the data:

- (X, B) is slc of dimension d with $K_X + B \sim_{\mathbb{Q}} 0$,

Moduli of stable Calabi-Yau pairs

Next we treat moduli of Calabi-Yau pairs.

Calabi-Yau pairs do not carry a natural ample divisor, so we need to add one.

Fix $d \in \mathbb{N}$ and $c, v \in \mathbb{Q}^{>0}$.

A (d, c, v)**-stable Calabi-Yau pair** $(X, B), A$ is defined by the data:

- (X, B) is slc of dimension d with $K_X + B \sim_{\mathbb{Q}} 0$,
- $B = cD$ for some integral divisor $D \geq 0$,

Moduli of stable Calabi-Yau pairs

Next we treat moduli of Calabi-Yau pairs.

Calabi-Yau pairs do not carry a natural ample divisor, so we need to add one.

Fix $d \in \mathbb{N}$ and $c, v \in \mathbb{Q}^{>0}$.

A (d, c, v)**-stable Calabi-Yau pair** $(X, B), A$ is defined by the data:

- (X, B) is slc of dimension d with $K_X + B \sim_{\mathbb{Q}} 0$,
- $B = cD$ for some integral divisor $D \geq 0$,
- $A \geq 0$ is an ample integral divisor with volume $\mathrm{vol}(A) = v$,

Moduli of stable Calabi-Yau pairs

Next we treat moduli of Calabi-Yau pairs.

Calabi-Yau pairs do not carry a natural ample divisor, so we need to add one.

Fix $d \in \mathbb{N}$ and $c, v \in \mathbb{Q}^{>0}$.

A (d, c, v)**-stable Calabi-Yau pair** $(X, B), A$ is defined by the data:

- (X, B) is slc of dimension d with $K_X + B \sim_{\mathbb{Q}} 0$,
- $B = cD$ for some integral divisor $D \geq 0$,
- $A \geq 0$ is an ample integral divisor with volume $\mathrm{vol}(A) = v$,
- $(X, B + uA)$ is slc for some $u \in \mathbb{Q}^{>0}$,

Moduli of stable Calabi-Yau pairs

Next we treat moduli of Calabi-Yau pairs.

Calabi-Yau pairs do not carry a natural ample divisor, so we need to add one.

Fix $d \in \mathbb{N}$ and $c, v \in \mathbb{Q}^{>0}$.

A (d, c, v)**-stable Calabi-Yau pair** $(X, B), A$ is defined by the data:

- (X, B) is slc of dimension d with $K_X + B \sim_{\mathbb{Q}} 0$,
- $B = cD$ for some integral divisor $D \geq 0$,
- $A \geq 0$ is an ample integral divisor with volume $\mathrm{vol}(A) = v$,
- $(X, B + uA)$ is slc for some $u \in \mathbb{Q}^{>0}$,

Moduli of stable Calabi-Yau pairs

Next we treat moduli of Calabi-Yau pairs.

Calabi-Yau pairs do not carry a natural ample divisor, so we need to add one.

Fix $d \in \mathbb{N}$ and $c, v \in \mathbb{Q}^{>0}$.

A (d, c, v)**-stable Calabi-Yau pair** $(X, B), A$ is defined by the data:

- (X, B) is slc of dimension d with $K_X + B \sim_{\mathbb{Q}} 0$,
- $B = cD$ for some integral divisor $D \geq 0$,
- $A \geq 0$ is an ample integral divisor with volume $\mathrm{vol}(A) = v$,
- $(X, B + uA)$ is slc for some $u \in \mathbb{Q}^{>0}$,

Example: X an elliptic curve, $B = 0$, $A \in X$ a point.

Moduli of stable Calabi-Yau pairs

Next we treat moduli of Calabi-Yau pairs.

Calabi-Yau pairs do not carry a natural ample divisor, so we need to add one.

Fix $d \in \mathbb{N}$ and $c, v \in \mathbb{Q}^{>0}$.

A (d, c, v)**-stable Calabi-Yau pair** $(X, B), A$ is defined by the data:

- (X, B) is slc of dimension d with $K_X + B \sim_{\mathbb{Q}} 0$,
- $B = cD$ for some integral divisor $D \geq 0$,
- $A \geq 0$ is an ample integral divisor with volume $\mathrm{vol}(A) = v$,
- $(X, B + uA)$ is slc for some $u \in \mathbb{Q}^{>0}$,

Example: X an elliptic curve, $B = 0$, $A \in X$ a point. Then $(X, B), A$ is a $(1, 1, 1)$-stable Calabi-Yau pair.

Moduli of stable Calabi-Yau pairs

Next we treat moduli of Calabi-Yau pairs.

Calabi-Yau pairs do not carry a natural ample divisor, so we need to add one.

Fix $d \in \mathbb{N}$ and $c, v \in \mathbb{Q}^{>0}$.

A (d, c, v)**-stable Calabi-Yau pair** $(X, B), A$ is defined by the data:

- (X, B) is slc of dimension d with $K_X + B \sim_{\mathbb{Q}} 0$,
- $B = cD$ for some integral divisor $D \geq 0$,
- $A \geq 0$ is an ample integral divisor with volume $\mathrm{vol}(A) = v$,
- $(X, B + uA)$ is slc for some $u \in \mathbb{Q}^{>0}$,

Example: X an elliptic curve, $B = 0$, $A \in X$ a point. Then $(X, B), A$ is a $(1, 1, 1)$-stable Calabi-Yau pair.

Moduli of stable Calabi-Yau pairs

Moduli of stable Calabi-Yau pairs

Example: $X = \mathbb{P}^2$, $B \subset X$ an elliptic curve, $A \subset X$ a conic.

Moduli of stable Calabi-Yau pairs

Example: $X = \mathbb{P}^2$, $B \subset X$ an elliptic curve, $A \subset X$ a conic. Then $(X, B), A$ is a $(2, 1, 4)$-stable Calabi-Yau pair.

Moduli of stable Calabi-Yau pairs

Example: $X = \mathbb{P}^2$, $B \subset X$ an elliptic curve, $A \subset X$ a conic. Then $(X, B), A$ is a $(2, 1, 4)$-stable Calabi-Yau pair.

Birkar: There is a projective coarse moduli space for (d, c, v)-stable Calabi-Yau pairs.

Moduli of stable Calabi-Yau pairs

Example: $X = \mathbb{P}^2$, $B \subset X$ an elliptic curve, $A \subset X$ a conic. Then $(X, B), A$ is a $(2, 1, 4)$-stable Calabi-Yau pair.

Birkar: There is a projective coarse moduli space for (d, c, v)-stable Calabi-Yau pairs.

This is a consequence of the moduli theory of KSBA stable pairs together with the following.

Moduli of stable Calabi-Yau pairs

Example: $X = \mathbb{P}^2$, $B \subset X$ an elliptic curve, $A \subset X$ a conic. Then $(X, B), A$ is a $(2, 1, 4)$-stable Calabi-Yau pair.

Birkar: There is a projective coarse moduli space for (d, c, v)-stable Calabi-Yau pairs.

This is a consequence of the moduli theory of KSBA stable pairs together with the following.

Birkar: The (d, c, v)-stable Calabi-Yau pairs form a bounded family.

Moduli of stable Calabi-Yau pairs

Example: $X = \mathbb{P}^2$, $B \subset X$ an elliptic curve, $A \subset X$ a conic. Then $(X, B), A$ is a $(2, 1, 4)$-stable Calabi-Yau pair.

Birkar: There is a projective coarse moduli space for (d, c, v)-stable Calabi-Yau pairs.

This is a consequence of the moduli theory of KSBA stable pairs together with the following.

Birkar: The (d, c, v)-stable Calabi-Yau pairs form a bounded family.

Given a (d, c, v)-stable Calabi-Yau pair $(X, B), A$, the boundedness implies $(X, B + tA)$ is a KSBA stable pair, for some fixed $t \in \mathbb{Q}^{>0}$.

Moduli of stable Calabi-Yau pairs

Example: $X = \mathbb{P}^2$, $B \subset X$ an elliptic curve, $A \subset X$ a conic. Then $(X, B), A$ is a $(2, 1, 4)$-stable Calabi-Yau pair.

Birkar: There is a projective coarse moduli space for (d, c, v)-stable Calabi-Yau pairs.

This is a consequence of the moduli theory of KSBA stable pairs together with the following.

Birkar: The (d, c, v)-stable Calabi-Yau pairs form a bounded family.

Given a (d, c, v)-stable Calabi-Yau pair $(X, B), A$, the boundedness implies $(X, B + tA)$ is a KSBA stable pair, for some fixed $t \in \mathbb{Q}^{>0}$.

Restricting the family of (d, c, v)-stable Calabi-Yau pairs to special situations gives many interesting examples of moduli spaces, e.g. Fano varieties.

Generalised pairs

Generalised pairs

The theory of generalised pairs has played a fundamental role in recent progress in birational geometry.

Generalised pairs

The theory of generalised pairs has played a fundamental role in recent progress in birational geometry.

It has been applied to a wide range of problems on pluricanonical systems, Fano varieties and complements, Calabi-Yau varieties and fibrations, existence of minimal models, termination of flips, connectedness of singular loci, moduli, varieties with nef anti-canonical divisor, etc.

Generalised pairs

The theory of generalised pairs has played a fundamental role in recent progress in birational geometry.

It has been applied to a wide range of problems on pluricanonical systems, Fano varieties and complements, Calabi-Yau varieties and fibrations, existence of minimal models, termination of flips, connectedness of singular loci, moduli, varieties with nef anti-canonical divisor, etc.

A **generalised pair** is roughly a pair together with a nef divisor on some birational model.

Generalised pairs

The theory of generalised pairs has played a fundamental role in recent progress in birational geometry.

It has been applied to a wide range of problems on pluricanonical systems, Fano varieties and complements, Calabi-Yau varieties and fibrations, existence of minimal models, termination of flips, connectedness of singular loci, moduli, varieties with nef anti-canonical divisor, etc.

A **generalised pair** is roughly a pair together with a nef divisor on some birational model.

A generalised pair consists of

Generalised pairs

The theory of generalised pairs has played a fundamental role in recent progress in birational geometry.

It has been applied to a wide range of problems on pluricanonical systems, Fano varieties and complements, Calabi-Yau varieties and fibrations, existence of minimal models, termination of flips, connectedness of singular loci, moduli, varieties with nef anti-canonical divisor, etc.

A **generalised pair** is roughly a pair together with a nef divisor on some birational model.

A generalised pair consists of

- a normal projective variety X equipped with a contraction $X \to Z$,

Generalised pairs

The theory of generalised pairs has played a fundamental role in recent progress in birational geometry.

It has been applied to a wide range of problems on pluricanonical systems, Fano varieties and complements, Calabi-Yau varieties and fibrations, existence of minimal models, termination of flips, connectedness of singular loci, moduli, varieties with nef anti-canonical divisor, etc.

A **generalised pair** is roughly a pair together with a nef divisor on some birational model.

A generalised pair consists of

- a normal projective variety X equipped with a contraction $X \to Z$,
- a $\mathbb{Q}$-divisor $B \geq 0$ on X, and

Generalised pairs

The theory of generalised pairs has played a fundamental role in recent progress in birational geometry.

It has been applied to a wide range of problems on pluricanonical systems, Fano varieties and complements, Calabi-Yau varieties and fibrations, existence of minimal models, termination of flips, connectedness of singular loci, moduli, varieties with nef anti-canonical divisor, etc.

A **generalised pair** is roughly a pair together with a nef divisor on some birational model.

A generalised pair consists of

- a normal projective variety X equipped with a contraction $X \to Z$,
- a $\mathbb{Q}$-divisor $B \geq 0$ on X, and
- a birational contraction $\phi \colon X' \to X$ and a nef$/Z$ $\mathbb{Q}$-divisor M' on X'

Generalised pairs

The theory of generalised pairs has played a fundamental role in recent progress in birational geometry.

It has been applied to a wide range of problems on pluricanonical systems, Fano varieties and complements, Calabi-Yau varieties and fibrations, existence of minimal models, termination of flips, connectedness of singular loci, moduli, varieties with nef anti-canonical divisor, etc.

A **generalised pair** is roughly a pair together with a nef divisor on some birational model.

A generalised pair consists of

- a normal projective variety X equipped with a contraction $X \to Z$,
- a $\mathbb{Q}$-divisor $B \geq 0$ on X, and
- a birational contraction $\phi\colon X' \to X$ and a nef$/Z$ $\mathbb{Q}$-divisor M' on X'

such that $K_X + B + M$ is $\mathbb{Q}$-Cartier where $M := \phi_* M'$.

Generalised pairs

The theory of generalised pairs has played a fundamental role in recent progress in birational geometry.

It has been applied to a wide range of problems on pluricanonical systems, Fano varieties and complements, Calabi-Yau varieties and fibrations, existence of minimal models, termination of flips, connectedness of singular loci, moduli, varieties with nef anti-canonical divisor, etc.

A **generalised pair** is roughly a pair together with a nef divisor on some birational model.

A generalised pair consists of

- a normal projective variety X equipped with a contraction $X \to Z$,
- a $\mathbb{Q}$-divisor $B \geq 0$ on X, and
- a birational contraction $\phi\colon X' \to X$ and a nef$/Z$ $\mathbb{Q}$-divisor M' on X'

such that $K_X + B + M$ is $\mathbb{Q}$-Cartier where $M := \phi_* M'$.

Generalised pairs

Generalised pairs

Assuming ϕ is a log resolution of (X, B) we can write

$$K_{X'} + B' + M' = \phi^*(K_X + B + M)$$

and define singularities.

Generalised pairs

Assuming ϕ is a log resolution of (X, B) we can write

$$K_{X'} + B' + M' = \phi^*(K_X + B + M)$$

and define singularities.
We say $(X, B + M)$ is generalised lc if each coefficient of B' is ≤ 1.

Generalised pairs

Assuming ϕ is a log resolution of (X, B) we can write

$$K_{X'} + B' + M' = \phi^*(K_X + B + M)$$

and define singularities.
We say $(X, B + M)$ is generalised lc if each coefficient of B' is ≤ 1.

One of the natural ways generalised pairs appear is through canonical bundle formulae.

Generalised pairs

Assuming ϕ is a log resolution of (X, B) we can write

$$K_{X'} + B' + M' = \phi^*(K_X + B + M)$$

and define singularities.
We say $(X, B + M)$ is generalised lc if each coefficient of B' is ≤ 1.

One of the natural ways generalised pairs appear is through canonical bundle formulae.

Suppose (V, Δ) is a projective lc pair and $f\colon V \to X$ is a contraction.

Generalised pairs

Assuming ϕ is a log resolution of (X, B) we can write

$$K_{X'} + B' + M' = \phi^*(K_X + B + M)$$

and define singularities.
We say $(X, B + M)$ is generalised lc if each coefficient of B' is ≤ 1.

One of the natural ways generalised pairs appear is through canonical bundle formulae.

Suppose (V, Δ) is a projective lc pair and $f\colon V \to X$ is a contraction.

Suppose $K_V + \Delta \sim_{\mathbb{Q}} f^*L$ for some $\mathbb{Q}$-divisor L.

Generalised pairs

Assuming ϕ is a log resolution of (X, B) we can write

$$K_{X'} + B' + M' = \phi^*(K_X + B + M)$$

and define singularities.
We say $(X, B + M)$ is generalised lc if each coefficient of B' is ≤ 1.

One of the natural ways generalised pairs appear is through canonical bundle formulae.

Suppose (V, Δ) is a projective lc pair and $f\colon V \to X$ is a contraction.

Suppose $K_V + \Delta \sim_{\mathbb{Q}} f^*L$ for some $\mathbb{Q}$-divisor L.

Then the canonical bundle formula says

$$K_V + \Delta \sim_{\mathbb{Q}} f^*(K_X + B + M).$$

Usually M is not nef but it is nef on some birational model of X.

Generalised pairs

Assuming ϕ is a log resolution of (X, B) we can write

$$K_{X'} + B' + M' = \phi^*(K_X + B + M)$$

and define singularities.
We say $(X, B + M)$ is generalised lc if each coefficient of B' is ≤ 1.

One of the natural ways generalised pairs appear is through canonical bundle formulae.

Suppose (V, Δ) is a projective lc pair and $f\colon V \to X$ is a contraction.

Suppose $K_V + \Delta \sim_{\mathbb{Q}} f^*L$ for some $\mathbb{Q}$-divisor L.

Then the canonical bundle formula says

$$K_V + \Delta \sim_{\mathbb{Q}} f^*(K_X + B + M).$$

Usually M is not nef but it is nef on some birational model of X.

Then we can consider $(X, B + M)$ as a generalised pair.

Generalised pairs

Assuming ϕ is a log resolution of (X, B) we can write

$$K_{X'} + B' + M' = \phi^*(K_X + B + M)$$

and define singularities.
We say $(X, B + M)$ is generalised lc if each coefficient of B' is ≤ 1.

One of the natural ways generalised pairs appear is through canonical bundle formulae.

Suppose (V, Δ) is a projective lc pair and $f \colon V \to X$ is a contraction.

Suppose $K_V + \Delta \sim_{\mathbb{Q}} f^*L$ for some $\mathbb{Q}$-divisor L.

Then the canonical bundle formula says

$$K_V + \Delta \sim_{\mathbb{Q}} f^*(K_X + B + M).$$

Usually M is not nef but it is nef on some birational model of X.

Then we can consider $(X, B + M)$ as a generalised pair.

Example of application: effective Iitaka fibrations

Example of application: effective Iitaka fibrations

Let W be a smooth projective variety of Kodaira dimension $\kappa(W) \geq 0$.

Example of application: effective Iitaka fibrations

Let W be a smooth projective variety of Kodaira dimension $\kappa(W) \geq 0$.

The Kodaira dimension $\kappa(W)$ is the largest number $\kappa \in \{-\infty, 0, 1, \ldots, \dim W\}$ such that

$$\limsup_{m \in \mathbb{N}} \frac{h^0(X, mK_W)}{m^\kappa} > 0.$$

Example of application: effective Iitaka fibrations

Let W be a smooth projective variety of Kodaira dimension $\kappa(W) \geq 0$.

The Kodaira dimension $\kappa(W)$ is the largest number $\kappa \in \{-\infty, 0, 1, \ldots, \dim W\}$ such that

$$\limsup_{m \in \mathbb{N}} \frac{h^0(X, mK_W)}{m^\kappa} > 0.$$

By Iitaka, for sufficiently divisible $m \in \mathbb{N}$, the system $|mK_W|$ defines the **Iitaka fibration** $W \dashrightarrow X$.

Example of application: effective Iitaka fibrations

Let W be a smooth projective variety of Kodaira dimension $\kappa(W) \geq 0$.

The Kodaira dimension $\kappa(W)$ is the largest number $\kappa \in \{-\infty, 0, 1, \ldots, \dim W\}$ such that

$$\limsup_{m \in \mathbb{N}} \frac{h^0(X, mK_W)}{m^\kappa} > 0.$$

By Iitaka, for sufficiently divisible $m \in \mathbb{N}$, the system $|mK_W|$ defines the **Iitaka fibration** $W \dashrightarrow X$.

The very general fibres F of $W \dashrightarrow X$ have Kodaira dimension zero and $\dim X = \kappa(W)$.

Example of application: effective Iitaka fibrations

Let W be a smooth projective variety of Kodaira dimension $\kappa(W) \geq 0$.

The Kodaira dimension $\kappa(W)$ is the largest number $\kappa \in \{-\infty, 0, 1, \ldots, \dim W\}$ such that

$$\limsup_{m \in \mathbb{N}} \frac{h^0(X, mK_W)}{m^\kappa} > 0.$$

By Iitaka, for sufficiently divisible $m \in \mathbb{N}$, the system $|mK_W|$ defines the **Iitaka fibration** $W \dashrightarrow X$.

The very general fibres F of $W \dashrightarrow X$ have Kodaira dimension zero and $\dim X = \kappa(W)$.

B.-Zhang: we can choose m bounded if certain invariants of F are bounded.

Example of application: effective Iitaka fibrations

Let W be a smooth projective variety of Kodaira dimension $\kappa(W) \geq 0$.

The Kodaira dimension $\kappa(W)$ is the largest number $\kappa \in \{-\infty, 0, 1, \ldots, \dim W\}$ such that

$$\limsup_{m \in \mathbb{N}} \frac{h^0(X, mK_W)}{m^\kappa} > 0.$$

By Iitaka, for sufficiently divisible $m \in \mathbb{N}$, the system $|mK_W|$ defines the **Iitaka fibration** $W \dashrightarrow X$.

The very general fibres F of $W \dashrightarrow X$ have Kodaira dimension zero and $\dim X = \kappa(W)$.

B.-Zhang: we can choose m bounded if certain invariants of F are bounded.

A canonical bundle type formula gives divisors $B \geq 0$ and nef M s.t. can assume

$$H^0(W, mK_W) \simeq H^0(X, m(K_X + B + M))$$

for m divisible by some fixed number.

Example of application: effective Iitaka fibrations

Example of application: effective Iitaka fibrations

So it is enough to find bounded m s.t. $|m(K_X + B + M)|$ defines a birational map.

Example of application: effective Iitaka fibrations

So it is enough to find bounded m s.t. $|m(K_X + B + M)|$ defines a birational map.

This follows from the next more general statement.

Example of application: effective Iitaka fibrations

So it is enough to find bounded m s.t. $|m(K_X + B + M)|$ defines a birational map.

This follows from the next more general statement.

B.-Zhang: Let $d, r \in \mathbb{N}$ and let $\Phi \subset \mathbb{R}^{\geq 0}$ be a DCC set. Assume

Example of application: effective Iitaka fibrations

So it is enough to find bounded m s.t. $|m(K_X + B + M)|$ defines a birational map.

This follows from the next more general statement.

B.-Zhang: Let $d, r \in \mathbb{N}$ and let $\Phi \subset \mathbb{R}^{\geq 0}$ be a DCC set. Assume

- (X, B) is a projective lc pair of dimension d,

Example of application: effective Iitaka fibrations

So it is enough to find bounded m s.t. $|m(K_X + B + M)|$ defines a birational map.

This follows from the next more general statement.

B.-Zhang: Let $d, r \in \mathbb{N}$ and let $\Phi \subset \mathbb{R}^{\geq 0}$ be a DCC set. Assume

- (X, B) is a projective lc pair of dimension d,
- the coefficients of B are in Φ,

Example of application: effective Iitaka fibrations

So it is enough to find bounded m s.t. $|m(K_X + B + M)|$ defines a birational map.

This follows from the next more general statement.

B.-Zhang: Let $d, r \in \mathbb{N}$ and let $\Phi \subset \mathbb{R}^{\geq 0}$ be a DCC set. Assume

- (X, B) is a projective lc pair of dimension d,
- the coefficients of B are in Φ,
- rM is a nef Cartier divisor, and

Example of application: effective Iitaka fibrations

So it is enough to find bounded m s.t. $|m(K_X + B + M)|$ defines a birational map.

This follows from the next more general statement.

B.-Zhang: Let $d, r \in \mathbb{N}$ and let $\Phi \subset \mathbb{R}^{\geq 0}$ be a DCC set. Assume

- (X, B) is a projective lc pair of dimension d,
- the coefficients of B are in Φ,
- rM is a nef Cartier divisor, and
- $K_X + B + M$ is big.

Example of application: effective Iitaka fibrations

So it is enough to find bounded m s.t. $|m(K_X + B + M)|$ defines a birational map.

This follows from the next more general statement.

B.-Zhang: Let $d, r \in \mathbb{N}$ and let $\Phi \subset \mathbb{R}^{\geq 0}$ be a DCC set. Assume

- (X, B) is a projective lc pair of dimension d,
- the coefficients of B are in Φ,
- rM is a nef Cartier divisor, and
- $K_X + B + M$ is big.

Then $|m(K_X + B + M)|$ defines a birational map for some bounded $m \in \mathbb{N}$.

Example of application: effective Iitaka fibrations

So it is enough to find bounded m s.t. $|m(K_X + B + M)|$ defines a birational map.

This follows from the next more general statement.

B.-Zhang: Let $d, r \in \mathbb{N}$ and let $\Phi \subset \mathbb{R}^{\geq 0}$ be a DCC set. Assume

- (X, B) is a projective lc pair of dimension d,
- the coefficients of B are in Φ,
- rM is a nef Cartier divisor, and
- $K_X + B + M$ is big.

Then $|m(K_X + B + M)|$ defines a birational map for some bounded $m \in \mathbb{N}$.

For usual pairs, that is when $M = 0$, this was proved by Hacon-McKernan-Xu.